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Bergman spaces with exponential type weights

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Abstract

For $1 \leq p < \infty$, let A_ω^p be the weighted Bergman space associated with an exponential type weight ω satisfying

$$\int_{\mathbb{D}} |K_z(\xi)| \omega(\xi)^{1/2} \, dA(\xi) \leq C\omega(z)^{-1/2}, \quad z \in \mathbb{D},$$

where K_z is the reproducing kernel of A_ω^2. This condition allows us to obtain some interesting reproducing kernel estimates and more estimates on the solutions of the $\overline{\partial}$-equation (Theorem 2.5) for more general weight ω_*. As an application, we prove the boundedness of the Bergman projection on L_ω^p, identify the dual space of A_ω^p, and establish an atomic decomposition for it. Further, we give necessary and sufficient conditions for the boundedness and compactness of some operators acting from A_ω^p into A_ω^q, $1 \leq p, q < \infty$, such as Toeplitz and (big) Hankel operators.

MSC: 32A36; 30H35; 47B38; 41A65; 46J15

Keywords: Bergman space; Bergman kernel; Bergman projection; Atomic decomposition

1 Introduction and results

Let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$. A weight is a positive function $\omega \in L^1(\mathbb{D}, dA)$, where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized area measure on $\mathbb{D}$. For $0 < p < \infty$, the weighted Bergman space $A^p(\omega)$ consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{A^p(\omega)} = \left(\int_{\mathbb{D}} |f(z)|^p \omega(z) \, dA(z) \right)^{1/p} < \infty.$$

For $p = \infty$, we introduce the growth space $L^\infty(\omega^{1/2})$ of all measurable functions f on $\mathbb{D}$ with

$$\|f\|_{L^\infty(\omega^{1/2})} := \operatorname*{ess\,sup}_{z \in \mathbb{D}} |f(z)| \omega(z)^{1/2} < \infty,$$

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and we let

$$A^\infty(\omega^{1/2}) := L^\infty(\omega^{1/2}) \cap H(\mathbb{D}).$$

In this paper we study Bergman spaces with weights belonging to a certain class $\mathcal{W}$, which we describe now. Decreasing weights ω satisfying conditions will be specified in what follows. The class $\mathcal{W}$, considered previously in [24] and [12], consists of the radial decreasing weights of the form $\omega(z) = e^{-2\varphi(z)}$, where $\varphi \in C^2(\mathbb{D})$ is a radial function such that $(\Delta\varphi(z))^{-1/2} \asymp \tau(z)$ for some positive radial function $\tau(z)$ that decreases to 0 as $|z| \to 1^-$ and satisfies $\lim_{r\to 1^-} \tau'(r) = 0$. Here Δ denotes the standard Laplace operator. Furthermore, we assume that either there exists a constant $C > 0$ such that $\tau(r)(1-r)^{-C}$ increases for r close to 1 or

$$\lim_{r\to 1^-} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

The prototype is the exponential weight

$$\omega_\sigma(z) = \exp\left(\frac{-b}{(1-|z|^2)^\sigma}\right), \quad b, \sigma > 0. \tag{1.1}$$

For the weights ω in our class, the point evaluations $L_z : f \longmapsto f(z)$ are bounded linear functionals on $A^2(\omega)$ for each $z \in \mathbb{D}$. In particular, the space $A^2(\omega)$ is a reproducing kernel Hilbert space: for each $z \in \mathbb{D}$, there are functions $K_z \in A^2(\omega)$ with $\|L_z\| = \|K_z\|_{A^2(\omega)}$ such that $L_z f = f(z) = \langle f, K_z \rangle_\omega$, where

$$\langle f, g \rangle_\omega = \int_{\mathbb{D}} f(z)\overline{g(z)}\omega(z)\,dA(z).$$

The function K_z is called the reproducing kernel for the Bergman space $A^2(\omega)$ and has the property that $K_z(\xi) = \overline{K_\xi(z)}$. The Bergman spaces with exponential type weights have attracted a lot of attention in recent years [9, 12, 16, 17, 24, 25] since new techniques, different from those used for standard Bergman spaces, are required. Consider the class $\mathcal{E}$ that consists of the weights $\omega \in \mathcal{W}$ satisfying

$$\int_{\mathbb{D}} |K_z(\xi)|\omega(\xi)^{1/2}\,dA(\xi) \le C\omega(z)^{-1/2}, \quad z \in \mathbb{D}. \tag{1.2}$$

It has been proved in [7] that the exponential type weights ω_σ given by (1.1) with $\sigma = 1$ satisfy the previous condition and, therefore, they are in the class $\mathcal{E}$. Note that following the proof given in [7] with nontrivial modifications, we are able to show that every weight of the form (1.1), with $0 < \sigma < \infty$, is in the class $\mathcal{E}$. Recently, Hu, Xiaofen, and Schuster proved that the prototype weights, considered in (1.1), satisfy (1.2) (see [14, Corollary 3.2]). This integral estimate allows to study other properties and operators, such as the Bergman projection which is given by

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\xi)\overline{K_z(\xi)}\omega(\xi)\,dA(\xi), \quad z \in \mathbb{D}.$$

The boundedness of the Bergman projection P_ω on $L^2(\omega)$ is trivial from the general theory of Hilbert spaces. In contrast with the case of standard Bergman spaces (where the Bergman projection is bounded for $1 < p < \infty$) in the case of exponential type weights, it turns out that the natural Bergman projection is not bounded on $L^p(\omega)$ unless $p = 2$ (see [8] and [35]). At first glance, this may look surprising, but when one takes into account the similarities with Fock spaces, this seems to be more natural. It turns out that, similarly to the Fock space setting, when studying problems where reproducing kernels are involved, the most convenient setting is provided by the spaces $A^p(\omega^{p/2})$. As a consequence of condition (1.2), we get the right estimates for the norm of reproducing kernels in $A^p(\omega^{p/2})$ for $1 \le p \le \infty$. Also, we prove in Theorem 3.2 that, for weights in the class $\mathcal{E}$, the Bergman projection $P_\omega : L^p(\omega^{p/2}) \to A^p(\omega^{p/2})$ is bounded for $1 \le p \le \infty$. A consequence of that result will be the identification of the dual space of $A^p(\omega^{p/2})$ with the space $A^{p'}(\omega^{p'/2})$ and $A^1(\omega^{1/2})$ with $A^\infty(\omega^{1/2})$ under the natural integral pairing $\langle \cdot, \cdot \rangle_\omega$, where p' denotes the conjugate exponent of p. Afterwards, by using the duality and the estimates for the p-norms of reproducing kernels, we are going to obtain an atomic decomposition for Bergman spaces with exponential type weights: for weights $\omega \in \mathcal{E}$, every function in the weighted Bergman space $A^p(\omega^{p/2})$, $1 \le p < \infty$, can be decomposed into a series of very nice functions (called atoms). These atoms are defined in terms of kernel functions and in some sense act as a basis for the space $A^p(\omega^{p/2})$. The atomic decomposition for Bergman space with standard weights was obtained by Coifman and Rochberg [6], and it has become a powerful tool in the study of weighted Bergman spaces. We refer to the books [26, 37, 38] for a modern proof of these results. The norm estimates for the reproducing kernels in $A^p(\omega^{p/2})$ for $1 \le p < \infty$ permit to extend the results on the boundedness and compactness of Toeplitz operators T_μ, to consider the action of T_μ between different large weighted Bergman spaces, and to find a general description of when $T_\mu : A^p(\omega^{p/2}) \longrightarrow A^q(\omega^{q/2})$ is bounded or compact for all values of $1 \le p, q < \infty$. Furthermore, we also generalize the results obtained in [12] on the boundedness and compactness of big Hankel operators $H_{\bar{g}}$ with conjugate analytic symbols to the non-Hilbert space setting, characterizing for all $1 < p, q < \infty$ the operators

$$H_{\bar{g}} : A^p\left(\omega^{p/2}\right) \longrightarrow L^q\left(\omega^{q/2}\right)$$

that are bounded or compact. As mentioned earlier, one of the key tools consists in using the estimates for the $\bar{\partial}$-equation obtained in Sect. 2.

In what follows we use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ with $a \le Cb$; and the notation $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$. Also, respectively the expressions L^p_ω and A^p_ω mean $L^p(\mathbb{D}, \omega^{p/2}\, dA)$ and $A^p(\omega^{p/2})$ for $1 \le p < \infty$.

2 Preliminaries and basic properties

A positive function τ on $\mathbb{D}$ is said to be of class $\mathcal{L}$ if it satisfies the following two properties:

(A) There is a constant c_1 such that $\tau(z) \le c_1(1 - |z|)$ for all $z \in \mathbb{D}$;

(B) There is a constant c_2 such that $|\tau(z) - \tau(\zeta)| \le c_2|z - \zeta|$ for all $z, \zeta \in \mathbb{D}$.

We also use the notation

$$m_\tau := \frac{\min(1, c_1^{-1}, c_2^{-1})}{4},$$

where c_1 and c_2 are the constants appearing in the previous definition. For $a \in \mathbb{D}$ and $\delta > 0$, we use $D(\delta\tau(a))$ to denote the Euclidean disc centered at a and having radius $\delta\tau(a)$. It is easy to see from conditions (A) and (B) (see [24, Lemma 2.1]) that if $\tau \in \mathcal{L}$ and $z \in D(\delta\tau((a))$, then

$$\frac{1}{2}\tau(a) \leq \tau(z) \leq 2\tau(a) \tag{2.1}$$

for sufficiently small $\delta > 0$, that is, for $\delta \in (0, m_\tau)$. This fact will be used many times in this paper.

Definition 2.1 We say that a weight ω is of *class* $\mathcal{L}^*$ if it is of the form $\omega = e^{-2\varphi}$, where $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$ and $(\Delta\varphi(z))^{-1/2} \asymp \tau(z)$ with $\tau(z)$ is a function in the class $\mathcal{L}$.

It is straightforward to see that $\mathcal{W} \subset \mathcal{L}^*$. The following result is from [24, Lemma 2.2] and gives the boundedness of the point evaluation functional on A_ω^2.

Lemma A *Let $\omega \in \mathcal{L}^*$, $0 < p < \infty$, and $z \in \mathbb{D}$. If $\beta \in \mathbb{R}$, there exists $M \geq 1$ such that*

$$\left|f(z)\right|^p \omega(z)^\beta \leq \frac{M}{\delta^2\tau(z)^2} \int_{D(\delta\tau(z))} \left|f(\xi)\right|^p \omega(\xi)^\beta \, dA(\xi)$$

for all $f \in H(\mathbb{D})$ and all sufficiently small $\delta > 0$.

We also need a similar estimate for the gradient of $|f|\omega^{1/2}$.

Lemma 2.2 *Let $\omega \in \mathcal{L}^*$ and $0 < p < \infty$. For any $\delta_0 > 0$ sufficiently small, there exists a constant $C(\delta_0) > 0$ such that*

$$\left|\nabla\left(|f|\omega^{1/2}\right)(z)\right| \leq \frac{C(\delta_0)}{\tau(z)^{1+\frac{2}{p}}} \left(\int_{D(\delta_0\tau(z)/2)} \left|f(\xi)\right|^p \omega(\xi)^{p/2} \, dA(\xi)\right)^{1/p}$$

for all $f \in H(\mathbb{D})$.

Proof We follow the method used in [22]. Without loss of generality we can assume $z = 0$. Then, applying the Riesz decomposition (see for example [29]) of the subharmonic function φ in $D(0, \frac{\delta_0}{2}\tau(0))$, we obtain

$$\varphi(\xi) = u(\xi) + \int_{D(\frac{r}{2})} G(\xi, \eta)\Delta\varphi(\eta) \, dA(\eta), \tag{2.2}$$

where $r = \delta_0\tau(0)$, u is the least harmonic majorant of φ in $D(0, \frac{r}{2})$, and G is the Green function defined for every $\xi, \eta \in D(0, r)$, $\xi \neq \eta$, by

$$G(\xi, \eta) := \log\left|\frac{r(\xi - \eta)}{r^2 - \bar{\eta}\xi}\right|^2.$$

For $\xi, \eta \in D(0, \frac{r}{2})$, we have

$$
\begin{aligned}
\left| \frac{\partial G}{\partial \xi}(\xi, \eta) \right| &= \frac{r^2 - |\eta|^2}{|\xi - \eta||r^2 - \eta\bar{\xi}|} \\
&\leq \frac{r^2}{|\xi - \eta| \cdot |r^2 - |\eta||\xi||} \leq \frac{4}{3|\xi - \eta|}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\left| \frac{\partial \varphi(0)}{\partial \xi} - \frac{\partial u(0)}{\partial \xi} \right| &\leq \int_{D(\frac{r}{2})} \left| \frac{\partial G}{\partial \xi}(0, \eta) \right| \Delta\varphi(\eta) \, dA(\eta) \\
&\lesssim \frac{1}{\tau(0)^2} \int_{D(\frac{r}{2})} \frac{dA(\eta)}{|\eta|} = \frac{\delta_0}{\tau(0)}.
\end{aligned}
\tag{2.3}
$$

We pick a function $h \in H(\mathbb{D})$ such that $\mathrm{Re}(h) = u$. Also,

$$
\begin{aligned}
\left| \nabla \left(|f| e^{-\varphi} \right)(\xi) \right| &= 2 \cdot \left| \frac{1}{2} \frac{f'(\xi) \overline{f(\xi)}}{|f(\xi)|} - \frac{\partial \varphi}{\partial \xi}(\xi) |f(\xi)| \right| e^{-\varphi(\xi)} \\
&= \left| f'(\xi) - 2 \frac{|f(\xi)|^2}{\overline{f(\xi)}} \frac{\partial \varphi}{\partial \xi}(\xi) \right| e^{-\varphi(\xi)}.
\end{aligned}
$$

Therefore, since $h'(0) = 2\frac{\partial u}{\partial \xi}(0)$, we get

$$
\begin{aligned}
\left| \nabla \left(|f| e^{-\varphi} \right)(0) \right| &= \left| f'(0) - 2f(0) \frac{\partial \varphi}{\partial \xi}(0) \right| e^{-\varphi(0)} \\
&\leq \left| f'(0) - 2f(0) \frac{\partial u}{\partial \xi}(0) \right| e^{-\varphi(0)} + 2 \left| \frac{\partial u}{\partial \xi}(0) - \frac{\partial \varphi}{\partial \xi}(0) \right| |f(0)| e^{-\varphi(0)} \\
&\lesssim \left| \frac{\partial (f e^{-h})(0)}{\partial \xi} \right| e^{u(0)-\varphi(0)} + \left| \frac{\partial u}{\partial \xi}(0) - \frac{\partial \varphi}{\partial \xi}(0) \right| |f(0)| e^{-\varphi(0)}.
\end{aligned}
$$

By (2.3) we have

$$
\left| \frac{\partial u}{\partial \xi}(0) - \frac{\partial \varphi}{\partial \xi}(0) \right| |f(0)| e^{-\varphi(0)} \lesssim \frac{\delta_0}{\tau(0)} |f(0)| e^{-\varphi(0)}.
$$

This gives

$$
\left| \nabla \left(|f| e^{-\varphi} \right)(0) \right| \lesssim \left| \frac{\partial (f e^{-h})(0)}{\partial \xi} \right| e^{u(0)-\varphi(0)} + \frac{|f(0)|}{\tau(0)} e^{-\varphi(0)}.
\tag{2.4}
$$

It follows from Lemma A that

$$
\frac{|f(0)|}{\tau(0)} e^{-\varphi(0)} \lesssim \frac{1}{\tau(0)^{1+\frac{2}{p}}} \left(\int_{D(\delta_0 \tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} \, dA(z) \right)^{1/p}.
\tag{2.5}
$$

To deal with the other term appearing in (2.4), notice that if we use identity (2.2) with the function $\phi(\xi) = |\xi|^2 - (r/2)^2$ (since $\Delta\phi(\xi) = 4$ and its least harmonic majorant is $u_\phi = 0$),

we obtain

$$\int_{D(\frac{r}{2})} G(\xi, \eta) \, dA(\eta) = \frac{1}{4} \big(|\xi|^2 - (r/2)^2 \big).$$

