Boundedness of Toeplitz operators in Bergman-type spaces

Boundedness of Toeplitz Operators in Bergman-type Spaces

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Abstract. The characterization of the bounded Toeplitz operators T_a in Bergman spaces is an open problem even in the simplest case of the unweighted Bergman-Hilbert space $A^2(\mathbb{D})$. We consider here recent partial results on the topic. These include sufficient conditions for the boundedness and compactness of T_a in terms of weak Carleson-types condition for the symbol a. The results were recently generalized to the case of spaces on the unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$. The second approach is based on certain results on the structure of the Bergman-spaces, namely, representations of their weighted norms using finite-dimensional decompositions of the spaces. This approach provides a characterization of the boundedness and compactness in the case of operators in spaces with weighted sup-norms.

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1. Introduction: the spaces and operators

The focus of this article is on recent results on the boundedness of Toeplitz operators on weighted Bergman spaces of holomorphic functions, mainly on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, although some of the results are also formulated on the unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$, $N = 2, 3, \ldots$. The related question on the compactness is only considered when it can be dealt with parallel to boundedness, and certain more special recent results for compactness will remain out of this review.

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We will concentrate on two circles of ideas. First, we deal with Toeplitz operators with oscillating symbols and weak Carleson-type sufficient conditions for boundedness. The starting point of this direction of research is the article [30]. The second approach applies to operators with radial symbols, and it is based on the results on the structure of weighted Bergman spaces which were pioneered in the works of W.Lusky, [17], [18], [19] and adapted to the study of Toeplitz operators recently in the papers [4], [5]. This led to a characterization of the boundedness and compactness of Toeplitz operators in weighted H^∞-spaces.

Let us present the basic notation and definitions. The notation concerning the spaces on the unit ball $\mathbb{B}_N$ will only be needed and thus given at the end of Section 2. The normalized area measure on $\mathbb{D}$ is denoted by $dA = \pi^{-1} r dr d\theta$, where r and θ are the polar coordinates of $z = re^{i\theta} \in \mathbb{C}$. Given $1 \le p < \infty$ and the real parameter $\alpha > -1$ we define the weighted area measure by $dA_\alpha(z) = (1+\alpha)(1-r^2)^\alpha dA(z)$ and set

$$L^p_\alpha(\mathbb{D}) = \Big\{ g : \mathbb{D} \to \mathbb{C} \text{ measurable } : \|g\|^p_{p,\alpha} := \int_{\mathbb{D}} |g|^p dA_\alpha < \infty \Big\} \text{ and}$$

$$A^p_\alpha(\mathbb{D}) = \{ g \in L^p_v(\mathbb{D}) : g \text{ holomorphic } \};$$

in the case $\alpha = 0$ these spaces are denoted by $L^p(\mathbb{D})$ and $A^p(\mathbb{D})$, respectively. Here, $v(z) = (1-|z|^2)^\alpha$ are called standard weights.

We will also consider more general weighted Bergman spaces and their analogue, weighted Hardy space H^∞_v corresponding to $p = \infty$. In general, by a weight v we mean a continuous function $\mathbb{D} \to]0,\infty[$ which is radial, vanishing on the boundary and decreasing with the radius, i.e. there holds $v(z) = v(|z|)$ for all $z \in \mathbb{D}$, $\lim_{|z|\to 1} v(z) = 0$ and $v(r) \ge v(s)$ if $1 > s > r > 0$. We denote $v dA = dA_v$ and, for $1 \le p < \infty$,

$$L^p_v(\mathbb{D}) = \Big\{ g : \mathbb{D} \to \mathbb{C} \text{ measurable } : \|g\|^p_{p,v} := \int_{\mathbb{D}} |g|^p dA_v < \infty \Big\} \text{ and}$$

$$A^p_v(\mathbb{D}) = \{ g \in L^p_v(\mathbb{D}) : g \text{ holomorphic } \},$$

and

$$h^\infty_v(\mathbb{D}) = \{ g : \mathbb{D} \to \mathbb{C} : g \text{ harmonic}, \ \|g\|_v := \sup_{z\in\mathbb{D}} |g(z)| v(|z|) < \infty \}$$

and

$$H^\infty_v(\mathbb{D}) = \{ g \in h^\infty_v : g \text{ holomorphic } \};$$

we use the standard notation $H^\infty(\mathbb{D}) = (H^\infty(\mathbb{D}), \|\cdot\|_\infty)$ in the non-weighted case. In all of the above cases, the subspaces of holomorphic and harmonic functions are closed subspaces of the their superspaces.

We write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Given α, the Bergman projection P_α is the orthogonal projection from the Hilbert space $L^2_\alpha(\mathbb{D})$ onto the closed subspace $A^2_\alpha(\mathbb{D})$. Given a function $a \in L^1(\mathbb{D})$, we also denote by M_a the pointwise multiplier $M_a : f \mapsto af$, where $f : \mathbb{D} \to \mathbb{C}$ is a measurable function (which is usually holomorphic or harmonic in the sequel). If $1 \le p < \infty$, then a Toeplitz operator T_a on $A^p_\alpha(\mathbb{D})$, with symbol a, is in principle defined as the composition

$$T_a f = P_\alpha M_a f, \tag{1.1}$$

but the assumptions made so far do not always suffice to guarantee that (1.1) makes sense, since M_a might map f outside $L^2_\alpha(\mathbb{D})$. In the case a is a bounded function, there is no problem with the definition, since P_α can be written with the help of the Bergman kernel as the intergal operator

$$P_\alpha f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^{2+\alpha}} dA_\alpha(w)\,,$$

hence

$$P_\alpha M_a f = \int_{\mathbb{D}} \frac{a(w) f(w)}{(1 - z\overline{w})^{2+\alpha}} dA_\alpha(w), \tag{1.2}$$

and for every $z \in \mathbb{D}$, these integrals converge for all $f \in L^1_\alpha(\mathbb{D})$. Moreover, it is known that P_α is a bounded operator in the space $L^p_\alpha(\mathbb{D})$, when $1 < p < \infty$, which yields the boundedness of $T_a : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ for bounded symbols.