Therefore, since $\Delta\varphi(\eta) = \frac{1}{\tau(\eta)^2} \lesssim \frac{1}{\tau(0)^2} = \Delta\varphi(0)$ and the Green function $G \leq 0$, we obtain, for every $\xi \in D(0, \frac{r}{2})$,

$$\begin{aligned}
u(\xi) - \varphi(\xi) &= -\int_{D(\frac{r}{2})} G(\xi, \eta) \Delta\varphi(\eta) \, dA(\eta) \\
&\lesssim \frac{\Delta\varphi(0)}{4} \big((r/2)^2 - |\xi|^2 \big) = \frac{1}{4\tau(0)^2} \big((r/2)^2 - |\xi|^2 \big).
\end{aligned}$$

This gives

$$e^{u(0)-\varphi(0)} \leq e^{C\delta_0^2}.$$

Therefore

$$\left| \frac{\partial (fe^{-h})(0)}{\partial \xi} \right| e^{u(0)-\varphi(0)} \lesssim \left| \frac{\partial (fe^{-h})(0)}{\partial \xi} \right|. \tag{2.6}$$

On the other hand, using Cauchy's inequality, the fact that $\varphi - u \leq 0$, and Lemma A, we get

$$\begin{aligned}
\left| \frac{\partial (fe^{-h})}{\partial \xi}(0) \right| &\lesssim \left| \int_{|\eta|=\frac{\delta_0 \tau(0)}{4}} \frac{f(\eta)e^{-h(\eta)}}{\eta^2} \, d\eta \right| \\
&\lesssim \frac{1}{\delta_0^2 \tau(0)^2} \int_{|\eta|=\frac{\delta_0 \tau(0)}{4}} |f(\eta)| \big| e^{-\varphi(\eta)} e^{\varphi(\eta)-u(\eta)} \big| \, |d\eta| \\
&\lesssim \frac{1}{\tau(0)^2} \int_{|\eta|=\frac{\delta_0 \tau(0)}{4}} \left(\frac{1}{\tau(\eta)^2} \int_{D(\delta_0 \tau(\eta)/4)} |f(z)|^p e^{-p\varphi(z)} \, dA(z) \right)^{1/p} |d\eta|.
\end{aligned}$$

Finally, using $\tau(\eta) \asymp \tau(0)$, we obtain

$$\begin{aligned}
\left| \frac{\partial fe^{-h}}{\partial \xi}(0) \right| &\lesssim \frac{1}{\tau(0)^2} \int_{|\eta|=\frac{\delta_0 \tau(0)}{4}} \left(\frac{1}{\tau(0)^2} \int_{D(\delta_0 \tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} \, dA(z) \right)^{1/p} |d\eta| \\
&\lesssim \frac{1}{\tau(0)^{1+\frac{2}{p}}} \left(\int_{D(\delta_0 \tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} \, dA(z) \right)^{1/p}.
\end{aligned}$$

Bearing in mind (2.6) this gives

$$\left| \frac{\partial (fe^{-h})(0)}{\partial \xi} \right| e^{u(0)-\varphi(0)} \lesssim \frac{1}{\tau(0)^{1+\frac{2}{p}}} \left(\int_{D(\delta_0 \tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} \, dA(z) \right)^{1/p}.$$

Plugging this and (2.5) into (2.4), we get the desired result.   □

The following lemma on coverings is due to Oleinik, see [21].

Lemma B *Let τ be a positive function on $\mathbb{D}$ of class $\mathcal{L}$, and let $\delta \in (0, m_\tau)$. Then there exists a sequence of points $\{z_n\} \subset \mathbb{D}$ such that the following conditions are satisfied*:

(i) $z_n \notin D(\delta\tau(z_k))$, $n \neq k$;

(ii) $\bigcup_n D(\delta\tau(z_n)) = \mathbb{D}$;

(iii) $\widetilde{D}(\delta\tau(z_n)) \subset D(3\delta\tau(z_n))$, *where* $\widetilde{D}(\delta\tau(z_n)) = \bigcup_{z \in D(\delta\tau(z_n))} D(\delta\tau(z))$, $n = 1, 2, 3, \ldots$;

(iv) $\{D(3\delta\tau(z_n))\}$ *is a covering of $\mathbb{D}$ of finite multiplicity N.*

The multiplicity N in Lemma B is independent of δ, and it is easy to see that one can take, for example, $N = 256$. Any sequence satisfying the conditions in Lemma B will be called a (δ, τ)-*lattice*. Note that $|z_n| \to 1^-$ as $n \to \infty$. In what follows, the sequence $\{z_n\}$ will always refer to the sequence chosen in Lemma B.

2.1 Integral estimates for reproducing kernels

We use the notation k_z for the normalized reproducing kernels in A_ω^p, that is,

$$k_z = \frac{K_z}{\|K_z\|_{A_\omega^p}}.$$

The next result (see [4, 17, 24] for (a) when $p = 2$ and [18, Lemma 3.6] for part (b)) provides useful estimates involving reproducing kernels.

Theorem A *Let K_z be the reproducing kernel of A_ω^2. Then*

(a) *For $\omega \in \mathcal{W}$, one has*

$$\|K_z\|_{A_\omega^2} \asymp \omega(z)^{-1/2}\tau(z)^{-1}, \quad z \in \mathbb{D}. \tag{2.7}$$

(b) *For all sufficiently small $\delta \in (0, m_\tau)$ and $\omega \in \mathcal{W}$, one has*

$$\left|K_z(\zeta)\right| \asymp \|K_z\|_{A_\omega^2} \cdot \|K_\zeta\|_{A_\omega^2}, \quad \zeta \in D_\delta(z). \tag{2.8}$$

Lemma 2.3 *Let $\omega \in \mathcal{E}$. For each $z \in \mathbb{D}$, we have*

$$\|K_z\|_{A^\infty(\omega^{1/2})} \lesssim \omega(z)^{-1/2}\tau(z)^{-2}.$$

Proof By Lemma A and condition (1.2), we have

$$\begin{aligned}
\omega(\xi)^{1/2}\left|K_z(\xi)\right| &= \omega(\xi)^{1/2}\left|K_\xi(z)\right| \\
&\lesssim \frac{\omega(\xi)^{1/2}}{\omega(z)^{1/2}\tau(z)^2} \int_{D(\delta\tau(z))} \left|K_\xi(s)\right|\omega(s)^{1/2}\, dA(s) \\
&\lesssim \omega(z)^{-1/2}\tau(z)^{-2},
\end{aligned}$$

which finishes the proof. □

Lemma 2.3 together with condition (1.2) allows us to obtain the following estimate for the norm of the reproducing kernel in A_ω^p.

Lemma 2.4 *Let $1 \leq p < \infty$, $\omega \in \mathcal{E}$, and $z \in \mathbb{D}$. Then*

$$\|K_z\|_{A_\omega^p} \asymp \omega(z)^{-1/2}\tau(z)^{-2(p-1)/p}.$$

Proof By (1.2) and Lemma 2.3, we have

$$\|K_z\|_{A_\omega^p}^p = \int_{\mathbb{D}} \left|K_z(\xi)\right|^p \omega(\xi)^{p/2}\, dA(\xi) = \int_{\mathbb{D}} \left|K_z(\xi)\right|\omega(\xi)^{1/2}\left(\left|K_z(\xi)\right|\omega(\xi)^{1/2}\right)^{p-1}\, dA(\xi)$$

$$\leq \|K_z\|_{A^\infty(\omega^{1/2})}^{p-1} \int_{\mathbb{D}} \left|K_z(\xi)\right|\omega(\xi)^{1/2}\, dA(\xi) \lesssim \|K_z\|_{A^\infty(\omega^{1/2})}^{p-1}\omega(z)^{-1/2}$$

$$\lesssim \omega(z)^{-p/2}\tau(z)^{-2(p-1)}.$$

On the other hand, by using statement (b) of Theorem A, we have

$$\|K_z\|_{A_\omega^p}^p \geq \int_{D(\delta\tau(z))} \left|K_z(\xi)\right|^p \omega(\xi)^{p/2}\, dA(\xi)$$

$$\geq C\|K_z\|_{A_\omega^2}^p \int_{D(\delta\tau(z))} \|K_\xi\|_{A_\omega^2}^p \omega(\xi)^{p/2}\, dA(\xi)$$

$$\geq C\omega(z)^{-p/2}\tau(z)^{-2(p-1)}.$$

This completes the proof.　　　　　　　　　　　　　　　　　　　　　　　　$\square$

2.2 Estimates for the $\overline{\partial}$-equation

The following result, which provides more estimates on the solutions of the $\overline{\partial}$-equation, will play a crucial role in describing the bounded Hankel operators acting from A_ω^p to A_ω^q when $1 \leq p \leq q < \infty$. Also, it can be of independent interest.

Theorem 2.5 *Let $\omega \in \mathcal{W}$, and consider the associated weight $\omega_*(z) := \omega(z)\tau(z)^\alpha$, $z \in \mathbb{D}$, and $\alpha \in \mathbb{R}$. Then there exists a solution u of the equation $\overline{\partial}u = f$ such that*

$$\int_{\mathbb{D}} \left|u(z)\right|^p \omega_*(z)^{p/2}\, dA(z) \leq C \int_{\mathbb{D}} \left|f(z)\right|^p \omega_*(z)^{p/2}\tau(z)^p\, dA(z)$$

for all $1 \leq p < \infty$, provided the right-hand side is finite. Moreover, one also has the L^∞-estimate

$$\sup_{z \in \mathbb{D}} \left|u(z)\right|\omega_*(z)^{1/2} \leq C \sup_{z \in \mathbb{D}} \left|f(z)\right|\omega_*(z)^{1/2}\tau(z).$$

Proof We follow the method used in [5] where the case $\alpha = 0$ was proved. By Lemma 3.1 in [24], there are holomorphic functions F_a and some $\delta_0 \in (0, m_\tau)$ such that

　(i)　$\left|F_a(\xi)\right| \asymp \omega(\xi)^{-1/2}$,　$\xi \in D(\delta_0\tau(a))$.

　(ii)　$\left|F_a(\xi)\right| \leq C\omega(\xi)^{-1/2}\left(\dfrac{\min(\tau(\xi), \tau(a))}{|a - \xi|}\right)^M$,　$(a, \xi) \in \mathbb{D} \times \mathbb{D}$.　　　　(2.9)

Let $\delta_1 < \delta_0$. Then there is a sequence $\{z_n\}_{n \geq 1}$ such that $\{D(\delta_1\tau(z_n))\}$ is a covering of $\mathbb{D}$ of finite multiplicity N and satisfies the other statements of Lemma B. Let χ_n be a partition

of unity subordinate to the covering $D(\delta_1\tau(z_n))$. Consider

$$S_nf(z) = F_{z_n}(z)\int_{\mathbb{D}}\frac{f(\xi)\chi_n(\xi)}{(\xi-z)F_{z_n}(\xi)}\,dA(\xi).$$

Since F_{z_n} are holomorphic functions on $\mathbb{D}$, by the Cauchy–Pompeiu formula, we have

$$\overline{\partial}S_nf(z) = f(z)\chi_n(z), \quad n = 1,2....$$

Then

$$Sf(z) = \sum_{n=1}^{\infty}S_nf(z) = \omega_*(z)^{-1/2}\int_{\mathbb{D}}G(z,\xi)f(\xi)\omega_*(\xi)^{1/2}\,dA(\xi),$$

where

$$G(z,\xi) = \sum_{n=1}^{\infty}\frac{F_{z_n}(z)}{\xi-z}\frac{\chi_n(\xi)}{F_{z_n}(\xi)}\omega_*(\xi)^{-1/2}\omega_*(z)^{1/2}.$$

Since χ_n is a partition of the unity, we have

$$\overline{\partial}(Sf) = \sum_{n=1}^{\infty}\overline{\partial}(S_nf) = f\sum_{n=1}^{\infty}\chi_n = f$$

on $\mathbb{D}$, so that Sf solves the equation $\overline{\partial}Sf(z) = f(z)$.

On the other hand, assume that

$$\int_{\mathbb{D}}\left|G(z,\xi)\right|\frac{dA(\xi)}{\tau(\xi)} \lesssim 1 \tag{2.10}$$

and

$$\int_{\mathbb{D}}\frac{|F_{z_n}(z)|}{|\xi-z|}\omega_*(z)^{1/2}\,dA(z) \lesssim \tau(\xi)^{1+\alpha/2}, \quad \xi\in D\big(\delta_1\tau(z_n)\big). \tag{2.11}$$

Then, by (2.13), it is straightforward that the L^{∞}-estimate holds. Our next goal is to prove the inequality

$$\int_{\mathbb{D}}\left|Sf(z)\right|^p\omega_*(z)^{p/2}\,dA(z) \lesssim \int_{\mathbb{D}}\left|f(z)\right|^p\omega_*(z)^{p/2}\tau(z)^p\,dA(z).$$

Consider $g(\xi) := f(\xi)\omega_*(\xi)^{1/2}$ and $Tg(z) := \int_{\mathbb{D}}G(z,\xi)g(\xi)\,dA(\xi)$. Then the last inequality takes the form

$$\int_{\mathbb{D}}\left|Tg(z)\right|^p\,dA(z) \lesssim \int_{\mathbb{D}}\left|g(z)\right|^p\tau(z)^p\,dA(z).$$

Therefore, using Hölder's inequality and (2.13), we have

$$\begin{aligned}\left|Tg(z)\right|^p &\le \int_{\mathbb{D}}\left|g(\xi)\right|^p\tau(\xi)^{p-1}\left|G(z,\xi)\right|dA(\xi)\left(\int_{\mathbb{D}}\left|G(z,\xi)\right|\frac{dA(\xi)}{\tau(\xi)}\right)^{p-1}\\ &\lesssim \int_{\mathbb{D}}\left|g(\xi)\right|^p\tau(\xi)^{p-1}\left|G(z,\xi)\right|dA(\xi).\end{aligned}$$

These estimates and Fubini's theorem give

$$\int_{\mathbb{D}} \left| Tg(z) \right|^p dA(z) \lesssim \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \left| g(\xi) \right|^p \tau(\xi)^{p-1} \left| G(z,\xi) \right| dA(\xi) \right) dA(z)$$

$$\lesssim \int_{\mathbb{D}} \left| g(\xi) \right|^p \tau(\xi)^{p-1} \left(\int_{\mathbb{D}} \left| G(z,\xi) \right| dA(z) \right) dA(\xi).$$

Now, using the expression of the kernel $G(z,\xi)$ and the fact that χ_n are supported in $D(\delta_1 \tau(z_n))$, we obtain

$$\int_{\mathbb{D}} \left| Tg(z) \right|^p dA(z)$$

$$\lesssim \int_{\mathbb{D}} \left| g(\xi) \right|^p \tau(\xi)^{p-1} \left(\int_{\mathbb{D}} \sum_{n=1}^{\infty} \frac{|F_{z_n}(z)|}{|\xi - z|} \frac{\chi_n(\xi)}{|F_{z_n}(\xi)|} \frac{\omega_*(z)^{1/2}}{\omega_*(\xi)^{1/2}} dA(z) \right) dA(\xi)$$

$$\lesssim \sum_{n=1}^{\infty} \int_{\mathbb{D}} \left| g(\xi) \right|^p \tau(\xi)^{p-1} \left(\frac{\omega_*(\xi)^{-1/2}}{|F_{z_n}(\xi)|} \int_{\mathbb{D}} \frac{|F_{z_n}(z)|}{|\xi - z|} \omega_*(z)^{1/2} dA(z) \right) \chi_n(\xi) dA(\xi)$$

$$\lesssim \sum_{n=1}^{\infty} \int_{D(\delta_1 \tau(z_n))} \left| g(\xi) \right|^p \tau(\xi)^{p-1} \left(\frac{\omega_*(\xi)^{-1/2}}{|F_{z_n}(\xi)|} \int_{\mathbb{D}} \frac{|F_{z_n}(z)|}{|\xi - z|} \omega_*(z)^{1/2} dA(z) \right) dA(\xi).$$

By (2.11) and using the fact that $|F_{z_n}(\xi)| \asymp \omega(\xi)^{-1/2}$, $\xi \in D(\delta_1 \tau(z_n))$, it follows that

$$\int_{\mathbb{D}} \left| Tg(z) \right|^p dA(z)$$

$$\lesssim \sum_{n=1}^{\infty} \int_{D(\delta_1 \tau(z_n))} \left| g(\xi) \right|^p \tau(\xi)^{p-1-\alpha/2} \left(\int_{\mathbb{D}} \frac{|F_{z_n}(z)|}{|\xi - z|} \omega_*(z)^{1/2} dA(z) \right) dA(\xi)$$

$$\lesssim \sum_{n=1}^{\infty} \int_{D(\delta_1 \tau(z_n))} \left| g(\xi) \right|^p \tau(\xi)^p dA(\xi) \tag{2.12}$$

$$\lesssim \int_{\mathbb{D}} \left| g(\xi) \right|^p \tau(\xi)^p dA(\xi),$$

where the last inequality above follows from the fact that $\{D(\delta_\tau(z_n))\}$ is a covering of $\mathbb{D}$ of finite multiplicity.

Now we are going to prove that

$$\int_{\mathbb{D}} \left| G(z,\xi) \right| \frac{dA(\xi)}{\tau(\xi)} \lesssim 1. \tag{2.13}$$

First we consider the covering of $\{\xi \in \mathbb{D} : |z - \xi| > \delta_2 \tau(z)\}$ given by

$$R_k(z) = \left\{ \xi \in \mathbb{D} : 2^{k-1} \delta_2 \tau(z) < |z - \xi| \leq 2^k \delta_2 \tau(z) \right\}, \quad k = 1, 2, \dots$$

Let $4\delta_1 < \delta_2 < \frac{\delta_0}{5}$ and $z \in \mathbb{D}$ be fixed. If $\xi \in D(\delta_2 \tau(z)) \cap D(\delta_1 \tau(z_n))$, using (2.1), we have

$$|z - z_n| \leq |z - \xi| + |\xi - z_n| \leq \delta_2 \tau(z) + \delta_1 \tau(z_n)$$

$$\leq 4\delta_2 \tau(z_n) + \delta_1 \tau(z_n) < \delta_0 \tau(z_n),$$

that implies $z \in D(\delta_0 \tau(z_n))$. Using (2.1) and property (i) of (2.9), it follows

$$\left|G(z,\xi)\right| \lesssim \frac{\omega(z)^{-1/2}}{|\xi - z|} \frac{\omega_*(\xi)^{-1/2}\omega_*(z)^{1/2}}{\omega(\xi)^{-1/2}} \sum_{n=1}^{\infty} \chi_n(\xi) \lesssim \frac{1}{|\xi - z|}.$$

Therefore, using again (2.1) and polar coordinates, we get

$$\int_{D(\delta_2 \tau(z))} \left|G(z,\xi)\right| \frac{dA(\xi)}{\tau(\xi)} \lesssim \frac{1}{\tau(z)} \int_{D(\delta_2 \tau(z))} \frac{1}{|\xi - z|} \, dA(\xi) \lesssim 1. \qquad (2.14)$$

If $\xi \in (\mathbb{D} \smallsetminus D(\delta_2 \tau(z))) \cap D(\delta_1 \tau(z_n))$, we show that $z \notin D(\delta_1 \tau(z_n))$. In fact, if not, $z \in D(\delta_1 \tau(z_n))$, then using (2.1) we have

$$|z - z_n| > |z - \xi| - |\xi - z_n| > \delta_2 \tau(z) - \delta_1 \tau(z_n) \geq (\delta_2/2 - \delta_1)\tau(z_n) \geq \delta_1 \tau(z_n),$$

this implies a contradiction with our assumption. Thus,

$$|z - \xi| \leq |z - z_n| + |z_n - \xi| \leq |z - z_n| + \delta_1 \tau(z_n)$$

$$\leq 2|z - z_n|.$$

Also, using $\tau(\xi) \asymp \tau(z_n)$, we get

$$\frac{|z - z_n|}{\min(\tau(z), \tau(z_n))} \geq C \frac{|z - \xi|}{\min(\tau(z), \tau(\xi))}.$$

Then, again using $\tau(\xi) \asymp \tau(z_n)$ and property (ii) of (2.9) with

$$M > \max(1; -\alpha/2; 1 + \alpha/2),$$

we have

$$\begin{aligned}
\left|G(z,\xi)\right| &\leq C \sum_{n=1}^{\infty} \frac{|F_{z_n}(z)|}{|\xi - z|} \frac{\chi_n(\xi)}{|F_{z_n}(\xi)|} \omega_*(\xi)^{-1/2} \omega_*(z)^{1/2} \\
&\leq C \frac{\omega(z)^{-1/2}}{|\xi - z|} \frac{\omega_*(\xi)^{-1/2}\omega_*(z)^{1/2}}{\omega(\xi)^{-1/2}} \left(\frac{\min(\tau(z), \tau(\xi))}{|\xi - z|}\right)^M \sum_{n=1}^{\infty} \chi_n(\xi) \qquad (2.15) \\
&\leq C \frac{\tau(z)^{\frac{\alpha}{2}}}{\tau(\xi)^{\frac{\alpha}{2}} |\xi - z|} \left(\frac{\min(\tau(z), \tau(\xi))}{|\xi - z|}\right)^M.
\end{aligned}$$

Then

$$\int_{\mathbb{D} \smallsetminus D(\delta_2 \tau(z))} \left|G(z,\xi)\right| \frac{dA(\xi)}{\tau(\xi)}$$

$$\leq C\tau(z)^{\alpha/2} \int_{\mathbb{D} \smallsetminus D(\delta_2 \tau(z))} \frac{\tau(\xi)^{\frac{-(2+\alpha)}{2}}}{|\xi - z|} \left(\frac{\min(\tau(z), \tau(\xi))}{|\xi - z|}\right)^M dA(\xi).$$

• If $2 + \alpha > 0$.

$$\int_{\mathbb{D} \smallsetminus D(\delta_2 \tau(z))} \big| G(z, \xi) \big| \frac{dA(\xi)}{\tau(\xi)} \le C\tau(z)^{M-1} \sum_{k=1}^{\infty} \int_{R_k(z)} \frac{1}{|\xi - z|^{M+1}} \, dA(\xi)$$

$$\le C\tau(z)^{M-1} \sum_{k=1}^{\infty} \int_{R_k(z)} \frac{1}{(2^k \tau(z))^{M+1}} \, dA(\xi)$$

$$\le C \sum_{k=1}^{\infty} \frac{1}{2^{k(M-1)}} \lesssim 1.$$

• If $2 + \alpha \le 0$. Using condition (B) in the definition of the class $\mathcal{L}$, it follows that

$$\tau(\xi) \le C2^k \delta_2 \tau(z), \quad \xi \in R_k(z), k = 0, 1, 2, \dots.$$

So,

$$\int_{\mathbb{D} \smallsetminus D(\delta_2 \tau(z))} \big| G(z, \xi) \big| \frac{dA(\xi)}{\tau(\xi)} \le C\tau(z)^{\alpha/2} \int_{\mathbb{D} \smallsetminus D(\delta_2 \tau(z))} \frac{\tau(\xi)^{-(2+\alpha)/2} \tau(z)^M}{|\xi - z|^{M+1}} \, dA(\xi)$$

$$\le C\tau(z)^{M+\alpha/2} \sum_{k=1}^{\infty} \int_{R_k(z)} \frac{(2^k \tau(z))^{-(2+\alpha)/2}}{(2^k \tau(z))^{M+1}} \, dA(\xi)$$

$$\le C \sum_{k=1}^{\infty} \frac{1}{2^{k(M+\alpha/2)}} \lesssim 1.$$

This together with (2.14) establishes (2.13).

Finally, it remains to prove inequality (2.11). We split this integral in two parts: one integrating over the disk $D(\delta_2 \tau(\xi))$ and the other one over $\mathbb{D} \smallsetminus D(\delta_2 \tau(\xi))$. We compute the first integral using (i) of (2.9), $\tau(\xi) \asymp \tau(z_n) \asymp \tau(z)$, $z \in D(\delta_2 \tau(\xi))$, and by using polar coordinates, we obtain

$$\int_{D(\delta_2 \tau(\xi))} \frac{|F_{z_n}(z)|}{|\xi - z|} \omega_*(z)^{1/2} \, dA(z) \lesssim \tau(\xi)^{\alpha/2} \int_{D(\delta_2 \tau(\xi))} \frac{dA(z)}{|z - \xi|} \lesssim \tau(\xi)^{1+\alpha/2}. \tag{2.16}$$

Now we consider

$$I(\xi) := \int_{\mathbb{D} \smallsetminus D(\delta_2 \tau(\xi))} \frac{|F_{z_n}(z)|}{|\xi - z|} \omega_*(z)^{1/2} \, dA(z), \quad \xi \in D\big(\delta_1 \tau(z_n)\big).$$

For $z \notin D(\delta_2 \tau(\xi))$,

$$|z - \xi| \le |z - z_n| + |z_n - \xi|$$

$$\le |z - z_n| + \delta_1 \tau(z_n)$$

$$\le |z - z_n| + \frac{2\delta_1}{\delta_2} |z - z_n|$$

$$\le \left(1 + \frac{2\delta_1}{\delta_2}\right) |z - z_n|.$$

Then, again by using $\tau(\xi) \asymp \tau(z_n)$, we obtain

$$\frac{\min(\tau(z_n), \tau(z))}{|z - z_n|} \leq C \frac{\min(\tau(\xi), \tau(z))}{|z - \xi|}.$$

This together with (2.9) taking $M > \max(1, 2 + \alpha/2)$ gives

$$I(\xi) \lesssim \int_{\mathbb{D} \smallsetminus D(\delta_2 \tau(\xi))} \frac{\tau(z)^{\alpha/2}}{|\xi - z|} \frac{(\min(\tau(z_n), \tau(z)))^M}{|z - z_n|^M} \, dA(z)$$

$$\lesssim \int_{\mathbb{D} \smallsetminus D(\delta_2 \tau(\xi))} \tau(z)^{\alpha/2} \frac{(\min(\tau(\xi), \tau(z)))^M}{|z - \xi|^{M+1}} \, dA(z).$$

• Suppose first that $\alpha \geq 0$. By $\tau(z) \leq C2^k \tau(\xi)$, for every $z \in R_k(\xi)$, $k = 1, 2, \ldots$, we get

$$I(\xi) \lesssim \sum_{k=1}^{\infty} \int_{R_k(\xi)} \frac{(2^k \tau(\xi))^{\alpha/2} \tau(\xi)^M}{(2^k \tau(\xi))^{M+1}} \, dA(z)$$

$$\lesssim \tau(\xi)^{1+\alpha/2} \sum_{k=1}^{\infty} \frac{1}{2^{k(M-2-\alpha/2)}} \lesssim \tau(\xi)^{1+\alpha/2}.$$

• If $\alpha < 0$, then

$$I(\xi) \lesssim \int_{\mathbb{D} \smallsetminus D(\delta_2 \tau(\xi))} \tau(z)^{\alpha/2} \frac{(\min(\tau(\xi), \tau(z)))^M}{|z - \xi|^{M+1}} \, dA(z)$$

$$\lesssim \sum_{k=1}^{\infty} \int_{R_k(\xi)} \frac{\tau(\xi)^{M+\alpha/2}}{|z - \xi|^{M+1}} \, dA(z)$$

$$\lesssim \tau(\xi)^{1+\alpha/2} \sum_{k=1}^{\infty} \frac{1}{2^{k(M-1)}} \lesssim \tau(\xi)^{1+\alpha/2}.$$

This together with (2.16) establishes (2.11). The proof is complete. □

2.3 Carleson type measures

We are going to define (vanishing) q-Carleson measures for A_ω^p, $0 < p, q < \infty$, for weights ω in the class $\mathcal{W}$ and give some essential theorems.

Definition 2.6 Given $\omega \in \mathcal{W}$ and $0 < p, q \leq \infty$, let μ be a positive measure on $\mathbb{D}$. We say that μ is a q-Carleson measure for A_ω^p if there exists a positive constant C such that

$$\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \leq C \|f\|_{A_\omega^p}^q$$

for all $f \in A_\omega^p$. Thus, by the definition, μ is q-Carleson for A_ω^p when the inclusion $I_\mu : A_\omega^p \longrightarrow L^q(\mathbb{D}, d\mu)$ is bounded.

Next, the following theorems were essentially proved in [24, Theorem 1]. They established necessary and sufficient conditions for $I_\mu : A_\omega^p \longrightarrow L^q(\mathbb{D}, d\mu)$ to be bounded (compact) when $0 < p, q < \infty$.

Theorem B *Given* $\omega \in \mathcal{W}$ *and* $0 < p \leq q < \infty$, *let* μ *be a finite positive Borel measure on* $\mathbb{D}$. *Then* $I_\mu : A_\omega^p \longrightarrow L^q(\mathbb{D}, d\mu)$ *is bounded if and only if, for each sufficiently small* $\delta > 0$,

$$K_{\mu,\omega} := \sup_{a \in \mathbb{D}} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} \omega(\xi)^{-q/2} \, d\mu(\xi) < \infty. \tag{2.17}$$

Moreover, in that case, $K_{\mu,\omega} \asymp \|I_\mu\|^q_{A_\omega^p \to L^q(\mathbb{D}, d\mu)}$.

Theorem C *Given* $\omega \in \mathcal{W}$ *and* $0 < p \leq q < \infty$, *let* μ *be a finite positive Borel measure on* $\mathbb{D}$. *Then* $I_\mu : A_\omega^p \longrightarrow L^q(\mathbb{D}, d\mu)$ *is compact if and only if, for each sufficiently small* $\delta > 0$,

$$\lim_{r \to 1^-} \sup_{|a| > r} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} \omega(\xi)^{-q/2} \, d\mu(\xi) = 0. \tag{2.18}$$

Theorem D *Given* $\omega \in \mathcal{W}$ *and* $0 < q < p < \infty$, *let* μ *be a finite positive Borel measure on* $\mathbb{D}$. *The following conditions are equivalent*:

(a) $I_\mu : A_\omega^p \longrightarrow L^q(\mathbb{D}, d\mu)$ *is compact.*
(b) $I_\mu : A_\omega^p \longrightarrow L^q(\mathbb{D}, d\mu)$ *is bounded.*
(c) *For each sufficiently small* $\delta > 0$, *the function*

$$F_\mu(z) = \frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \omega(\xi)^{-q/2} \, d\mu(\xi)$$

belongs to $L^{\frac{p}{p-q}}(\mathbb{D}, dA)$.
Moreover, one has

$$\|I_\mu\|^q_{A_\omega^p \to L^q(\mu)} \asymp \|F_\mu\|_{L^{\frac{p}{p-q}}(\mathbb{D})}.$$

3 Bounded projections

The boundedness of Bergman projection is a fact of fundamental importance. In the case of the unit disc, the boundedness of Bergman projections is studied in [13, 38], and it immediately gives the duality between the Bergman spaces. The natural Bergman projection is not necessarily bounded on L_ω^p unless $p = 2$ (see [8] and [35] for more details). However, we are going to see next that P_ω is bounded on L_ω^p for weights ω in the class $\mathcal{E}$. We first prove the boundedness of the sublinear operator P_ω^+ defined as

$$P_\omega^+ f(z) = \int_{\mathbb{D}} f(\xi) |K_z(\xi)| \omega(\xi) \, dA(\xi).$$

We mention here that, for the case of the exponential weight with $\sigma = 1$, the results of this section have been obtained recently in [7].

Theorem 3.1 *Let* $1 \leq p < \infty$ *and* $\omega \in \mathcal{E}$. *The operator* P_ω^+ *is bounded on* L_ω^p *and on* $L^\infty(\omega^{1/2})$.