It is not difficult to construct unbounded symbols a which still induce bounded Toeplitz operators, but the characterization of symbols $a \in L^1(\mathbb{D})$ such that $T_a : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ is well-defined and bounded is a well-known open problem. Let us mention some partial results on it. The characterization of boundedness and compactness of Toeplitz operators with nonnegative symbols in terms of Carleson type measures first appeared in [24]

D.Luecking [15] proved that a Toeplitz operator T_a with a nonnegative symbol $a \in L^1(\mathbb{D})$ is bounded in $A^2(\mathbb{D})$, if and only if the average

$$|B(z,r)|^{-1} \int_{B(z,r)} a(w)\, dA(w)$$

is a bounded function of z. Here $B(z,r)$ denotes a disk in the Bergman metric, with center z and some fixed radius $r > 0$. Toeplitz operators with radial symbols in the space $A^2_\alpha(\mathbb{D})$ and analogues on higher dimensional domains were thoroughly considered in [9]: in this case the operator is unitarily equivalent with a sequence space multiplier, see also (5.1) below, and thus the boundedness properties can be determined. A partial generalization to the case $p \neq 2$ was established in [21]. The Berezin transform

$$B(f)(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1 - z\bar{w}|^4} dA(w), \quad z \in \mathbb{D}, \tag{1.3}$$

is a useful tool for the theory of Toeplitz operators, although it will not be used in this article. N.Zorboska proved in [38] for symbols a of bounded mean oscillation that the Toeplitz operator $T_a : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ is bounded if and only if $B(a)$ is bounded. The results of [15] and [38] generalize to other $A^p(\mathbb{D})$-spaces, $1 < p < \infty$, as well, see e.g. [30]. Here is a non-exhaustive list of other works dealing with the boundedness and compactness of Toeplitz operators in Bergman-type spaces: [8], [9], [10], [11], [12], [15], [16], [21], [22], [25], [27], [28], [29], [30], [34], [35], [39], [36], [38]. The monograph [37] is a standard reference for the topic, and we also mention the survey article [31].

In this article we will review in Section 2 the results of [30], [33], [11]. These consist of sufficient, weak Carleson-type conditions for the boundedness and compactness of Toeplitz operators in reflexive Bergman spaces with standard weights, both on the unit disk and the unit ball. Sections 3–6 are mainly based on the recent works [4], [5], which deal with operators on $H^\infty_v(\mathbb{D})$-spaces with quite general classes of weights. Theorem 4.1 of Section 4 states that there is a bounded harmonic symbol f for which T_f is unbounded in $H^\infty_v(\mathbb{D})$ for any radial weight v satisfying our general assumptions. The main result of Section 5, Theorem 5.3 contains a necessary and sufficient condition for the boundedness of T_f in $H^\infty_v(\mathbb{D})$, as well as the corresponding result for the compactness. These conditions are slightly abstract, and thus in Section 6 we derive some more concrete, easily formulated sufficient conditions based on the results of Section 5.

We conclude this section by a remark on the definition of Toeplitz operators as an improper integral. Here, we fix $\alpha > -1$ and assume the symbol a is radial. Formula (1.4) will be considered in detail in Section 2 even for more general, non-radial symbols. The proof of Proposition 1.1 is taken here from [14], although some versions of it have probably been known for specialists for a long time.

Proposition 1.1. *Let a be a radial symbol, i.e. $a(z) = a(|z|)$ for almost all $z \in \mathbb{D}$, belonging to $L^1_\alpha(\mathbb{D})$, $\alpha > -1$, and let $g(z) = \sum_{n=0}^\infty g_n z^n$ be a holomorphic function on $\mathbb{D}$. Then, the defining integral* (1.2) *of $T_a g$ exists in the improper sense as the limit*

$$T_a g(z) = \lim_{\rho\to 1} \int\limits_{|w|<\rho} \frac{a(w)g(w)}{(1-z\overline{w})^{2+\alpha}} dA_\alpha(w), \tag{1.4}$$

convergent for every $z \in \mathbb{D}$. Moreover,

$$T_a g = \sum_{n=0}^\infty \frac{\beta_{a,\alpha}(1,n) g_n}{(\alpha+1)B(n+1,\alpha+1)} z^n \tag{1.5}$$

and in particular the power series on the right converges for all $z \in \mathbb{D}$.

Here and in the next we denote by B and Γ Euler's beta- and gamma-functions,

$$B(n+1,c) \quad = \quad \frac{n!\Gamma(c)}{\Gamma(n+1+c)}, \quad c > 0,$$

and for $0 < \rho \le 1$

$$\beta_{a,\alpha}(\rho,n) \quad = \quad (\alpha+1) \int\limits_0^{\sqrt{\rho}} t^n (1-t)^\alpha a(\sqrt{t}) dt, \tag{1.6}$$

where the integral converges by the assumptions that a is radial and belongs to $L^1_\alpha(\mathbb{D})$.