Proof We first consider the easiest case $p = 1$. By Fubini's theorem and condition (1.2), we obtain

$$
\begin{aligned}
\left\| P_\omega^+ f \right\|_{A_\omega^1} &= \int_{\mathbb{D}} \left| P_\omega^+ f(z) \right| \omega(z)^{1/2} \, dA(z) \\
&\leq \int_{\mathbb{D}} \int_{\mathbb{D}} \left| f(z) \right| \left| K_z(\xi) \right| \omega(\xi) \omega(z)^{1/2} \, dA(\xi) \, dA(z) \\
&= \int_{\mathbb{D}} \left| f(\xi) \right| \omega(\xi) \left(\int_{\mathbb{D}} \left| K_\xi(z) \right| \omega(z)^{1/2} \, dA(z) \right) dA(\xi) \\
&\lesssim \int_{\mathbb{D}} \left| f(\xi) \right| \omega(\xi)^{1/2} \, dA(\xi) = \| f \|_{L_\omega^1}.
\end{aligned}
$$

Next, we consider the case $1 < p < \infty$. Let p' denote the conjugate exponent of p. By Hölder's inequality and (1.2), we get

$$
\begin{aligned}
\left| P_\omega^+ f(z) \right|^p &\leq \left(\int_{\mathbb{D}} \left| f(\xi) \right|^p \left| K_z(\xi) \right| \omega(\xi)^{\frac{p+1}{2}} \, dA(\xi) \right) \left(\int_{\mathbb{D}} \left| K_z(\xi) \right| \omega(\xi)^{1/2} \, dA(\xi) \right)^{p-1} \\
&\lesssim \left(\int_{\mathbb{D}} \left| f(\xi) \right|^p \left| K_z(\xi) \right| \omega(\xi)^{\frac{p+1}{2}} \, dA(\xi) \right) \omega(z)^{-\frac{(p-1)}{2}}.
\end{aligned}
$$

This together with Fubini's theorem and another application of (1.2) gives

$$
\begin{aligned}
\left\| P_\omega^+ f \right\|_{A_\omega^p}^p &= \int_{\mathbb{D}} \left| P_\omega^+ f(z) \right|^p \omega(z)^{p/2} \, dA(z) \\
&\lesssim \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \left| f(\xi) \right|^p \left| K_z(\xi) \right| \omega(\xi)^{\frac{p+1}{2}} \, dA(\xi) \right) \omega(z)^{1/2} \, dA(z) \\
&= \int_{\mathbb{D}} \left| f(\xi) \right|^p \omega(\xi)^{\frac{p+1}{2}} \left(\int_{\mathbb{D}} \left| K_\xi(z) \right| \omega(z)^{1/2} \, dA(z) \right) dA(\xi) \\
&\lesssim \| f \|_{L_\omega^p}^p.
\end{aligned}
$$

Finally, if $f \in L^\infty(\omega^{1/2})$, by condition (1.2) we get

$$
\begin{aligned}
\omega(z)^{1/2} \left| P_\omega^+(f)(z) \right| &\leq \omega(z)^{1/2} \int_{\mathbb{D}} \left| f(\xi) \right| \left| K_z(\xi) \right| \omega(\xi) \, dA(\xi) \\
&\leq \| f \|_{L^\infty(\omega^{1/2})} \omega(z)^{1/2} \int_{\mathbb{D}} \left| K_z(\xi) \right| \omega(\xi)^{1/2} \, dA(\xi) \\
&\lesssim \| f \|_{L^\infty(\omega^{1/2})}.
\end{aligned}
$$

This shows that P_ω^+ is bounded on $L^\infty(\omega^{1/2})$. The proof is complete. □

Theorem 3.2 *Let $1 \leq p < \infty$ and $\omega \in \mathcal{E}$. The Bergman projection $P_\omega : L_\omega^p \longrightarrow A_\omega^p$ is bounded. Moreover, $P_\omega : L^\infty(\omega^{1/2}) \to A^\infty(\omega^{1/2})$ is also bounded.*

Proof In view of Theorem 3.1, it remains to see that $P_\omega f$ defines an analytic function on $\mathbb{D}$. This follows easily by the density of the polynomials, the boundedness of P_ω^+, and the completeness of A_ω^p. □

Corollary 3.3 *Let $\omega \in \mathcal{E}$. The reproducing formula $f = P_\omega f$ holds for each $f \in A_\omega^1$.*

Proof This is an immediate consequence of the boundedness of the Bergman projection and the density of the polynomials. $\square$

4 Duality

As in the case of the standard Bergman spaces, one can use the result just proved on the boundedness of the Bergman projection P_ω in L_ω^p to identify the dual spaces of A_ω^p. As usual, if X is a Banach space, we denote its dual by X^*. Next two results (Theorems 4.1 and 4.2) on the duality of Bergman spaces with exponential type weights appear also on [7].

Theorem 4.1 *Let $\omega \in \mathcal{E}$ and $1 < p < \infty$. The dual space of A_ω^p can be identified (with equivalent norms) with $A_\omega^{p'}$ under the integral pairing*

$$\langle f, g \rangle_\omega = \int_{\mathbb{D}} f(z)\overline{g(z)}\omega(z)\,dA(z).$$

Here p' denotes the conjugate exponent of p, that is, $p' = p/(p-1)$.

Proof Let $1 < p < \infty$ and let $p' = p/(p-1)$ be its dual exponent. Given a function $g \in A_\omega^{p'}$, Hölder's inequality implies that the linear functional $\psi_g : A_\omega^p \longrightarrow \mathbb{C}$ defined by

$$\psi_g(f) := \int_{\mathbb{D}} f(\xi)\overline{g(\xi)}\omega(\xi)\,dA(\xi), \quad f \in A_\omega^p,$$

is bounded with $\|\psi_g\| \leq \|g\|_{A_\omega^{p'}}$.

Conversely, let $T \in (A_\omega^p)^*$. By the Hahn–Banach theorem, we can extend T to an element $\widetilde{T} \in (L_\omega^p)^*$ such that $\|\widetilde{T}\| = \|T\|$. By the Riesz representation theorem, there exists $H \in L^{p'}(\mathbb{D}, \omega^{p/2}\,dA)$ with $\|H\|_{L^{p'}(\omega^{p/2})} = \|\widetilde{T}\| = \|T\|$ such that

$$\widetilde{T}(f) = \int_{\mathbb{D}} f(\xi)\overline{H(\xi)}\omega(\xi)^{p/2}\,dA(\xi)$$

for every $f \in A_\omega^p$. Consider the function $h(\xi) = H(\xi)\omega(\xi)^{\frac{p}{2}-1}$. Then $h \in L_\omega^{p'}$ with

$$\|h\|_{L_\omega^{p'}} = \|H\|_{L^{p'}(\omega^{p/2})} = \|T\|$$

and

$$T(f) = \widetilde{T}(f) = \int_{\mathbb{D}} f(\xi)\overline{h(\xi)}\omega(\xi)\,dA(\xi), \quad f \in A_\omega^p.$$

Let $g = P_\omega h$. By Theorem 3.2, the Bergman projection $P_\omega : L_\omega^{p'} \longrightarrow A_\omega^{p'}$ is bounded. Thus $g \in A_\omega^{p'}$ with

$$\|g\|_{A_\omega^{p'}} = \|P_\omega h\|_{A_\omega^{p'}} \lesssim \|h\|_{L_\omega^{p'}} = \|T\|.$$

From Fubini's theorem it is easy to see that P_ω is self-adjoint. Indeed,

$$
\begin{aligned}
\langle P_\omega f, g \rangle_\omega &= \int_{\mathbb{D}} P_\omega f(\xi) \overline{g(\xi)} \omega(\xi) \, dA(\xi) \\
&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} f(s) \overline{K_\xi(s)} \omega(s) \, dA(s) \right) \overline{g(\xi)} \omega(\xi) \, dA(\xi) \\
&= \int_{\mathbb{D}} \overline{\left(\int_{\mathbb{D}} g(\xi) \overline{K_s(\xi)} \omega(\xi) \, dA(\xi) \right)} f(s) \omega(s) \, dA(s) \\
&= \int_{\mathbb{D}} \overline{P_\omega g(s)} f(s) \omega(s) \, dA(s) = \langle f, P_\omega g \rangle_\omega.
\end{aligned}
$$

The interchange of the order of integration is well justified, because of the boundedness of the operator P_ω^+ (see Theorem 3.1) given by

$$
P_\omega^+ f(z) = \int_{\mathbb{D}} f(\zeta) |K_z(\zeta)| \omega(\zeta) \, dA(\zeta).
$$

Therefore, since $f = P_\omega f$ for every $f \in A_\omega^p$, according to Corollary 3.3, we get

$$
\begin{aligned}
T(f) &= \int_{\mathbb{D}} f(\xi) \overline{h(\xi)} \omega(\xi) \, dA(\xi) \\
&= \langle f, P_\omega h \rangle_\omega = \langle f, g \rangle_\omega = \psi_g(f).
\end{aligned}
$$

Finally, the function g is unique. Indeed, if there is another function $\widetilde{g} \in A_\omega^{p'}$ with $T(f) = \psi_g(f) = \psi_{\widetilde{g}}(f)$ for every $f \in A_\omega^p$, then by taking $f = K_a$ for each $a \in \mathbb{D}$ (that belongs to A_ω^p due to Lemma 2.4) and using the reproducing formula, we obtain

$$
g(a) = \overline{\psi_g(K_a)} = \overline{\psi_{\widetilde{g}}(K_a)} = \widetilde{g}(a), \quad a \in \mathbb{D}.
$$

Thus, any bounded linear functional T is of the form $T = \psi_g$ for some unique $g \in A_\omega^{p'}$ and, furthermore,

$$
\|T\| \asymp \|g\|_{A_\omega^{p'}}.
$$

The proof is complete. $\qquad\square$

Theorem 4.2 *Let $\omega \in \mathcal{E}$. The dual space of A_ω^1 can be identified* (*with equivalent norms*) *with $A^\infty(\omega^{1/2})$ under the integral pairing $\langle f, g \rangle_\omega$.*

Proof Let $g \in A_\omega^\infty$. The linear functional $\psi_g : A_\omega^1 \longrightarrow \mathbb{C}$ defined by $\psi_g(f) := \langle f, g \rangle_\omega$ is bounded with $\|\psi_g\| \leq \|g\|_{L^\infty(\omega^{1/2})}$ since, for every $f \in A_\omega^1$,

$$
|\psi_g(f)| \leq \|g\|_{L^\infty(\omega^{1/2})} \|f\|_{A_\omega^1}.
$$

Conversely, let $T \in (A_\omega^1)^*$. Consider the space X that consists of the functions of the form $h = f \omega^{1/2}$ with $f \in A_\omega^1$. Clearly, X is a subspace of $L^1(\mathbb{D}, dA)$ and $F(h) := T(h\omega^{-1/2}) = T(f)$

defines a bounded linear functional on X with $\|F\| = \|T\|$. By the Hahn–Banach theorem, F has an extension $\widetilde{F} \in (L^1(\mathbb{D}, dA))^*$ with $\|\widetilde{F}\| = \|F\|$. Hence, there is a function $G \in L^\infty(\mathbb{D}, dA)$ with $\|G\|_{L^\infty(\mathbb{D}, dA)} = \|F\|$ such that

$$F(h) = \widetilde{F}(h) = \int_{\mathbb{D}} h(\xi)\overline{G(\xi)}\, dA(\xi), \quad h \in X,$$

or

$$T(f) = \int_{\mathbb{D}} f(\xi)\overline{G(\xi)}\omega(\xi)^{1/2}\, dA(\xi), \quad f \in A_\omega^1.$$

Consider the function $H(z) = \omega(z)^{-1/2}G(z)$. Then $H \in L^\infty(\omega^{1/2})$ with

$$\|H\|_{L^\infty(\omega^{1/2})} = \|G\|_{L^\infty(\mathbb{D}, dA)} = \|F\| = \|T\|.$$

By Theorem 3.2, the function $g = P_\omega H$ is in A_ω^∞ with

$$\|g\|_{A^\infty(\omega^{1/2})} \lesssim \|H\|_{L^\infty(\omega^{1/2})} = \|T\|.$$

Also, for $f \in A_\omega^1$, by the reproducing formula, we have

$$T(f) = \int_{\mathbb{D}} f(\xi)\overline{H(\xi)}\omega(\xi)\, dA(\xi) = \langle P_\omega f, H\rangle_\omega = \langle f, P_\omega H\rangle_\omega = \psi_g(f).$$

Finally, as in the proof of Theorem 4.1, the function g is unique. $\qquad\square$

Corollary 4.3 *Let $\omega \in \mathcal{E}$. The set E of finite linear combinations of reproducing kernels is dense in A_ω^p, $1 \le p < \infty$.*

Proof Since E is a linear subspace of A_ω^p, by standard functional analysis and the duality results in Theorems 4.1 and 4.2, it is enough to prove that $g \equiv 0$ if $g \in A_\omega^{p'}$ satisfies $\langle f, g\rangle_\omega = 0$ for each f in E (with p' being the conjugate exponent of p, and $g \in A^\infty(\omega^{1/2})$ if $p = 1$). But, taking $f = K_z$ for each $z \in \mathbb{D}$ and using the reproducing formula, we get $g(z) = P_\omega g(z) = \langle g, K_z\rangle_\omega = 0$ for each $z \in \mathbb{D}$. This finishes the proof. $\qquad\square$

Our next goal is to identify the predual of A_ω^1. For a given weight v, we introduce the space $A_0(v)$ consisting of those functions $f \in A^\infty(v)$ with $\lim_{|z| \to 1^-} v(z)|f(z)| = 0$. Clearly, $A_0(v)$ is a closed subspace of $A^\infty(v)$.

Theorem 4.4 *Let $\omega \in \mathcal{E}$. Under the integral pairing $\langle f, g\rangle_\omega$, the dual space of $A_0(\omega^{1/2})$ can be identified (with equivalent norms) with A_ω^1.*

Proof If $g \in A_\omega^1$, clearly $\Lambda_g(f) = \langle f, g\rangle_\omega$ defines a bounded linear functional in $A_0(\omega^{1/2})$ with $\|\Lambda_g\| \le \|g\|_{A_\omega^1}$. Conversely, assume that $\Lambda \in (A_0(\omega^{1/2}))^*$. Consider the space X that consists of functions of the form $h = f\omega^{1/2}$ with $f \in A_0(\omega^{1/2})$. Clearly, X is a subspace of $C_0(\mathbb{D})$ (the space of all continuous functions vanishing at the boundary) and $T(h) = \Lambda(\omega^{-1/2}h) = \Lambda(f)$ defines a bounded linear functional on X with $\|T\| = \|\Lambda\|$. By the Hahn–Banach theorem,

T has an extension $\widetilde{T} \in (C_0(\mathbb{D}))^*$ with $\|\widetilde{T}\| = \|T\|$. Hence, by the Riesz representation theorem, there is a measure $\mu \in \mathcal{M}(\mathbb{D})$ (the Banach space of all complex Borel measures μ equipped with the variation norm $\|\mu\|_{\mathcal{M}}$) with $\|\mu\|_{\mathcal{M}} = \|T\|$ such that

$$T(h) = \widetilde{T}(h) = \int_{\mathbb{D}} h(\zeta)\, d\mu(\zeta), \quad h \in X,$$

or

$$\Lambda(f) = \int_{\mathbb{D}} f(\zeta)\omega(\zeta)^{1/2}\, d\mu(\zeta), \quad f \in A_0(\omega^{1/2}).$$

Consider the function g defined on the unit disk by

$$\overline{g(z)} = \int_{\mathbb{D}} K_z(\zeta)\omega(\zeta)^{1/2}\, d\mu(\zeta), \quad z \in \mathbb{D}.$$

Clearly, g is analytic on $\mathbb{D}$ and, by Fubini's theorem and condition (1.2), we have

$$
\begin{aligned}
\|g\|_{A^1_\omega} &\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |K_z(\zeta)|\omega(\zeta)^{1/2}\, d|\mu|(\zeta) \right) \omega(z)^{1/2}\, dA(z) \\
&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |K_\zeta(z)|\omega(z)^{1/2}\, dA(z) \right) \omega(\zeta)^{1/2}\, d|\mu|(\zeta) \\
&\lesssim |\mu|(\mathbb{D}) = \|\mu\|_{\mathcal{M}} = \|\Lambda\|,
\end{aligned}
$$

proving that g belongs to A^1_ω. Now, since $A_0(\omega^{1/2}) \subset A^2_\omega$, the reproducing formula $f(\zeta) = \langle f, K_\zeta \rangle_\omega$ holds for all $f \in A_0(\omega^{1/2})$. This and Fubini's theorem yield

$$
\begin{aligned}
\Lambda_g(f) = \langle f, g \rangle_\omega &= \int_{\mathbb{D}} f(z) \left(\int_{\mathbb{D}} K_z(\zeta)\omega(\zeta)^{1/2}\, d\mu(\zeta) \right) \omega(z)\, dA(z) \\
&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} f(z)\overline{K_\zeta(z)}\omega(z)\, dA(z) \right) \omega(\zeta)^{1/2}\, d\mu(\zeta) \\
&= \int_{\mathbb{D}} f(\zeta)\omega(\zeta)^{1/2}\, d\mu(\zeta) = \Lambda(f).
\end{aligned}
$$

By the reproducing formula, the function g is uniquely determined by the identity $g(z) = \overline{\Lambda(K_z)}$. This completes the proof. $\qquad\square$

For the case of normal weights, the analogues of Theorems 4.2 and 4.4 were obtained by Shields and Williams in [32]. They also asked what happens with the exponential weights, problem that is solved in the present paper.

5 Atomic decomposition

For $1 \leq p < \infty$, in this section we are going to obtain an atomic decomposition for the large weighted Bergman space A^p_ω, that is, we show that every function in the Bergman spaces A^p_ω with ω in the class $\mathcal{E}$ can be decomposed into a series of kernel functions. With the help of the duality results and the estimates for the p-norm of the reproducing kernels K_z, we can reach our goal. Before stating the main theorem of this section, we need two auxiliary lemmas as follows.

Lemma 5.1 *Let $\omega \in \mathcal{W}$. There is a sequence $\{z_n\} \subset \mathbb{D}$ such that*

$$\sum_{k=1}^{\infty} |f(z_k)|^p \omega(z_k)^{p/2} \tau(z_k)^2 \gtrsim \|f\|_{A_\omega^p}^p$$

for all $f \in A_\omega^p$ and $1 \le p < \infty$.

Proof Let $\{z_k\}$ be a (ε, τ)-lattice on $\mathbb{D}$ (that exists by Lemma B) with $\varepsilon > 0$ small enough to be specified later. Let $f \in A_\omega^p$. We consider

$$I_f(k) := \sum_{k=1}^{\infty} |f(z_k)|^p \omega(z_k)^{p/2} \tau(z_k)^2.$$

We have

$$\|f\|_{A_\omega^p}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z)^{p/2} \, dA(z)$$

$$\le C \left[\sum_{k=1}^{\infty} \int_{D(\varepsilon\tau(z_k))} \left(|f(z)| \omega(z)^{1/2} - |f(z_k)| \omega(z_k)^{1/2} \right)^p \, dA(z) + C\varepsilon^2 I_f(k) \right].$$

For $z \in D(\varepsilon\tau(z_k))$, there exists $\xi_{k,z} \in [z, z_k]$ such that

$$\left(|f(z)| \omega(z)^{1/2} - |f(z_k)| \omega(z_k)^{1/2} \right)^p \le \left| \nabla \left(|f| \omega^{1/2} \right)(\xi_{k,z}) \right|^p |z - z_k|^p$$

$$\le \varepsilon^p \tau(z_k)^p \left| \nabla \left(|f| \omega^{1/2} \right)(\xi_{k,z}) \right|^p.$$

This together with Lemma 2.2, with $\delta_0 \in (0, m_\tau)$ fixed, yields

$$\int_{D(\varepsilon\tau(z_k))} \left(|f(z)| \omega(z)^{1/2} - |f(z_k)| \omega(z_k)^{1/2} \right)^p \, dA(z)$$

$$\le C\varepsilon^p \tau(z_k)^p \int_{D(\varepsilon\tau(z_k))} \left(\frac{1}{\tau(\xi_{k,z})^{p+2}} \int_{D(\delta_0\tau(\xi_{k,z}))} |f(\eta)|^p \omega(\eta)^{p/2} \, dA(\eta) \right) dA(z).$$

Using that $\tau(\xi_{k,z}) \asymp \tau(z_k)$ and $D(\delta_0\tau(\xi_{k,z})) \subset D(3\delta_0\tau(z_k))$ for $z \in D(\varepsilon\tau(z_k))$, we obtain

$$\int_{D(\varepsilon\tau(z_k))} \left(|f(z)| \omega(z)^{1/2} - |f(z_k)| \omega(z_k)^{1/2} \right)^p \, dA(z)$$

$$\le C\varepsilon^{p+2} \left(\int_{D(3\delta_0\tau(z_k))} |f(\eta)|^p \omega(\eta)^{p/2} \, dA(\eta) \right).$$

Therefore,

$$\|f\|_{A_\omega^p}^p \le C\varepsilon^{p+2} \sum_{k=1}^{\infty} \int_{D(3\delta_0\tau(z_k))} |f(\eta)|^p \omega(\eta)^{p/2} \, dA(\eta) + C\varepsilon^2 I_f(k).$$

By Lemma B, every point $z \in \mathbb{D}$ belongs to at most $C\varepsilon^{-2}$ of the sets $D(3\delta_0\tau(z_k))$, and therefore

$$\|f\|_{A_\omega^p}^p \le C\varepsilon^p \|f\|_{A_\omega^p}^p + C\varepsilon^2 I_f(n).$$

Thus, taking $\varepsilon > 0$ so that $C\varepsilon^p < 1/2$, we get the desired result. $\square$

We may rephrase Lemma 5.1 by saying that every (ε, τ)-lattice, with $\varepsilon > 0$ small enough, is a sampling sequence for the Bergman space A_ω^p. Recall that $\{z_k\} \subset \mathbb{D}$ is a sampling sequence for the Bergman space A_ω^p if

$$\|f\|_{A_\omega^p}^p \asymp \sum_k |f(z_k)|^p \omega(z_k)^{p/2} \tau(z_k)^2$$

for any $f \in A_\omega^p$. Just note that Lemma 5.1 gives one inequality, and the other follows by standard methods using Lemma A and the lattice properties. Sampling sequences on the classical Bergman space were characterized by K. Seip [30] (see also the monographs [10] and [31]). For sampling sequences on large weighted Bergman spaces, we refer to [4].

Lemma 5.2 *Let $\omega \in \mathcal{E}$, $1 \le p < \infty$, and $(z_k)_{k \in \mathbb{N}} \subset \mathbb{D}$ be the sequence defined in Lemma B. The function given by*

$$F(z) := \sum_{k=1}^\infty \lambda_k \omega(z_k)^{1/2} \tau(z_k)^{2(\frac{p-1}{p})} K_{z_k}(z) \tag{5.1}$$

belongs to A_ω^p for every sequence $\lambda = \{\lambda_k\} \in \ell^p$. Moreover,

$$\|F\|_{A_\omega^p} \lesssim \|\lambda\|_{\ell^p}.$$

Proof By Hölder's inequality, Lemma A, and Lemma 2.4, it is easy to see that the partial sums of the series in (5.1) converge uniformly on compact subsets of $\mathbb{D}$. Thus, F defines an analytic function on $\mathbb{D}$. Furthermore, for p = 1, using (1.2) we have

$$\|F\|_{A_\omega^1} \le \sum_{k=1}^\infty |\lambda_k| \omega(z_k)^{1/2} \|K_{z_k}\|_{A_\omega^1} \lesssim \|\lambda\|_{\ell^1}.$$

For the case $p > 1$, consider

$$M(z) := \sum_{k=1}^\infty \tau(z_k)^2 \omega(z_k)^{1/2} |K_{z_k}(z)|.$$

By Hölder's inequality, we have

$$\|F\|_{A_\omega^p}^p \le \int_{\mathbb{D}} \left(\sum_{k=0}^\infty |\lambda_k| \omega(z_k)^{1/2} \tau(z_k)^{2(\frac{p-1}{p})} |K_{z_k}(z)| \right)^p \omega(z)^{p/2} \, dA(z).$$

$$\lesssim \int_{\mathbb{D}} \left(\sum_{k=1}^\infty |\lambda_k|^p \omega(z_k)^{1/2} |K_{z_k}(z)| \right) M(z)^{p-1} \omega(z)^{p/2} \, dA(z).$$

On the other hand, using Lemma A, Lemma B, and (1.2), we have

$$M(z) := \sum_{k=1}^\infty \tau(z_k)^2 \omega(z_k)^{1/2} |K_{z_k}(z)|$$

$$\lesssim \sum_{k=1}^\infty \int_{D(\delta\tau((z_k))} |K_z(\xi)| \omega(\xi)^{1/2} \, dA(\xi)$$

$$\lesssim \int_{\mathbb{D}} \left|K_z(\xi)\right| \omega(\xi)^{1/2} \, dA(\xi) \lesssim \omega(z)^{-1/2}.$$

Therefore, applying again (1.2), we obtain

$$\|F\|_{A_\omega^p}^p \lesssim \int_{\mathbb{D}} \left(\sum_{k=0}^{\infty} |\lambda_k|^p \omega(z_k)^{1/2} \left|K_{z_k}(z)\right| \right) \omega(z)^{1/2} \, dA(z)$$

$$\lesssim \sum_{k=1}^{\infty} |\lambda_k|^p \omega(z_k)^{1/2} \int_{\mathbb{D}} \left|K_{z_k}(z)\right| \omega(z)^{1/2} \, dA(z)$$

$$\lesssim \sum_{k=1}^{\infty} |\lambda_k|^p,$$

which completes the proof.   □

Now we are ready to state our main result related to an atomic decomposition of large weighted Bergman spaces A_ω^p for $1 \le p < \infty$. Recall that $k_{p,z}$ is the normalized reproducing kernel in A_ω^p, that is,

$$k_{p,z} = K_z / \|K_z\|_{A_\omega^p}, \quad z \in \mathbb{D}.$$

Theorem 5.3 *Let $\omega \in \mathcal{E}$ and $1 \le p < \infty$. There exists a τ-lattice $\{z_n\} \subset \mathbb{D}$ such that*:
  (i) *For any $\lambda = \{\lambda_n\} \in \ell^p$, the function*

$$f(z) = \sum_n \lambda_n k_{p,z_n}(z)$$

   is in A_ω^p with $\|f\|_{A_\omega^p} \le C\|\lambda\|_{\ell^p}$.
  (ii) *For every $f \in A_\omega^p$, there exists $\lambda = \{\lambda_n\} \in \ell^p$ such that*

$$f(z) = \sum_n \lambda_n k_{p,z_n}(z)$$

   and $\|\lambda\|_{\ell^p} \le C\|f\|_{A_\omega^p}$.

Proof On the one hand, statement (i) is exactly Lemma 5.2. On the other hand, in order to prove (ii), we define a linear operator $S : \ell^p \longrightarrow A_\omega^p$ given by

$$S(\{\lambda_n\}) := \sum_{n=0}^{\infty} \lambda_n k_{p,z_n}.$$

By (i), the operator S is bounded. By the duality results obtained in the previous section, when $1 < p < \infty$, the adjoint operator $S^* : A_\omega^{p'} \to \ell^{p'}$, where p' is the conjugate exponent of p, is defined by

$$\langle Sx, f \rangle_\omega = \langle x, S^*f \rangle_\ell = \sum_n x_n \overline{(S^*f)}_n$$

for every $x \in \ell^p$ and $f \in A_\omega^{p'}$. To compute S^*, let e_n denote the vector that equals 1 at the nth coordinate and equals 0 at the other coordinates. Then $Se_n = k_{p,z_n}$, and using the reproducing formula, we get

$$
\begin{aligned}
\overline{\left(S^*f\right)_n} &= \langle e_n, S^*f \rangle_\ell = \langle Se_n, f \rangle_\omega = \langle k_{p,z_n}, f \rangle_\omega \\
&= \|K_{z_n}\|_{A_\omega^p}^{-1} \cdot \langle K_{z_n}, f \rangle_\omega = \frac{\overline{f(z_n)}}{\|K_{z_n}\|_{A_\omega^p}}.
\end{aligned}
$$

Hence, $S^* : A_\omega^{p'} \longrightarrow \ell^{p'}$ is given by

$$
S^*f = \left\{ \left(S^*f\right)_n \right\} = \left\{ \frac{f(z_n)}{\|K_{z_n}\|_{A_\omega^p}} \right\}_n .
$$

We must prove that S is surjective in order to finish the proof of this case. By a classical result in functional analysis, it is enough to show that S^* is bounded below. By Lemma 5.1 and Lemma 2.4, we obtain

$$
\left\| S^*f \right\|_{\ell^{p'}}^{p'} \asymp \sum_{n=1}^\infty \left| f(z_n) \right|^{p'} \omega(z_n)^{p'/2} \tau(z_n)^2 \gtrsim \|f\|_{A_\omega^{p'}}^{p'},
$$

which shows that S^* is bounded below. Finally, once the surjectivity is proved, the estimate $\|\lambda\|_{\ell^p} \lesssim \|f\|_{A_\omega^p}^p$ is a standard application of the open mapping theorem. When $p = 1$, then $S^* : A^\infty(\omega^{1/2}) \longrightarrow \ell^\infty$ is given by

$$
\left\{ \left(S^*f\right)_n \right\} = \left\{ \frac{f(z_n)}{\|K_{z_n}\|_{A_\omega^1}} \right\}_n .
$$

Hence we must show that

$$
\sup_{z \in \mathbb{D}} \omega(z)^{1/2} \left| f(z) \right| = \|f\|_{A^\infty(\omega^{1/2})} \lesssim \left\| S^*f \right\|_{\ell^\infty} \asymp \sup_n \omega(z_n)^{1/2} \left| f(z_n) \right|
$$

for $f \in A^\infty(\omega^{1/2})$. However, this can be proved with the same method as Lemma 5.1. Indeed, let $z \in \mathbb{D}$. Then there is a point z_n with $z \in D(\varepsilon\tau(z_n))$. By Lemma A, we have

$$
\begin{aligned}
\omega(z)^{1/2} \left| f(z) \right| \leq{} & \frac{C_1}{\varepsilon^2 \tau(z)^2} \int_{D(\varepsilon\tau(z))} \left(\left| f(\zeta) \right| \omega(\zeta)^{1/2} - \left| f(z_n) \right| \omega(z_n)^{1/2} \right) dA(\zeta) \\
& + C_1 \left| f(z_n) \right| \omega(z_n)^{1/2}.
\end{aligned}
$$

As done in the proof of Lemma 5.1, we have

$$
\begin{aligned}
\left| \left| f(\zeta) \right| \omega(\zeta)^{1/2} - \left| f(z_n) \right| \omega(z_n)^{1/2} \right| &\leq C_2 \varepsilon \frac{1}{\tau(z_n)^2} \int_{D(3\delta_0\tau(z_n))} \left| f(\xi) \right| \omega(\xi)^{1/2} \, dA(\xi) \\
&\leq C_3 \varepsilon \|f\|_{A^\infty(\omega^{1/2})}.
\end{aligned}
$$

Thus, putting this in the previous estimate, we obtain

$$
\omega(z)^{1/2} \left| f(z) \right| \leq C_4 \varepsilon \|f\|_{A^\infty(\omega^{1/2})} + C_1 \sup_n \omega(z_n)^{1/2} \left| f(z_n) \right|.
$$

Finally, taking the supremum on z and $\varepsilon > 0$ small enough so that $C_4 \varepsilon \leq 1/2$, we have

$$\|f\|_{A^\infty(\omega^{1/2})} \lesssim \sup_n \omega(z_n)^{1/2} |f(z_n)|.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

6 Toeplitz operators

In this section we are going to extend the results given in [3, Theorem 1.1] to the non-Hilbert space setting, when the weight ω is in the class $\mathcal{E}$. Concretely, we characterize the bounded and compact operators T_μ acting from A_ω^p to A_ω^q when $1 \leq p, q < \infty$. Recall that the Toeplitz operator T_μ is defined by

$$T_\mu f(z) = \int_{\mathbb{D}} f(\xi) \overline{K_z(\xi)} \omega(\xi) \, d\mu(\xi).$$

Note that T_μ is very loosely defined here, because it is not clear when the integrals above will converge, even if the measure μ is finite. We suppose that μ is a finite positive Borel measure that satisfies the condition

$$\int_{\mathbb{D}} |K_z(\xi)|^2 \omega(\xi) \, d\mu(\xi) < \infty. \tag{6.1}$$

Then the Toeplitz operator T_μ is well defined on a dense subset of A_ω^p, $1 \leq p < \infty$. In fact, by Corollary 4.3 and Theorem 5.3, the set E of finite linear combinations of reproducing kernels is dense in A_ω^p. Therefore, it follows from condition (6.1) and the Cauchy–Schwarz inequality that $T_\mu(f)$ is well defined for any $f \in E$. Also, recall that, for $\delta \in (0, m_\tau)$, the averaging function of μ on $\mathbb{D}$ is given by

$$\widehat{\mu}_\delta(z) := \frac{\mu(D(\delta\tau(z))}{\tau(z)^2}, \quad z \in \mathbb{D}.$$

Theorem 6.1 *Let $\omega \in \mathcal{E}$, $1 \leq p \leq q < \infty$, and μ be a finite positive Borel measure on $\mathbb{D}$ satisfying (6.1). Then $T_\mu : A_\omega^p \to A_\omega^q$ is bounded if and only if, for each $\delta \in (0, m_\tau)$ sufficiently small,*

$$E(\mu) = \sup_{z \in \mathbb{D}} \tau(z)^{2(\frac{1}{q} - \frac{1}{p})} \widehat{\mu}_\delta(z) < \infty. \tag{6.2}$$

Moreover,

$$\|T_\mu\|_{A_\omega^p \to A_\omega^q} \asymp E(\mu).$$

Proof Since we have the estimate $\|K_z\|_{A_\omega^p} \asymp \omega(z)^{-1/2} \tau(z)^{-2(p-1)/p}$, if we assume that the Toeplitz operator $T_\mu : A_\omega^p \longrightarrow A_\omega^q$ is bounded, then we obtain (6.2) with the same argument as in the proof of Theorem 1.1 in [3].