Proof of Proposition 1.1. We start by the remark that for all $m \in \mathbb{N}_0$, the integral

$$\int_{\mathbb{D}} g(w)\overline{w}^m a(w) dA_\alpha(w)$$

exists in the improper sense for every holomorphic g on the disk $\mathbb{D}$. Namely, the rotational symmetry of a and the usual orthogonality relations of trigonometric polynomials yield for all $m \in \mathbb{N}_0$

$$\int_{|w|<\rho} g(w)\overline{w}^m a(w) dA_\alpha(w) = 2(\alpha+1) g_m \int_0^\rho r^{2m+1} a(r)(1-r^2)^\alpha dr \tag{1.7}$$

Clearly, the limit exists, when $\rho \to 1$. For every $0 < \rho < 1$, $z \in \mathbb{D}$, we obtain by (1.7)

$$\begin{aligned} &\int_{|w|<\rho} \frac{a(w)g(w)}{(1-z\overline{w})^{2+\alpha}} dA_\alpha(w) \\ =& \int_{|w|<\rho} g(w) \left(\sum_{n=0}^\infty \frac{(z\overline{w})^n}{(\alpha+1)B(n+1,\alpha+1)} \right) a(w) dA_\alpha(w) \\ =& \sum_{n=0}^\infty \frac{\beta_{a,\alpha}(\rho,n) g_n}{(\alpha+1)B(n+1,\alpha+1)} z^n. \end{aligned} \tag{1.8}$$

Let $L \in \mathbb{N}$ be such that $L \geq |\alpha| + 1$. Then,

$$B(n+1,\alpha+1) \geq \frac{n!\Gamma(\alpha+1)}{(n+L)!} \geq C_L n^{-L} \tag{1.9}$$

for some constant $C_L > 0$. We also have

$$\beta_{a,\alpha}(\rho,n) \leq \beta_{a,\alpha}(1,n) = 2(\alpha+1)\int_0^1 t^{2n}(1-t^2)^\alpha a(t) dt \leq C_\alpha \tag{1.10}$$

for another constant $C_\alpha > 0$, for all ρ and n, since $a \in L^1_\alpha(\mathbb{D})$. Moreover, since g is a holomorphic function on $\mathbb{D}$, we have $\limsup_{n\to\infty} |g_n|^{\frac{1}{n}} \leq 1$, hence,

$$\limsup_{n\to\infty} \left| \frac{\beta_{a,\alpha}(1,n) g_n}{(\alpha+1)B(n+1,\alpha+1)} \right|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left(C_L C_\alpha n^L \right)^{\frac{1}{n}} \cdot \limsup_{n\to\infty} |g_n|^{\frac{1}{n}} \leq 1.$$

The same estimate holds, independently of ρ, when $\beta_{a,\alpha}(1,n)$ is replaced by $\beta_{1,\alpha}(\rho,n)$. Hence, by the elementary theory of power series, (1.5), (1.8) converge uniformly on compact subsets of the disk and define holomorphic functions. Moreover, we have $\beta_{a,\alpha}(\rho,n) \to \beta_{a,\alpha}(1,n)$ for every n as $\rho \to 1$, hence, considering truncated series (1.5), (1.8) shows that the limit on the right of (1.4) exists for every $z \in \mathbb{D}$ and coincides with (1.5). □

2. Toeplitz operators with oscillating symbols

If an unbounded, measurable function a is strongly oscillating, it may give rise to a Toeplitz operator via the improper integral (1.4), and the operator may even be bounded with respect to a Bergman norm. A sufficient condition for oscillating symbols to induce a bounded T_a was presented in the paper [30]. More precisely, in the reference it was shown that T_a is bounded under an averaging condition for the symbol itself rather than for its modulus. The result needs a generalized definition of Toeplitz operators, which, however, eventually coincides with the improper integral. The result also extends to little Hankel operators.

We will next review the mentioned approach. It is based on a decomposition of the disk into an infinite family of $(D_n)_{n=1}^{\infty}$ subdomains, which have essentially constant area with respect to the hyperbolic geometry. The geometry of the subdomains needs to be specified carefully, since an explicit integration by parts -argument is a crucial step in the argument. Here, the sets D_n are rectangles in the polar coordinates, but they could also be chosen differently, see the discussion below.

Let us consider a symbol $a : \mathbb{D} \to \mathbb{C}$, which is at least locally Lebesgue-integrable on $\mathbb{D}$. We also fix the parameter $\alpha > -1$.

Definition 2.1. Denote by $\mathcal{D}$ the family of the sets $D := D(r, \theta)$, where

$$D = \{\rho e^{i\phi} \mid r \le \rho \le 1 - \frac{1}{2}(1-r) \ , \ \theta \le \phi \le \theta + \pi(1-r)\} \tag{2.1}$$

for all $0 < r < 1$, $\theta \in [0, 2\pi]$. Let $|D| := \int_D dA$ and, for $w = \rho e^{i\phi} \in D(r, \theta)$, let

$$\hat{a}_D(w) := \frac{1}{|D|} \int_r^{\rho} \int_{\theta}^{\phi} a(\varrho e^{i\varphi}) \varrho d\varphi d\varrho. \tag{2.2}$$

We will study symbols a for which there exists a constant $C > 0$ such that

$$|\hat{a}_D(w)| \le C \tag{2.3}$$

for all $D \in \mathcal{D}$ and all $w \in D$.