Conversely, we suppose that (6.2) holds. We first prove that

$$\int_{\mathbb{D}} |K_z(\xi)| \omega(\xi)^{1/2} \tau(\xi)^{-2(\frac{1}{p} - \frac{1}{q})} \, d\mu(\xi) \lesssim E(\mu) \omega(z)^{-1/2}. \tag{6.3}$$

Indeed, by Lemma A, we have

$$\left|K_z(\xi)\right|\omega(\xi)^{1/2} \lesssim \frac{1}{\tau(\xi)^2} \int_{D(\frac{\delta}{2}\tau(\xi))} \left|K_z(s)\right|\omega(s)^{1/2}\,dA(s).$$

Then, by Fubini's theorem, the fact that $\tau(s) \asymp \tau(\xi)$ for $s \in D(\delta\tau(\xi))$, and condition (1.2), we get

$$\int_{\mathbb{D}} \left|K_z(\xi)\right|\omega(\xi)^{1/2}\tau(\xi)^{-2(\frac{1}{p}-\frac{1}{q})}\,d\mu(\xi) \lesssim \int_{\mathbb{D}} \left|K_z(s)\right|\omega(s)^{1/2}\tau(s)^{2(\frac{1}{q}-\frac{1}{p})}\widehat{\mu}_\delta(s)\,dA(s)$$

$$\lesssim E(\mu)\int_{\mathbb{D}} \left|K_z(s)\right|\omega(s)^{1/2}\,dA(s)$$

$$\lesssim E(\mu)\omega(z)^{-1/2}.$$

This establishes (6.3). Now we proceed to prove that T_μ is bounded. If $q > 1$, by Hölder's inequality, we obtain

$$\left|T_\mu f(z)\right|^q \le \left(\int_{\mathbb{D}} \left|f(\xi)\right|\left|K_z(\xi)\right|\omega(\xi)\,d\mu(\xi)\right)^q$$

$$\le \left(\int_{\mathbb{D}} \left|f(\xi)\right|^q\omega(\xi)^{\frac{q+1}{2}}\left|K_z(\xi)\right|\frac{d\mu(\xi)}{\tau(\xi)^{2(\frac{1}{p}-\frac{1}{q})(1-q)}}\right)$$

$$\times \left(\int_{\mathbb{D}} \left|K_z(\xi)\right|\omega(\xi)^{1/2}\frac{d\mu(\xi)}{\tau(\xi)^{2(\frac{1}{p}-\frac{1}{q})}}\right)^{q-1}.$$

Using (6.3), we have

$$\left|T_\mu f(z)\right|^q \lesssim E(\mu)^{q-1}\left(\int_{\mathbb{D}} \left|f(\xi)\right|^q\omega(\xi)^{\frac{q+1}{2}}\left|K_z(\xi)\right|\tau(\xi)^{2(\frac{1}{p}-\frac{1}{q})(q-1)}\,d\mu(\xi)\right)\omega(z)^{-\frac{(q-1)}{2}}.$$

If $q = 1$, this holds directly. By Fubini's theorem and condition (1.2), we obtain

$$\|T_\mu f\|_{A_\omega^q}^q = \int_{\mathbb{D}} \left|T_\mu f(z)\right|^q\omega(z)^{q/2}\,dA(z)$$

$$\lesssim E(\mu)^{q-1}\int_{\mathbb{D}} \left|f(\xi)\right|^q\omega(\xi)^{\frac{(q+1)}{2}}\tau(\xi)^{2(q-1)(\frac{1}{p}-\frac{1}{q})}\left(\int_{\mathbb{D}} \left|K_\xi(z)\right|\omega(z)^{1/2}\,dA(z)\right)d\mu(\xi)$$

$$\lesssim E(\mu)^{q-1}\int_{\mathbb{D}} \left|f(\xi)\right|^q\omega(\xi)^{q/2}\tau(\xi)^{2(q-1)(\frac{1}{p}-\frac{1}{q})}\,d\mu(\xi).$$

Consider the measure ν given by

$$d\nu(\xi) := \omega(\xi)^{q/2}\tau(\xi)^{2(q-1)(\frac{1}{p}-\frac{1}{q})}\,d\mu(\xi).$$

Since (6.2) holds, by Theorem B, the identity $I_\nu : A_\omega^p \longrightarrow L^q(\mathbb{D}, d\nu)$ is bounded. Moreover, $\|I_\nu\| \lesssim E(\mu)^{1/q}$. Therefore,

$$\|T_\mu f\|_{A_\omega^q}^q \lesssim E(\mu)^{q-1}\int_{\mathbb{D}} \left|f(z)\right|^q\,d\nu(z) \lesssim E(\mu)^q \cdot \|f\|_{A_\omega^p}^q. \tag{6.4}$$

This finishes the proof. $\square$

In order to describe the boundedness of $T_\mu : A^p_\omega \to A^q_\omega$ when $1 \le q < p < \infty$, we need first an auxiliary result.

Proposition 6.2 *Let $\omega \in \mathcal{E}$ and $1 < q < p < \infty$. If $\widehat{\mu}_\delta \in L^{\frac{pq}{p-q}}(\mathbb{D}, dA)$, then*

$$J_{\delta,q} := \int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \left| f(\xi) \right| \omega(\xi)^{1/2} \, d\mu(\xi) \right)^q dA(z)$$

$$\lesssim \|\widehat{\mu}_\delta\|^q_{L^{\frac{pq}{p-q}}(\mathbb{D})} \cdot \|f\|^q_{A^p_\omega}$$

for any $f \in A^p_\omega$.

Proof For $z \in \mathbb{D}$ and $\xi \in D(\delta\tau(z))$, by Lemma A, Lemma B, and (2.1), we obtain

$$\left| f(\xi) \right| \omega(\xi)^{1/2} \lesssim \left(\frac{1}{\tau(\xi)^2} \int_{D(\frac{\delta}{3}\tau(\xi))} \left| f(s) \right|^p \omega(s)^{p/2} \, dA(s) \right)^{1/p}$$

$$\lesssim \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \left| f(s) \right|^p \omega(s)^{p/2} \, dA(s) \right)^{1/p}.$$

This gives

$$\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \left| f(\xi) \right| \omega(\xi)^{1/2} \, d\mu(\xi)$$

$$\lesssim \widehat{\mu}_\delta(z) \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \left| f(s) \right|^p \omega(s)^{p/2} \, dA(s) \right)^{1/p}.$$

Therefore,

$$J_{\delta,q} \lesssim \int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \left| f(s) \right|^p \omega(s)^{p/2} \, dA(s) \right)^{q/p} \widehat{\mu}_\delta(z)^q \, dA(z).$$

Applying Hölder's inequality, we get

$$J_{\delta,q} \lesssim \left(\int_{\mathbb{D}} \frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \left| f(s) \right|^p \omega(s)^{p/2} \, dA(s) \, dA(z) \right)^{q/p} \|\widehat{\mu}_\delta\|^q_{L^{\frac{pq}{p-q}}(\mathbb{D})}. \tag{6.5}$$

On the other hand, by Fubini's theorem and $\tau(z) \asymp \tau(s)$, for $s \in D(\delta\tau(z))$, we have

$$\int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \left| f(s) \right|^p \omega(s)^{p/2} \, dA(s) \right) dA(z) \lesssim \|f\|^p_{A^p_\omega}.$$

Combining this with (6.5), we get

$$J_{\delta,q} \lesssim \|f\|^q_{A^p_\omega} \cdot \|\widehat{\mu}_\delta\|^q_{L^{\frac{pq}{p-q}}(\mathbb{D})}.$$

The proof is complete.   □

Theorem 6.3 *Let $\omega \in \mathcal{E}$, $1 \le q < p < \infty$, and μ be a finite positive Borel measure on $\mathbb{D}$ satisfying (6.1). The following conditions are equivalent:*

(i) *The Toeplitz operator $T_\mu : A_\omega^p \to A_\omega^q$ is bounded.*
(ii) *For each sufficiently small $\delta > 0$, $\widehat{\mu}_\delta \in L^{\frac{pq}{p-q}}(\mathbb{D}, dA)$.*

Moreover,

$$\|T_\mu\|_{A_\omega^p \to A_\omega^q} \asymp \|\widehat{\mu}_\delta\|_{L^{\frac{pq}{p-q}}(\mathbb{D})}.$$

Proof (i) $\Longrightarrow$ (ii) For an arbitrary sequence $\lambda = \{\lambda_k\} \in \ell^p$, we consider the function

$$G_t(z) = \sum_{k=1}^{\infty} \lambda_k r_k(t) \omega(z_k)^{1/2} \tau(z_k)^{2(\frac{p-1}{p})} K_{z_k}(z), \quad 0 < t < 1,$$

where $r_k(t)$ is a sequence of Rademacher functions (see [20] or Appendix A of [11]) and $\{z_k\}$ is the sequence given in Lemma B. Because of the norm estimate

$$\|K_z\|_{A_\omega^p} \asymp \omega(z)^{-1/2} \tau(z)^{-2(p-1)/p}$$

given in Lemma 2.4, by part (i) of Theorem 5.3, we obtain

$$\|G_t\|_{A_\omega^p} \lesssim \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p}.$$

Thus, the boundedness of $T_\mu : A_\omega^p \to A_\omega^q$ gives

$$\|T_\mu G_t\|_{A_\omega^q}^q \lesssim \|T_\mu\|^q \cdot \|\lambda\|_{\ell^p}^q.$$

In other words, we have

$$\int_{\mathbb{D}} \left| \sum_{k=1}^{\infty} \lambda_k r_k(t) \omega(z_k)^{1/2} \tau(z_k)^{2(\frac{p-1}{p})} T_\mu K_{z_k}(z) \right|^q \omega(z)^{q/2} \, dA(z) \lesssim \|T_\mu\|^q \cdot \|\lambda\|_{\ell^p}^q.$$

Integrating with respect to t from 0 to 1, applying Fubini's theorem, and invoking Khinchine's inequality (see [20]), we obtain

$$B := \int_{\mathbb{D}} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 \omega(z_k) \tau(z_k)^{4(\frac{p-1}{p})} \left| T_\mu K_{z_k}(z) \right|^2 \right)^{q/2} \omega(z)^{q/2} \, dA(z) \lesssim \|T_\mu\|^q \cdot \|\lambda\|_{\ell^p}^q.$$

Let χ_k denote the characteristic function of the set $D(3\delta\tau(z_k))$. Since the covering $\{D(3\delta\tau(z_k))\}$ of $\mathbb{D}$ has finite multiplicity N, we have

$$\sum_{k=1}^{\infty} |\lambda_k|^q \omega(z_k)^{q/2} \tau(z_k)^{2q(\frac{p-1}{p})} \int_{D(3\delta\tau(z_k))} \left| T_\mu K_{z_k}(z) \right|^q \omega(z)^{q/2} \, dA(z)$$

$$= \int_{\mathbb{D}} \sum_{k=1}^{\infty} |\lambda_k|^q \omega(z_k)^{q/2} \tau(z_k)^{2q(\frac{p-1}{p})} \left| T_\mu K_{z_k}(z) \right|^q \chi_k(z) \omega(z)^{q/2} \, dA(z) \tag{6.6}$$

$$\leq \max\{1, N^{1-\frac{q}{2}}\} B.$$

Now, using Lemma A yields

$$\sum_{k=1}^{\infty} |\lambda_k|^q \omega(z_k)^{q/2} \tau(z_k)^{2q(\frac{p-1}{p})+2} \big| T_\mu K_{z_k}(z_k) \big|^q \omega(z_k)^{q/2} \lesssim \|T_\mu\|^q \cdot \|\lambda\|_{\ell^p}^q.$$

On the other hand, since for small $\delta > 0$ we have $|K_{z_k}(z)| \asymp \|K_{z_k}\|_{A_\omega^2} \|K_z\|_{A_\omega^2}$ for every $z \in D(\delta\tau(z_k))$, applying statement (a) of Theorem A and (2.1), we have

$$\big| T_\mu K_{z_k}(z_k) \big| \geq \int_{D(\delta\tau(z_k))} \big| K_{z_k}(z) \big|^2 \omega(z) \, d\mu(z)$$

$$\asymp \|K_{z_k}\|_{A_\omega^2}^2 \int_{D(\delta\tau(z_k))} \|K_z\|_{A_\omega^2}^2 \omega(z) \, d\mu(z)$$

$$\asymp \frac{\omega(z_k)^{-1} \widehat{\mu}_\delta(z_k)}{\tau(z_k)^2}.$$

That is,

$$\big| T_\mu K_{z_k}(z_k) \big|^q \omega(z_k)^{q/2} \gtrsim \frac{\omega(z_k)^{-\frac{q}{2}} \widehat{\mu}_\delta(z_k)^q}{\tau(z_k)^{2q}}.$$

Therefore,

$$\sum_{k=1}^{\infty} |\lambda_k|^q \tau(z_k)^{2q(\frac{1}{q}-\frac{1}{p})} \widehat{\mu}_\delta(z_k)^q \lesssim \|T_\mu\|^q \cdot \|\lambda\|_{\ell^p}^q.$$

Then, using the duality between $\ell^{p/q}$ and $\ell^{\frac{p}{p-q}}$, we conclude that

$$\sum_{k=1}^{\infty} \big(\tau(z_k)^{2q(\frac{1}{q}-\frac{1}{p})} \widehat{\mu}_\delta(z_k)^q \big)^{\frac{p}{p-q}} \lesssim \|T_\mu\|^{\frac{pq}{p-q}},$$

that means

$$\sum_{k=1}^{\infty} \tau(z_k)^2 \widehat{\mu}_\delta(z_k)^{\frac{pq}{p-q}} \lesssim \|T_\mu\|^{\frac{pq}{p-q}}.$$

This is the discrete version of our condition. To obtain the continuous version, simply note that

$$\widehat{\mu}_\delta(z) \lesssim \widehat{\mu}_{4\delta}(z_k), \quad z \in D\big(\delta\tau(z_k)\big).$$

Then

$$\int_{\mathbb{D}} \widehat{\mu}_\delta(z)^{\frac{pq}{p-q}} \, dA(z) \leq \sum_{k=0}^{\infty} \int_{D(\delta\tau(z_k))} \widehat{\mu}_\delta(z)^{\frac{pq}{p-q}} \, dA(z) \lesssim \sum_{k=0}^{\infty} \tau(z_k)^2 \widehat{\mu}_{4\delta}(z_k)^{\frac{pq}{p-q}}.$$

This finishes the proof of this implication.