The sets $D\big(1-2^{-m+1}, 2\pi(k-1)2^{-m}\big) \in \mathcal{D}$, where $m \in \mathbb{N}, k = 1, \dots, 2^{-m}$, form a decomposition of the disk $\mathbb{D}$. Let us re-index them somehow into a family $(D_n)_{n=1}^{\infty}$ with

$$D_n = \quad \{ \quad z = re^{i\theta} \mid r_n < r \le r_n', \theta_n < \theta \le \theta_n' \} \tag{2.4}$$

where, for some m and k,

$$r_n = 1 - 2^{-m+1}, \ r_n' = 1 - 2^{-m}, \ \ \theta_n = \pi(k-1)2^{-m+1}, \ \theta_n' = \pi k 2^{-m+1} \tag{2.5}$$

Given $f \in A^p_\alpha(\mathbb{D})$, we write for all $n = n(m, k)$

$$F_n f(z) = \int_{D_n} \frac{a(w) f(w)}{(1 - z\bar{w})^{2+\alpha}} dA_\alpha(w) \ , \quad z \in \mathbb{D}, \tag{2.6}$$

so that F_n can actually be considered as a conventional, bounded Toeplitz operator on $A^p_\alpha(\mathbb{D})$.

The following theorem, in the case of $\alpha = 0$, is the main result Theorem 2.3 of [30]. The weighted case was included in [11].

Theorem 2.2. *Let $1 < p < \infty$ and assume that the locally integrable function a satisfies the condition* (2.3)*. Given $f \in A^p_\alpha(\mathbb{D})$, the series $\sum_{n=1}^\infty F_n f(z)$ converges pointwise, absolutely for almost all $z \in \mathbb{D}$, and the generalized Toeplitz operator $T_a : A^p \to A^p$, defined by*

$$T_a f(z) = \sum_{n=1}^{\infty} F_n f(z) \tag{2.7}$$

is bounded for all $1 < p < \infty$, and there is a constant C_α, independent of a, such that

$$\|T_a\| \le C_\alpha \sup_{D \in \mathcal{D}, w \in D} |\hat{a}_D(w)|. \tag{2.8}$$

The main step of the proof consists of writing the integral (2.6) in polar coordinates and performing a double integration by parts (once with respect to both coordinates) such that there appear integrals of a and derivatives of $f(w)(1-|w|^2)^\alpha(1-z\bar{w})^{2+\alpha}$. The former can be estimated by using the assumption (2.3) and the latter by using bounds for the maximal Bergman projection and well known arguments and estimates related with hyperbolic geometry. One obtains a representation for the integral (1.2) as a pointwise convergent sum of the integrals (2.6) as in (2.7). We refer to [30] for the details. Improved versions of the proof appear in [33] and [11], and they yield our next theorem, although we do not repeat the proof here. We remark that every Toeplitz operator

$$T_{a_\rho} f(z) = \int_{|w|<\rho} \frac{a(w) f(w)}{(1 - z\bar{w})^{2+\alpha}} dA_\alpha(w) \tag{2.9}$$

is bounded $A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$, since the support of the symbol is contained in a compact subset of $\mathbb{D}$.

Theorem 2.3. *Let $1 < p < \infty$ and $1/p + 1/q = 1$, and let the symbol a be as in Theorem 2.2. Then, the generalized Toeplitz operator $T_a : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$, defined in* (2.7)*, can be written as*

$$T_a f \quad = \quad \lim_{\rho \to 1} T_{a_\rho} f \quad , \tag{2.10}$$

for all $f \in A^p_\alpha(\mathbb{D})$. The limit converges with respect to the strong operator topology. Moreover, the transposed operator $T_a^* : A^q_\alpha(\mathbb{D}) \to A^q_\alpha(\mathbb{D})$ (with respect to the standard complex dual pairing) satisfies

$$T_a^* f \quad = \quad \lim_{\rho \to 1} T_{\bar{a}_\rho} f \tag{2.11}$$

for $f \in A^q_\alpha(\mathbb{D})$ and for almost all $z \in \mathbb{D}$, and the limit also converges in the strong operator topology.

The limits in (2.10), (2.11) cannot in general converge in the operator norm, since the operators T_{a_ρ} are compact. We mention that, when $\alpha = 0$, the above results are formulated in [33] also for little Hankel operators

$$h_a f(z) = \int\limits_{\mathbb{D}} \frac{a(w)f(w)}{(1-\bar{z}w)^2} dA(w) \ , \quad z \in \mathbb{D}. \tag{2.12}$$

Here, one also defines using the same decomposition of the unit disk as above

$$H_n f(z) = \int\limits_{D_n} \frac{a(w)f(w)}{(1-\bar{z}w)^2} dA(w) \ , \quad z \in \mathbb{D}, \tag{2.13}$$

and defines the generalized little Hankel operator $h_a f(z)$ as $\sum_{n=1}^{\infty} H_n f(z)$. Then, if (2.3) holds for the symbol a, one obtains that $h_a : A^p(\mathbb{D}) \to L^p(\mathbb{D})$ is bounded for all $1 < p < \infty$, the operator norm of h_a has the same bound as in (2.8), and finally, the operator h_a and its transpose have representations as improper integrals similar to those in (2.10), (2.11).

The definition (2.7) of a generalized Toeplitz operator depends on the geometry of the special decomposition (2.4) of the unit disk, but Theorem 2.3 largely removes this unsatisfacatory feature, since the improper integral in (2.10) is quite a natural one. We remark that in the literature there are versions of the result, which use different subdomains of the unit disk. In [39] the condition (2.3) is replaced by a similar one on Carleson squares

$$S_h^\alpha(e^{i\theta}) = \big\{\rho e^{i\phi} \, : \, 1-h < \rho < 1, |\phi - \theta| < \pi\alpha h\big\}$$

where $0 < h < 1$, $0 \leq \theta \leq 2\pi$, $0 < \alpha \leq 1$. The authors give a boundedness result for the Toeplitz operators and they also show that their sufficient condition is equivalent to that in Theorem 2.2. Finally, they also prove the important observation that the sufficient condition (2.3) is not necessary to the boundedness of $T_a : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$.

Another variant appears in [22], [23] where Toeplitz operators on Bergman spaces of simply connected planar domains are considered. In such domains any geometric symmetry is usually lost, and there does not exist a decomposition of the domain which is as natural as the one for the disk, see (2.4). However, the author uses a Whitney decomposition with Euclidean rectangles and obtains results which are analogous Theorem 2.2. The Whitney decomposition can of course be applied also in the case of the disk, and it yields another sufficient condition for the boundedness of the Toeplitz operator. We do not know, if the condition is equivalent to (2.3).

In [32], we generalized Theorem 2.2 to the setting of $A^1(\mathbb{D})$, while bounded Toeplitz operators T_μ on $A^1_\alpha(\mathbb{B}_N)$ were characterized in terms of the reproducing kernels in [6] under additional conditions on the measure μ. We skip a detailed discussion on the boundedness problem in A^1-spaces and only note that the previous approach has not been worked out in the non-locally convex cases $0 < p < 1$.

Theorems 2.2 and 2.3, first proved in [30] and [33], have been generalized to the case of Toeplitz operators on the Bergman space of the unit ball of $\mathbb{C}^N$

in the recent work [11], but even presenting the results leads to non-trivial technical challenges. We do not directly need the Euclidean space $\mathbb{R}^3$ here, but since that dimension is still within the capabilities of the human imagination, we ask the reader to think about a radially symmetric decomposition of the unit ball of $\mathbb{R}^3$: that is indeed a challenge, since decomposing the ball surface into finitely many identical squares in spherical coordinates (corresponding to intervals $[\theta_n, \theta_n']$ in (2.4)–(2.5)) is impossible. For example, starting to fill the ball surface from the equator with spherical squares with one side parallel to the meridians, one runs into difficulties at latest when trying to fill the north and south caps.

The results of [11] are formulated for measures with standard weights and thus the proofs contain new information even in the case $N = 1$, since the earlier papers only contained the unweighted case. The basic idea of the proof is the same as in in [30] and [33], but new non-trivial technical considerations are nevertheless needed. Let us review the approach of [11] superficially without going into all technical details. For $\alpha > -1$, we define the weighted Lebesgue measure dV_α on the unit ball $\mathbb{B}_N$, $N \in \mathbb{N}$, by $dV_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dV(z)$, where dV is the unweighted N-dimensional (real) Lebesgue measure and c_α is a normalizing constant such that $\int_{\mathbb{B}_N} dV_\alpha = 1$. For $1 \le p < \infty$, we denote by $L^p_\alpha(\mathbb{B}_N)$ the L^p-space with respect to the measure dV_α and by $A^p_\alpha(\mathbb{B}_N)$ the weighted Bergman space of all holomorphic functions in $L^p_\alpha(\mathbb{B}_N)$. We also denote by P_α the orthogonal projection from $L^2_\alpha(\mathbb{B}_N)$ onto $A^2_\alpha(\mathbb{B}_N)$. It is known to be a bounded operator $L^p_\alpha(\mathbb{B}_N)$ onto $A^p_\alpha(\mathbb{B}_N)$ for all $1 < p < \infty$.

In the following it is useful to work with real variables by identifying $\mathbb{C}^N$ with $\mathbb{R}^n$, $n = 2N$, so that $\mathbb{B}_N$ equals $\mathbb{B}_n$ in real coordinates. Accordingly, any point $x \in \mathbb{B}_n$ with modulus $|x| = r$ can be written as

$$\begin{aligned} x &= (r\cos\theta_2, r\sin\theta_2\cos\theta_3, r\sin\theta_2\sin\theta_3\cos\theta_4, \cdots, \\ &\qquad r\sin\theta_2\cdots\sin\theta_{n-1}\cos\theta_n, r\sin\theta_2\cdots\sin\theta_{n-1}\sin\theta_n) \end{aligned}$$

in the spherical coordinates

$$\xi = (r, \theta_2, \cdots, \theta_n) \in [0,1[\times \prod_{j=2}^{n-1} [0,\pi[\times[0,2\pi[=: \mathbb{Q}_n,$$

and these determine the coordinate transform $\sigma : \mathbb{Q}_n \to \mathbb{B}_n$ by $x = \sigma(\xi)$. As in the case of the unit ball one needs to specify a suitable decomposition of the unit ball $\mathbb{B}_n$, but it turns out to be unexpectedly difficult in higher dimensions. We skip the detailed choice of the sets at this point, referring to Section 1 of [11] and only mention that it is possible to choose for every $m \in \mathbb{N}$ finitely many subsets $B_{m,k}$, $k = 1, \dots, K_m$, which are images under the mapping σ of certain rectangles $Q_{m,k} \subset \mathbb{Q}_n$ in polar coordinates, such that

– the volume of every $B_{m,k}$ is proportional to 2^{-nm},
– the union of all sets $B_{m,k}$ when $m \in \mathbb{N}$ and $k = 1, \dots, K_m$, covers $\mathbb{B}_n$,

– there is a constant $N \in \mathbb{N}$ such that any point $x \in \mathbb{B}_n$ is contained in at most N of the sets $B_{m,k}$.