(ii) $\implies$ (i) First we begin with the easiest case $q = 1$. By Fubini's theorem and condition (1.2), we have

$$
\begin{aligned}
\|T_\mu f\|_{A^1_\omega} &= \int_{\mathbb{D}} |T_\mu f(z)| \omega(z)^{1/2}\, dA(z) \\
&\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |f(\xi)| |K_z(\xi)| \omega(\xi)\, d\mu(\xi) \right) \omega(z)^{1/2}\, dA(z) \\
&= \int_{\mathbb{D}} |f(\xi)| \left(\int_{\mathbb{D}} |K_\xi(z)| \omega(z)^{1/2}\, dA(z) \right) \omega(\xi)\, d\mu(\xi) \\
&\lesssim \int_{\mathbb{D}} |f(\xi)| \omega(\xi)^{1/2}\, d\mu(\xi).
\end{aligned}
\tag{6.7}
$$

Now, by using Theorem D with the measure given by

$$
d\nu(\xi) := \omega(\xi)^{1/2}\, d\mu(\xi),
$$

we obtain the desired result. Finally, we study the case $1 < q < \infty$. Let $\{z_j\}$ be the sequence given in Lemma B. Applying Lemma A and Lemma B, we obtain

$$
\begin{aligned}
|T_\mu f(z)| &\leq \sum_{j=1}^{\infty} \int_{D(\delta\tau(z_j))} |f(\xi)| |K_z(\xi)| \omega(\xi)\, d\mu(\xi) \\
&\lesssim \sum_{j=1}^{\infty} \int_{D(\delta\tau(z_j))} |f(\xi)| \omega(\xi)^{1/2} \left(\frac{1}{\tau(\xi)^2} \int_{D(\delta\tau(\xi))} |K_z(s)| \omega(s)^{1/2}\, dA(s) \right) d\mu(\xi) \\
&\lesssim \sum_{j=1}^{\infty} \left(\int_{D(\delta\tau(z_j))} |f(\xi)| \omega(\xi)^{1/2} \frac{d\mu(\xi)}{\tau(\xi)^2} \right) \int_{D(3\delta\tau(z_j))} |K_z(s)| \omega(s)^{1/2}\, dA(s).
\end{aligned}
$$

Applying Hölder's inequality, we get

$$
|T_\mu f(z)|^q \lesssim M(z) \times N(z),
$$

where

$$
M(z) := \sum_{j=1}^{\infty} \left(\int_{D(\delta\tau(z_j))} |f(\xi)| \omega(\xi)^{1/2} \frac{d\mu(\xi)}{\tau(\xi)^2} \right)^q \int_{D(3\delta\tau(z_j))} |K_z(s)| \omega(s)^{1/2}\, dA(s),
$$

and

$$
N(z) := \left(\sum_{j=1}^{\infty} \int_{D(3\delta\tau(z_j))} |K_z(s)| \omega(s)^{1/2}\, dA(s) \right)^{q-1}.
$$

Furthermore, by Lemma B and condition (1.2), we have

$$
N(z) \lesssim \left(\int_{\mathbb{D}} |K_z(s)| \omega(s)^{1/2}\, dA(s) \right)^{q-1} \lesssim \omega(z)^{\frac{1-q}{2}}.
$$

Thus

$$\big|T_\mu f(z)\big|^q \omega(z)^{q/2} \lesssim M(z)\omega(z)^{1/2}.$$

This gives

$$\|T_\mu f\|_{A_\omega^q}^q \lesssim \sum_{j=1}^\infty \left(\int_{D(\delta\tau(z_j))} \big|f(\xi)\big|\omega(\xi)^{1/2} \frac{d\mu(\xi)}{\tau(\xi)^2} \right)^q K(j),$$

where

$$K(j) := \int_{\mathbb{D}} \left(\int_{D(3\delta\tau(z_j))} \big|K_z(s)\big|\omega(s)^{1/2} \, dA(s) \right) \omega(z)^{1/2} \, dA(z),$$

which, by Fubini's theorem and condition (1.2), gives

$$K(j) \lesssim \tau(z_j)^2.$$

Combining this with using (2.1) and Proposition 6.2 shows that

$$\begin{aligned}
\|T_\mu f\|_{A_\omega^q}^q &\lesssim \sum_{j=1}^\infty \tau(z_j)^2 \left(\frac{1}{\tau(z_j)^2} \int_{D(\delta\tau(z_j))} \big|f(\xi)\big|\omega(\xi)^{1/2} \, d\mu(\xi) \right)^q \\
&\lesssim \int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(4\delta\tau(z))} \big|f(\xi)\big|\omega(\xi)^{1/2} \, d\mu(\xi) \right)^q dA(z) \\
&\lesssim \|\widehat{\mu}_{4\delta}\|_{L^{\frac{pq}{p-q}}(\mathbb{D})}^q \cdot \|f\|_{A_\omega^p}^q.
\end{aligned}$$

This proves the desired result.   □

Next we characterize compact Toeplitz operators on weighted Bergman spaces A_ω^p for weights ω in the class $\mathcal{E}$. We need first a lemma.

Lemma 6.4 *Let $1 < p < \infty$, and let $k_{p,z}$ be the normalized reproducing kernels in A_ω^p, with $\omega \in \mathcal{E}$. Then $k_{p,z} \to 0$ weakly in A_ω^p as $|z| \to 1^-$.*

Proof By duality and the reproducing kernel properties, we must show that $|g(z)|/\|K_z\|_{A_\omega^p}$ goes to zero as $|z| \to 1^-$ whenever g is in $A_\omega^{p'}$, where p' denotes the conjugate exponent of p, but this follows easily by the density of the polynomials and Lemma A.   □

Theorem 6.5 *Let $\omega \in \mathcal{E}$, $1 < p \le q < \infty$, and μ be a finite positive Borel measure on $\mathbb{D}$ satisfying (6.1). Then the Toeplitz operator $T_\mu : A_\omega^p \to A_\omega^q$ is compact if and only if, for each $\delta \in (0, m_\tau)$ small enough, one has*

$$\lim_{|a| \to 1^-} \tau(a)^{2(\frac{1}{q} - \frac{1}{p})} \widehat{\mu}_\delta(a) = 0. \tag{6.8}$$

Proof First we assume that T_μ is compact. Following the proof of the boundedness part and the fact that $\|K_a\|_{A_\omega^p}^{-1} \asymp \omega(a)^{1/2} \tau(a)^{\frac{2(p-1)}{p}}$, we get the estimate

$$\tau(a)^{2(\frac{1}{q}-\frac{1}{p})}\widehat{\mu}_\delta(a) \lesssim \frac{\omega(a)^{1/2}}{\tau(a)^{2(\frac{1}{p}-1)}}\|T_\mu K_a\|_{A_\omega^q} \lesssim \|T_\mu k_{p,a}\|_{A_\omega^q}, \tag{6.9}$$

where $k_{p,a}$ are the normalized reproducing kernels in A_ω^p. Since, by Lemma 6.4, $k_{p,a}$ tends to zero weakly in A_ω^p and T_μ is compact, the result follows.

Conversely, we suppose that (6.8) holds. Let $\{f_n\} \subset A_\omega^p$ be a bounded sequence converging to zero uniformly on compact subsets of $\mathbb{D}$. By (6.4), we have

$$\|T_\mu f_n\|_{A_\omega^q} \lesssim \int_{\mathbb{D}} |f_n(z)|^q \, d\nu(z) = \|I_\nu f_n\|_{L^q(\mathbb{D},d\nu)}, \tag{6.10}$$

where $I_\nu : A_\omega^p \longrightarrow L^q(\mathbb{D}, d\nu)$ with $d\nu(\xi) = \omega(\xi)^{q/2}\tau(\xi)^{2(q-1)(\frac{1}{p}-\frac{1}{q})}\, d\mu(\xi)$. By using $\tau(a) \asymp \tau(\xi)$, for $\xi \in D(\delta\tau(a))$, we have

$$\sup_{|a|>r} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} \omega(\xi)^{-q/2} \, d\nu(\xi) \lesssim \sup_{|a|>r} \tau(a)^{2(\frac{1}{q}-\frac{1}{p})}\widehat{\mu}_\delta(a).$$

By Theorem C, I_ν is compact, and in view of (6.10), T_μ is compact. $\qquad\square$

Theorem 6.6 *Let $\omega \in \mathcal{E}$, $1 \leq q < p < \infty$, and μ be a finite positive Borel measure on $\mathbb{D}$ satisfying* (6.1). *The following conditions are equivalent*:

(i) *The Toeplitz operator $T_\mu : A_\omega^p \to A_\omega^q$ is compact.*

(ii) *For each sufficiently small $\delta > 0$,*

$$\widehat{\mu}_\delta \in L^{\frac{pq}{p-q}}(\mathbb{D}, dA). \tag{6.11}$$

Proof If T_μ is compact, then it is bounded, and by Theorem 6.3 we get the desired result. Conversely, if (6.11) holds, then by Theorem 6.3 T_μ is bounded. Since by Theorem 5.3 the spaces A_ω^p and A_ω^q are isomorphic to ℓ^p, the result is a consequence of a general result of Banach space theory: it is known that, for $1 \leq q < p < \infty$, every bounded operator from ℓ^p to ℓ^q is compact (see [19, Theorem I.2.7, p. 31]). $\qquad\square$

7 Hankel operators

One of the most important classes of operators acting on spaces of analytic functions is the Hankel operators. When acting on the classical Hardy spaces, their study presents [23, 27] a broad range of applications such as to control theory, approximation theory, prediction theory, perturbation theory, and interpolation problems. Furthermore, one can find an extensive literature on Hankel operators acting on other classical function spaces in one or several complex variables, such as Bergman spaces [1, 2, 15, 36, 37], Fock spaces [28], or Dirichlet spaces [33, 34]. In this section, we are going to study big Hankel operators acting on our large weighted Bergman spaces.

Definition 7.1 Let M_g denote the multiplication operator induced by a function g, and P_ω be the Bergman projection, where ω is a weight in the class $\mathcal{E}$. The Hankel operator H_g

is given by

$$H_g = H_g^\omega := (I - P_\omega)M_g.$$

We assume that the function g satisfies

$$gK_z \in L_\omega^1, \quad z \in \mathbb{D}. \tag{7.1}$$

Under this assumption, the Hankel operator H_g^ω is well defined on the set E of all finite linear combinations of reproducing kernels and, therefore, is densely defined in the weighted Bergman space A_ω^p, $1 \le p < \infty$. Also, for $f \in E$, one has

$$H_g^\omega f(z) = \int_{\mathbb{D}} \big(g(z) - g(s)\big)f(s)\overline{K_z(s)}\omega(s)\,dA(s).$$

We are going to study the boundedness and compactness when the symbol is conjugate analytic. In the Hilbert space case A_ω^2, and for weights in the class $\mathcal{W}$, a characterization of the boundedness, compactness, and membership in Schatten classes of the Hankel operator $H_{\bar{g}} : A_\omega^2 \to L_\omega^2$ was obtained in [12]. In order to extend such results to the non-Hilbert space setting, we need estimates for the p-norm of the reproducing kernels, and it is here when condition (1.2) and the exponential type class $\mathcal{E}$ enters in action. Before going to study the boundedness of the Hankel operator on A_ω^p with conjugate analytic symbols, we need the following lemma.

Lemma C *Let* $1 \le p < \infty$, $g \in H(\mathbb{D})$, *and* $a \in \mathbb{D}$. *Then*

$$\tau(a)\big|g'(a)\big| \le C\left(\frac{1}{\tau(a)^2} \int_{D(\delta\tau(a))} \big|g(z) - g(a)\big|^p dA(z)\right)^{1/p}.$$

Proof See for example [12]. □

Now we are ready to characterize the boundedness of the Hankel operator with conjugate analytic symbols acting on large weighted Bergman spaces in terms of the growth of the maximum modulus of g'. We begin with the case $1 \le p \le q < \infty$.

Theorem 7.2 *Let* $\omega \in \mathcal{E}$, $1 \le p \le q < \infty$, *and* $g \in H(\mathbb{D})$ *satisfying* (7.1). *The Hankel operator* $H_{\bar{g}} : A_\omega^p \longrightarrow L_\omega^q$ *is bounded if and only if*

$$\sup_{z \in \mathbb{D}} \tau(z)^{1+2(\frac{1}{q} - \frac{1}{p})}\big|g'(z)\big| < \infty. \tag{7.2}$$

Proof Suppose first that $H_{\bar{g}} : A_\omega^p \longrightarrow L_\omega^q$ is bounded. Thus

$$\|H_{\bar{g}}K_a\|_{L^q(\omega^{q/2})} \le \|H_{\bar{g}}\| \, \|K_a\|_{A^p(\omega^{p/2})}.$$

For each $z \in \mathbb{D}$, consider the function $g_z(\xi) := (g(z) - g(\xi))K_z(\xi)$. Condition (7.1) ensures that $g_z \in A_\omega^1$, and by the reproducing formula in Corollary 3.3, one has

$$H_{\overline{g}}K_a(z) = \int_{\mathbb{D}} \overline{(g(z) - g(\xi))}K_a(\xi)\overline{K_z(\xi)}\omega(\xi)\,dA(\xi)$$

$$= \overline{\langle g_z, K_a \rangle_\omega} = \overline{g_z(a)}.$$

Now, for δ small enough, we have $|K_z(a)| \asymp \|K_z\|_{A_\omega^2}\|K_a\|_{A_\omega^2}$ for $z \in D(\delta\tau(a))$. Hence, by the statement (a) of Theorem A and (2.1), we have

$$\|H_{\overline{g}}K_a\|_{L_\omega^q}^q = \int_{\mathbb{D}} |g(z) - g(a)|^q |K_z(a)|^q \omega(z)^{q/2}\,dA(z)$$

$$\geq \int_{D(\delta\tau(a))} |g(z) - g(a)|^q |K_z(a)|^q \omega(z)^{q/2}\,dA(z)$$

$$\asymp \int_{D(\delta\tau(a))} |g(z) - g(a)|^q \|K_z\|_{A_\omega^2}^q \|K_a\|_{A_\omega^2}^q \omega(z)^{q/2}\,dA(z)$$

$$\asymp \frac{\|K_a\|_{A_\omega^2}^q}{\tau(a)^q} \int_{D(\delta\tau(a))} |g(z) - g(a)|^q\,dA(z).$$

Because of the boundedness of the Hankel operator $H_{\overline{g}}$, we have

$$\|H_{\overline{g}}\|^q \geq \frac{\|H_{\overline{g}}K_a\|_{L_\omega^q}^q}{\|K_a\|_{A_\omega^p}^q}$$

$$\gtrsim \|K_a\|_{A_\omega^p}^{-q} \frac{\|K_a\|_{A_\omega^2}^q}{\tau(a)^q} \int_{D(\delta\tau(a))} |g(z) - g(a)|^q\,dA(z).$$

Finally, by the estimates on the norm of K_z in Lemma 2.4 and Theorem A, we obtain

$$\|H_{\overline{g}}\| \gtrsim \tau(a)^{2(\frac{1}{q}-\frac{1}{p})} \left(\frac{1}{\tau(a)^2} \int_{D(\delta\tau(a))} |g(z) - g(a)|^q\,dA(z) \right)^{1/q}.$$

By Lemma C, this completes the proof of this implication.

Conversely, assume that (7.2) holds, and let $1 \leq p \leq q < \infty$. By Theorem 2.5, there exists a solution u of the equation $\overline{\partial}u = f\overline{g'}$ in $L^q(\omega^{q/2})$ such that

$$\|u\|_{L_\omega^q}^q \lesssim \int_{\mathbb{D}} |\overline{\partial}u(z)|^q \omega(z)^{q/2}\tau(z)^q\,dA(z).$$

Since any solution v of the $\overline{\partial}$-equation has the form $v = u - h$ with $h \in H(\mathbb{D})$, and because $H_{\overline{g}}f$ is also a solution of $\overline{\partial}$-equation, there is a function $h \in H(\mathbb{D})$ such that $H_{\overline{g}}f = u - h$. As a result of $P_\omega(H_{\overline{g}}f) = 0$, we have $H_{\overline{g}}f = (I - P_\omega)u$, where I is the identity operator. Therefore, by the boundedness of P_ω on L_ω^q (see Theorem 3.2), we obtain

$$\|H_{\overline{g}}f\|_{L_\omega^q}^q \leq \|I - P_\omega\|^q \|u\|_{L_\omega^q}^q \lesssim \|u\|_{L_\omega^q}^q$$

$$\lesssim \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q \omega(z)^{q/2}\tau(z)^q\,dA(z). \tag{7.3}$$

By our assumption (7.2), we have

$$\|H_{\overline{g}}f\|_{L_\omega^q}^q \lesssim \int_{\mathbb{D}} |f(z)|^q \omega(z)^{q/2} \tau(z)^{2q(\frac{1}{p}-\frac{1}{q})} \, dA(z).$$

On the other hand, by Lemma A,

$$|f(z)|\omega(z)^{1/2} \lesssim \tau(z)^{-2/p}\|f\|_{A_\omega^p}.$$

Using the last pointwise estimate, we have

$$\|H_{\overline{g}}f\|_{L_\omega^q}^q \lesssim \|f\|_{A_\omega^p}^{q-p} \int_{\mathbb{D}} |f(z)|^p \omega(z)^{p/2} \, dA(z) = \|f\|_{A_\omega^p}^q.$$

This completes the proof. □

Next we are going to characterize the boundedness of the Hankel operator with conjugate analytic symbols when $1 \le q < p < \infty$. Before that we prove the following lemma.