We enumerate the sets $Q_{m,k}$ and $B_{m,k}$ into sequences $(Q_j)_{j=1}^{\infty}$ and $(B_j)_{j=1}^{\infty}$. Then, we impose on $\mathbb{Q}_n$ the partial ordering

$$x \leqslant y \Longleftrightarrow x_1 \le y_1,\ |\tfrac{\pi}{2} - x_2| \ge |\tfrac{\pi}{2} - y_2|,\ \ldots,\ |\tfrac{\pi}{2} - x_{n-1}| \ge |\tfrac{\pi}{2} - y_{n-1}|,\\ x_n \le y_n. \tag{2.14}$$

On each Q_j we pick up the smallest and largest points $x^{(j)} = \big(x_1^{(j)}, \ldots, x_n^{(j)}\big)$ and $y^{(j)} = \big(y_1^{(j)}, \ldots, y_n^{(j)}\big)$ with respect to the given ordering, hence, there holds $Q_j = Q\big(x^{(j)}, y^{(j)}\big)$, where we denote, for $a, b \in \mathbb{Q}_n$ with $a \leqslant b$,

$$Q(a,b) = \big\{x \in \mathbb{R}^n \,:\, a \leqslant x \leqslant b\big\}, \qquad B(a,b) = \sigma\big(Q(a,b)\big). \tag{2.15}$$

Note that for $x, y \in [0,1) \times [0, \frac{\pi}{2}]^{n-2} \times [0, 2\pi]$ the order relation "$\leqslant$" coincides with the usual partial order of points in $\mathbb{R}^n$, which is then mirrored to all of $\mathbb{Q}_n$ to account for the construction of the sets Q_j and B_j. In particular, the $x^{(j)}$ and $y^{(j)}$ are two opposite corners of Q_j and we have $B_j = B(x^{(j)}, y^{(j)})$.

Let $a : \mathbb{B}_N \to \mathbb{C}$ be a locally integrable function and $1 < p < \infty$. The generalized Toeplitz operator is defined by

$$T_a f(z) := \sum_{j=1}^{\infty} T_a(\chi_j f)(z) = \sum_{j=1}^{\infty} P_\alpha(a\chi_j f)(z), \tag{2.16}$$

if the series converges for almost every $z \in \mathbb{B}_N$ and all $f \in A^p_\alpha(\mathbb{B}_N)$. Here χ_j denotes the characteristic function of the set B_j. The boundedness of the Bergman projection P_α in $L^p_\alpha(\mathbb{B}_N)$ implies that $T_a f = P_\alpha(af)$ whenever $af \in L^p_\alpha(\mathbb{B}_N)$. In particular, if a is bounded, then T_a is just the standard Toeplitz operator. As in the one-dimensional case, a "weak" Carleson-type condition (2.18) implies that T_a becomes a well-defined bounded linear operator and the definition coincides with the integral definition, when it is interpreted as an improper integral. Accordingly, given a locally integrable $a : \mathbb{B}_N \to \mathbb{C}$, we define for all $j \in \mathbb{N}$

$$\widehat{a}_j := \sup_{y \in B_j} \Bigg| \int\limits_{B(x^{(j)}, y)} a\, dV_\alpha \Bigg| \tag{2.17}$$

and denote $|B|_\alpha = \int_B dV_\alpha$ for all measurable subsets $B \subset \mathbb{B}_N$.

Theorem 2.4. *Let $a : \mathbb{B}_N \to \mathbb{C}$ be locally integrable, $1 < p < \infty$ and the family $(B_j)_{j\in\mathbb{N}}$ be as above. If there exists a constant $C_a > 0$ such that*

$$\widehat{a}_j \le C_a |B_j|_\alpha \tag{2.18}$$

for all $j \in \mathbb{N}$, then the series (2.16) converges almost everywhere and in $L^p_\alpha(\mathbb{B}_N)$ and defines a bounded linear operator $A^p_\alpha(\mathbb{B}_N) \to A^p_\alpha(\mathbb{B}_N)$ with $\|T_a\| \le C_\alpha C_a$, for some constant $C_\alpha > 0$ independent of a.

Given the symbol a as above and $0 < \rho < 1$, we define $a_\rho(z) = a(z)$ for $|z| \le \rho$ and $a_\rho(z) = 0$ for $\rho < |z| < 1$; then every operator T_{a_ρ} is bounded on $A^p_\alpha(\mathbb{B}_N)$, since the supports of the symbols are compact subsets of the unit ball, or also by the previous theorem. As in the one-dimensional case, the assumption (2.18) allows the following representation of the Toeplitz operator, which does not depend on the decomposition $(B_j)_{j\in\mathbb{N}}$.

Theorem 2.5. *Let* $1 < p < \infty$ *and* $1/p + 1/q = 1$*, and suppose that* $a \in L^1_{\rm loc}$ *satisfies* (2.18)*. Then*

$$T_a f = \lim_{\rho\to 1} T_{a_\rho} f$$

for all $f \in A^p_\alpha(\mathbb{B}_N)$ *and the transpose operator* $T_a^* : A^q_\alpha(\mathbb{B}_N) \to A^q_\alpha(\mathbb{B}_N)$ *can be expressed as*

$$T_a^* f = \lim_{\rho\to 1} T_{\overline{a}_\rho} f$$

for $f \in A^p_\alpha(\mathbb{B}_N)$*.*

The transpose is defined respect to the standard duality of $A^p_\alpha(\mathbb{B}_N)$-spaces.

It would probably be possible and technically easier to formulate and prove a result analogous to Theorem 2.4 by using a rectangular Whitney decomposition of $\mathbb{B}_N$ instead of the one described here, but there would then be the disadvantage that the spherical symmetry would be lost and the condition for the boundedness would depend on the particular choice of the decomposition. In particular, it might be difficult or impossible to prove Theorem 2.5 with that approach.