Lemma 7.3 *Let* $\delta_0 \in (0, m_\tau)$ *and* $0 < r < \infty$. *Then*

$$|f'(z)|^r \lesssim \frac{1}{\tau(z)^{r+2}} \int_{D(\delta_0\tau(z))} |f(s)|^r \, dA(s)$$

for $f \in H(\mathbb{D})$.

Proof By Cauchy's integral formula and Lemma A, we get

$$|f'(z)| \lesssim \left| \int_{|\eta-z|=\frac{\delta_0\tau(z)}{4}} \frac{f(\eta)}{(\eta-z)^2} \, d\eta \right| \lesssim \frac{1}{\tau(z)^2} \int_{|\eta-z|=\frac{\delta_0\tau(z)}{4}} |f(\eta)||d\eta|$$

$$\lesssim \frac{1}{\tau(z)^2} \int_{|\eta-z|=\frac{\delta_0\tau(z)}{4}} \left(\frac{1}{\tau(\eta)^2} \int_{D(\delta_0\tau(\eta)/4)} |f(s)|^r \, dA(s) \right)^{1/r} |d\eta|.$$

An application of $\tau(\eta) \asymp \tau(z)$, for $\eta \in D(\delta_0\tau(z)/2)$, gives

$$|f'(z)| \lesssim \frac{1}{\tau(z)^2} \int_{|\eta-z|=\frac{\delta_0\tau(z)}{4}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta_0\tau(z)/2)} |f(s)|^r \, dA(s) \right)^{1/r} |d\eta|$$

$$\lesssim \frac{1}{\tau(z)^{1+\frac{2}{r}}} \left(\int_{D(\delta_0\tau(z)/2)} |f(s)|^r \, dA(s) \right)^{1/r},$$

which proves the desired result. □

The following result gives the characterization of the boundedness of the Hankel operator going from A_ω^p into L_ω^q when $1 \le q < p < \infty$.

Theorem 7.4 *Let* $\omega \in \mathcal{E}$, $1 \le q < p < \infty$, *and let* $g \in H(\mathbb{D})$ *satisfy* (7.1). *Then the following statements are equivalent*:
 (a) *The Hankel operator* $H_{\overline{g}} : A_\omega^p \longrightarrow L_\omega^q$ *is bounded*.
 (b) *The function* $\tau g'$ *belongs to* $L^r(\mathbb{D}, dA)$, *where* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Proof Suppose that $\tau g' \in L^r(\mathbb{D}, dA)$. By (7.3), since $p/q > 1$, a simple application of Hölder's inequality, yields

$$\|H_{\bar{g}}f\|_{L_\omega^q}^q \lesssim \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q \omega(z)^{q/2} \tau(z)^q \, dA(z) \leq \|f\|_{A_\omega^p}^q \|\tau g'\|_{L^r(\mathbb{D}, dA)}^q.$$

This proves the boundedness of $H_{\bar{g}}$.

Conversely, pick $\varepsilon > 0$ and let $\{z_k\}$ be a (ε, τ)- lattice on $\mathbb{D}$. For a sequence $\lambda = \{\lambda_k\} \in \ell^p$, we consider the function

$$G_t(z) = \sum_{k=1}^{\infty} \lambda_k r_k(t) \omega(z_k)^{1/2} \tau(z_k)^{2(\frac{p-1}{p})} K_{z_k}(z), \quad 0 < t < 1,$$

where $r_k(t)$ is a sequence of Rademacher functions. Because of the norm estimate for reproducing kernels given in Lemma 2.4, by part (i) of Theorem 5.3, we obtain

$$\|G_t\|_{A_\omega^p} \lesssim \|\lambda\|_{\ell^p}.$$

Thus, the boundedness of $H_{\bar{g}}$ gives

$$\|H_{\bar{g}}G_t\|_{L_\omega^q}^q \lesssim \|\lambda\|_{\ell^p}^q.$$

Therefore,

$$\int_{\mathbb{D}} \left| \sum_{k=1}^{\infty} \lambda_k r_k(t) \omega(z_k)^{1/2} \tau(z_k)^{2(\frac{p-1}{p})} H_{\bar{g}} K_{z_k}(z) \right|^q \omega(z)^{q/2} \, dA(z) \lesssim \|\lambda\|_{\ell^p}^q.$$

Using the same method in (6.6), we obtain

$$\int_{\mathbb{D}} \sum_{k=1}^{\infty} |\lambda_k|^q \omega(z_k)^{q/2} \tau(z_k)^{2q(\frac{p-1}{p})} |H_{\bar{g}} K_{z_k}(z)|^q \chi_k(z) \omega(z)^{q/2} \, dA(z) \lesssim \|\lambda\|_{\ell^p}^q, \tag{7.4}$$

where χ_k is the characteristic function of the set $D(3\varepsilon\tau(z_k))$. Additionally, by applying both statements (a) and (b) of Theorem A and (2.1), we get

$$\left| H_{\bar{g}} K_{z_k}(z) \right|^q \omega(z)^{q/2} = \left| g(z) - g(z_k) \right|^q \left| K_{z_k}(z) \right|^q \omega(z)^{q/2} \gtrsim \frac{\omega(z_k)^{-q/2}}{\tau(z_k)^{2q}} \left| g(z) - g(z_k) \right|^q.$$

Putting this in (7.4) gives

$$\sum_{k=1}^{\infty} |\lambda_k|^q \tau(z_k)^{-2q/p} \int_{D(3\varepsilon\tau(z_k))} \left| g(z) - g(z_k) \right|^q \, dA(z) \lesssim \|\lambda\|_{\ell^p}^q.$$

Furthermore, by Lemma C, we obtain

$$\sum_{k=1}^{\infty} |\lambda_k|^q \tau(z_k)^{-2q/p+2} \left| \tau(z_k) g'(z_k) \right|^q \lesssim \|\lambda\|_{\ell^p}^q.$$

Moreover, by the duality between $\ell^{p/q}$ and $\ell^{\frac{p}{p-q}}$, it follows

$$I_{g'} := \sum_{k=1}^{\infty} \tau(z_k)^2 \left(\tau(z_k) \left| g'(z_k) \right| \right)^r < \infty.$$

In order to finish the proof, we will justify that $\| \tau g' \|^r_{L^r(\mathbb{D}, dA)} \lesssim I_{g'}$. For that, on the one hand, by Lemma 7.3 applied to g', we get

$$\left| g''(\xi) \right|^r \lesssim \frac{1}{\tau(\xi)^{r+2}} \int_{D(\delta_0 \tau(\xi))} \left| g'(s) \right|^r dA(s). \tag{7.5}$$

On the other hand, by Cauchy estimates, there exists $\xi \in [z, z_k]$ such that

$$\left| g'(z) - g'(z_k) \right| \leq \left| g''(\xi) \right| \left| z - z_k \right|.$$

Using $\tau(\xi) \asymp \tau(z) \asymp \tau(z_k)$, for $\xi, z \in D(\varepsilon \tau(z_k))$, we have

$$
\begin{aligned}
\left\| \tau g' \right\|^r_{L^r(\mathbb{D}, dA)} &\leq \sum_k \int_{D(\varepsilon \tau(z_k))} \tau(z)^r \left| g'(z) \right|^r dA(z) \\
&\leq C \sum_k \int_{D(\varepsilon \tau(z_k))} \tau(z_k)^r \left| g'(z) - g'(z_k) \right|^r dA(z) + C\varepsilon^2 I_{g'} \\
&\leq C\varepsilon^r \sum_k \int_{D(\varepsilon \tau(z_k))} \tau(z_k)^{2r} \left| g''(\xi) \right|^r dA(z) + C\varepsilon^2 I_{g'}.
\end{aligned}
$$

By (7.5) and using again (2.1), we obtain

$$
\begin{aligned}
\left\| \tau g' \right\|^r_{L^r(\mathbb{D}, dA)} &\leq C\varepsilon^r \sum_k \int_{D(\varepsilon \tau(z_k))} \frac{\tau(z_k)^{2r}}{\tau(\xi)^{r+2}} \int_{D(\delta_0 \tau(\xi))} \left| g'(s) \right|^r dA(s)\, dA(z) + \varepsilon^2 I_{g'} \\
&\leq C\varepsilon^{r+2} \sum_k \int_{D(3\delta_0 \tau(z_k))} \tau(s)^r \left| g'(s) \right|^r dA(s) + \varepsilon^2 I_{g'}.
\end{aligned}
$$

By Lemma B, every point $z \in \mathbb{D}$ belongs to at most $C\varepsilon^{-2}$ of the sets $D(3\delta_0 \tau(z_k))$. Hence

$$\left(1 - C\varepsilon^r \right) \left\| \tau g' \right\|^r_{L^r(\mathbb{D}, dA)} \lesssim I_{g'}.$$

Thus, taking ε so that $C\varepsilon^r < 1/2$, we get the desired result. □

Next we characterize the compactness of the Hankel operator with conjugate analytic symbol acting from A^p_ω into L^q_ω, $1 \leq p, q < \infty$. This characterization will be given in two theorems depending on the order of p and q. We begin with the case $1 \leq p \leq q < \infty$.

Theorem 7.5 *Let* $\omega \in \mathcal{E}$, $1 < p \leq q < \infty$, *and* $g \in H(\mathbb{D})$ *satisfying* (7.1). *Then, the Hankel operator* $H_{\bar{g}} : A^p_\omega \longrightarrow L^q_\omega$ *is compact if and only if*

$$\lim_{|a| \to 1^-} \tau(a)^{1 + 2(\frac{1}{q} - \frac{1}{p})} \left| g'(a) \right| = 0.$$

Proof Let $a \in \mathbb{D}$ and $1 < p \leq q < \infty$. Recall that $k_{a,p}$ is the normalized reproducing kernel in A_ω^p. By Lemma 6.4, $k_{a,p} \to 0$ weakly. Thus, if $H_{\bar{g}}$ is compact, then

$$\lim_{|a| \to 1^-} \|H_{\bar{g}} k_{a,p}\|_{L_\omega^q} = 0.$$

Let δ be small enough such that $|K_a(z)| \asymp \|K_a\|_{A_\omega^2} \|K_z\|_{A_\omega^2}$ for $z \in D(\delta\tau(a))$. By Theorem A, using (2.1) and Lemma 2.4, we have

$$\|H_{\bar{g}} k_{a,p}\|_{L_\omega^q}^q \geq \int_{D(\delta\tau(a))} |g(a) - g(z)|^q \frac{|K_a(z)|^q \omega(z)^{q/2}}{\|K_a\|_{A_\omega^p}^q} \, dA(z)$$

$$\gtrsim \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} |g(a) - g(z)|^q \, dA(z).$$

It follows from Lemma C that

$$\|H_{\bar{g}} k_{a,p}\|_{L_\omega^q} \gtrsim \tau(a)^{1+2(\frac{1}{q}-\frac{1}{p})} |g'(a)|.$$

This implies that

$$\lim_{|a| \to 1^-} \tau(a)^{1+2(\frac{1}{q}-\frac{1}{p})} |g'(a)| = 0,$$

which completes the proof of this implication.

Conversely, let $\{f_n\}$ be a bounded sequence in A_ω^p such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. To show compactness, it is standard to see that it is enough to prove that $\|H_{\bar{g}} f_n\|_{L_\omega^q} \to 0$. By the assumption, given any $\varepsilon > 0$, there is $0 < r_0 < 1$ such that

$$\tau(z)^{1+2(\frac{1}{q}-\frac{1}{p})} |g'(z)| < \varepsilon, \quad r_0 < |z| < 1.$$

Since $\{f_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, there exists an integer n_0 such that

$$|f_n(z)| < \varepsilon \quad \text{for } |z| \leq r_0 \text{ and } n \geq n_0.$$

According to (7.3), we have

$$\|H_{\bar{g}} f_n\|_{L_\omega^q}^q \lesssim \int_{\mathbb{D}} |f_n(z)|^q |g'(z)|^q \omega(z)^{q/2} \tau(z)^q \, dA(z)$$

$$= \left(\int_{|z| \leq r_0} + \int_{r_0 < |z| < 1} \right) |f_n(z)|^q |g'(z)|^q \omega(z)^{q/2} \tau(z)^q \, dA(z).$$

On the one hand, it is easy to see that

$$\int_{|z| \leq r_0} |f_n(z)|^q |g'(z)|^q \omega(z)^{q/2} \tau(z)^q \, dA(z) \lesssim \varepsilon^q. \tag{7.6}$$

On the other hand, by Lemma A, we have the pointwise estimate

$$|f_n(z)| \lesssim \omega(z)^{-1/2} \tau(z)^{-2/p} \|f_n\|_{A_\omega^p}.$$

Applying this together with our assumption, we get

$$
\int_{r_0 < |z| < 1} \left| f_n(z) \right|^q \left| g'(z) \right|^q \omega(z)^{q/2} \tau(z)^q \, dA(z)
$$

$$
< \varepsilon^q \int_{\mathbb{D}} \left| f_n(z) \right|^p \left| f_n(z) \right|^{q-p} \omega(z)^{q/2} \tau(z)^{2q\left(\frac{1}{p} - \frac{1}{q}\right)} \, dA(z)
$$

$$
\lesssim \varepsilon^q \| f_n \|_{A_\omega^p}^{q-p} \int_{\mathbb{D}} \left| f_n(z) \right|^p \omega(z)^{p/2} \, dA(z) = \varepsilon^q \| f_n \|_{A_\omega^p}^q.
$$

Combining this with (7.6) gives $\lim_{n \to \infty} \| H_{\overline{g}} f_n \|_{L_\omega^q} = 0$. This shows that the Hankel operator $H_{\overline{g}} : A_\omega^p \longrightarrow L_\omega^q$ is compact. $\qquad\square$

The next theorem contains a compactness criterion for Hankel operators when $1 \le q < p < \infty$.

Theorem 7.6 *Let $\omega \in \mathcal{E}$, $1 \le q < p < \infty$, and $g \in H(\mathbb{D})$ satisfying (7.1). The following conditions are equivalent*:
  (a) *The Hankel operator $H_{\overline{g}} : A_\omega^p \longrightarrow L_\omega^q$ is compact.*
  (b) *The function $\tau g'$ belongs to $L^r(\mathbb{D}, dA)$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.*

Proof $(a) \Longrightarrow (b)$ Assume that $H_{\overline{g}}$ is compact. Then $H_{\overline{g}}$ is bounded. Hence, by applying Theorem 7.4, we get the desired result.

$(b) \Longrightarrow (a)$ Suppose that $\tau g'$ belongs to $L^r(\mathbb{D}, dA)$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. By Theorem 7.4, the Hankel operator $H_{\overline{g}}$ is bounded and, as a result of Theorem 5.3, the space A_ω^p is isomorphic to ℓ^p. In this case, $H_{\overline{g}}$ is also compact, due to a general result of Banach space theory: For $1 \le q < p < \infty$, every bounded operator from ℓ^p to ℓ^q is compact (see [19, Theorem I.2.7]). This finishes the proof. $\qquad\square$

8 Concluding remarks

I believe that I have done a satisfactory work in order to get a better understanding of the function properties of large weighted Bergman spaces and the operators acting on them. I hope that this work is going to attract many other researchers to this area, and expect that the study of this function spaces is going to experience a period of intensive research in the next years. There is still plenty of work to do for a better understanding of the theory of large weighted Bergman spaces, and several natural problems are waiting for further study or a complete solution: atomic decomposition, coefficient multipliers, zero sets, etc. I hope that the methods developed here will be of some help in order to attach the previous mentioned problems.

Declarations

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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