3. Toeplitz operators in H^∞_v-spaces: introduction

From now on we will deal with Toeplitz operators in spaces on $\mathbb{D}$ with quite general weights v satisfying the basic assumptions of Section 1. A typical, important example of weights considered in this section is the exponentially decreasing $v(r) = \exp(-1/(1-r))$. Because of such examples we need again to pay attention to the definition of Toeplitz operators in the spaces $A^p_v(\mathbb{D})$ and $H^\infty_v(\mathbb{D})$, namely, there is the problem that the Bergman projection may not be bounded. Actually we will show that this is always the case for $p = \infty$ for any weight, see Theorem 4.1, but even in the reflexive case there may be problems in this respect: in [7] it was shown that for the above mentioned exponential weight $v(z)$, the orthogonal projection $L^2_v(\mathbb{D}) \to A^2_v(\mathbb{D})$ is bounded in L^p_v if and only if $p = 2$. Moreover, in [19] W.Lusky proved that the mere existence of a bounded projection from $L^\infty_v(\mathbb{D})$ onto $H^\infty_v(\mathbb{D})$ is equivalent to v satisfying condition (B) of Definition 5.1, below. For example, the exponential weight v satisfies (B), but there also exist natural weights which do not, like $v(z) = (1-\log(1-|z|))^{-1}$ (see the statement after Theorem 1.2. of [19] and Example 2.4. of the same paper for other examples).

Yet, even in the spaces $H^\infty_v(\mathbb{D})$ and $A^p_v(\mathbb{D})$ with general weights, the definition of the Toeplitz operator involves the orthogonal projection $P_v : L^2_v(\mathbb{D}) \to A^2_v(\mathbb{D})$. It will be useful to consider the integral kernel of P_v, the so called Bergman kernel. In the next we follow well-known arguments, see e.g.

[7]. We denote the inner product in the Hilbert spaces $L^2_v(\mathbb{D})$ and $A^2_v(\mathbb{D})$ by $\langle f, g\rangle = \int_{\mathbb{D}} f\overline{g}\, dA_v$. Then, the functions $e_k(z) = \Gamma_{2k}^{-1/2} z^k$, where $k \in \mathbb{N}_0$ and

$$\Gamma_k = 2\pi \int_0^1 r^{k+1} v(r) dr, \tag{3.1}$$

form an orthonormal basis of $A^2_v(\mathbb{D})$. We remark that the numbers Γ_k satisfy for all $0 < \varrho < 1$ and some constant $C_{v,\varrho} > 0$ the following lower bound

$$\Gamma_k \geq C_{v,\varrho} \varrho^k \tag{3.2}$$

for every $k \in \mathbb{N}_0$. This follows from (3.1) by considering the integral e.g. over the interval $[\varrho, 1 - (1-\varrho)/2]$ only.

Convergence in the space $A^p_v(\mathbb{D})$, $1 < p < \infty$, with respect to the norm $\| \cdot \|_{p,v}$ implies pointwise convergence (hence $A^p_v(\mathbb{D})$ is a closed subspace of $L^p_v(\mathbb{D})$), and thus the point evaluation functionals at any point of $\mathbb{D}$ are bounded functionals on $A^p_v(\mathbb{D})$. Consequently, we find the Bergman kernel by using the Riesz representation theorem, which allows us to choose the family of functions $K_z \in A^2_v(\mathbb{D})$, $z \in \mathbb{D}$, such that

$$g(z) = \langle g, K_z\rangle = \int_{\mathbb{D}} g(w)\overline{K_z(w)}\, dA_v(w) \tag{3.3}$$

for all $g \in A^2_v(\mathbb{D})$. The integral operator defined by the right hand side can be extended to $L^2_v(\mathbb{D})$, and it actually defines the orthogonal projection from $L^2_v(\mathbb{D})$ onto $A^2_v(\mathbb{D})$, i.e. the Bergman projection P_v. Using the orthonormal basis $(e_k)_{k=0}^{\infty}$ we can write for all $z \in \mathbb{D}$

$$P_v g(z) = \sum_{k=0}^{\infty} \langle g, e_k\rangle e_k(z) = \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{\Gamma_{2k}} g(w) dA_v(w). \tag{3.4}$$

Here, the order of the summation and the integral can be changed, because (3.2) leads for any fixed $z \in \mathbb{D}$ to the estimate

$$\Big| \frac{z^k \overline{w}^k}{\Gamma_{2k}} \Big| \leq c_{v,\varrho} \Big(\frac{|z|}{\varrho^2} \Big)^k, \tag{3.5}$$

and we can choose here $\varrho^2 > |z|$ so that the sum on the right-hand side of (3.4) converges well enough. Moreover, the estimate (3.5) implies that for every $z \in \mathbb{D}$ the Bergman kernel K_z is a bounded function:

$$|K_z(w)| \leq C_z \ \text{ for all } w \in \mathbb{D}. \tag{3.6}$$

We obtain the following inference.

Lemma 3.1. *Let $f \in L^1(\mathbb{D})$. The integral defining the Toeplitz operator T_f with symbol f on H^∞_v,*

$$T_f g = \int_{\mathbb{D}} f(w) g(w) \overline{K_z(w)}\, dA_v(w), \tag{3.7}$$

converges for all $z \in \mathbb{D}$ and for all $g \in H^\infty_v(\mathbb{D})$, Indeed, if $g \in H_v^\infty(\mathbb{D})$, then, by definition, $gv \in L^\infty(\mathbb{D})$. Hence, the result follows from (3.6).

We remark that the a priori assumption $f \in L^1(\mathbb{D})$ is usual also in the considerations on Toeplitz operators in the reflexive Bergman spaces, but in that case this assumption does not guarantee that the defining integral (3.7) converges for all $g \in A_v^p(\mathbb{D})$. From this point of view, the case $p = \infty$ is simpler. However, although $T_f g$ of (3.7) is a well-defined holomorphic function it might not be an element of $H_v^\infty(\mathbb{D})$ and T_f might not be a bounded operator $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$. Actually it is an elementary consequence of the closed graph theorem that T_f is a bounded operator $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ if and only if $T_f(H_v^\infty(\mathbb{D})) \subset H_v^\infty(\mathbb{D})$. We will soon turn to questions on the boundedness of the operator T_f.

If $g \in H_v^\infty(\mathbb{D})$ is such that $fg \in L_v^2(\mathbb{D})$, we also have

$$(T_f g)(z) \;=\; \sum_{n=0}^{\infty} \langle fg, e_n \rangle e_n(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{2n}} \int_{\mathbb{D}} f(w) g(w) \overline{w}^n v(w) dA \;, \tag{3.8}$$

where the series converges in $L_v^2(\mathbb{D})$. However, the formula also holds for all $g \in H_v^\infty(\mathbb{D})$ (since we are assuming $f \in L^1(\mathbb{D})$) and the product fgv thus belongs to $L^1(\mathbb{D})$, and one can commute the summation and integration in (3.8), due to (3.5). In the latter case, the sum (3.8) converges uniformly for z in compact subsets of the disk.

4. Toeplitz operators with harmonic symbols in $H_v^\infty(\mathbb{D})$-spaces

In this section we will consider Toeplitz operators T_f with harmonic symbols $f : \mathbb{D} \to \mathbb{C}$ in weighted spaces $H_v^\infty(\mathbb{D})$. We assume that the weight v satisfies the basic requirements introduced in Section 1. In addition, the following notions will be needed here and in subsequent sections. For for any function $g : \mathbb{D} \to \mathbb{C}$ and radius $0 \le r \le 1$ we will denote

$$M_\infty(g, r) = \sup_{|z|=r} |g(z)|. \tag{4.1}$$

Also, a weight v is called normal if

$$\sup_{n\in\mathbb{N}} \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty \quad \text{and} \quad \inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1. \tag{4.2}$$

For example, the standard weights $v(r) = (1-r^2)^\alpha$, $\alpha > 0$ are normal, whereas the weights of exponential type, $v(r) = \exp(-\alpha/(1-r)^\beta)$, $\alpha, \beta > 0$, are not. The Riesz projection P maps harmonic functions into holomorphic ones and it is defined by

$$P\Big(\sum_{k\in\mathbb{Z}} a_k r^{|k|} e^{ik\theta} \Big) = \sum_{k=0}^{\infty} a_k r^k e^{ik\theta}, \quad r \in [0,1),\ \theta \in [0, 2\pi]. \tag{4.3}$$

For every $m > 0$ we denote by r_m be a point where the function $r \mapsto r^m v(r)$ attains its absolute maximum on $[0,1]$. Due to the general assumptions on the weights it is easily seen that $r_n \geq r_m$ if $n \geq m$ and $\lim_{m\to\infty} r_m = 1$; see for example [17] for details.

We now turn to questions on the boundedness of Toeplitz operators T_f with harmonic symbols f. In the case f is even holomorphic, the operator T_f is just the multiplier M_f, and it is quite plain that T_f is bounded, if and only if $f \in H^\infty(\mathbb{D})$, i.e., f is a bounded function. Due to the generality of the weights, the details of this claim are exposed in [4], Section 2. Allowing the symbol to be just a harmonic function changes the situation dramatically. The basic reason for this is the unboundedness of the Riesz and Bergman projections with respect to the sup-norm, but one can develop this idea as far as the following result. We repeat that in all of our results the weights v must satisfy the general assumptions made in Section 1.

Theorem 4.1. *There is a bounded harmonic function $f : \mathbb{D} \to \mathbb{C}$ such that T_f is not a bounded operator $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ for any weight v on $\mathbb{D}$.*

This result implies the following conclusion.

Corollary 4.2. *For any weight v, the Bergman projection P_v is not a bounded mapping $L_v^\infty(\mathbb{D}) \to L_v^\infty(\mathbb{D})$.*

Namely, the pointwise multiplication with a bounded function f is always a bounded operator $H_v^\infty(\mathbb{D}) \to L_v^\infty(\mathbb{D})$. So, if P_v were bounded, this would imply $T_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is bounded for every $f \in L^\infty(\mathbb{D})$, which would contradict Theorem 4.1. We actually see that even the restriction of P_v onto $h_v^\infty(\mathbb{D})$ is unbounded.

In the sequel, the complex variable z will always be written in the polar coordinates as $z = re^{i\theta}$, unless otherwise indicated.

Proof of Theorem 4.1. Let us fix a weight v on $\mathbb{D}$ and define first the function $f_0 : \partial\mathbb{D} \to \mathbb{C}$ by

$$f_0(z) = \begin{cases} 1\,, & \text{if } -\pi/2 \leq \theta \leq \pi/2 \\ 0\,, & \text{if } -\pi \leq \theta < -\pi/2 \text{ or } \pi/2 < \theta \leq \pi. \end{cases}$$

The symbol f is defined as the harmonic extension of f_0 on $\mathbb{D}$ obtained from the Poisson integral, hence, we have $f \in h^\infty(\mathbb{D})$. Calculating the Fourier coefficients of f_0 we observe that

$$f(z) = \frac{1}{2} + \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left(z^{2k+1} + \bar{z}^{2k+1} \right), \qquad z \in \mathbb{D}. \tag{4.4}$$

Indeed, let a_k, $k \in \mathbb{Z}$, be. Then we have

$$\begin{aligned} a_k &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-ikt} dt = \frac{e^{ik\pi/2} - e^{-ik\pi/2}}{2k\pi i} = \frac{e^{i|k|\pi/2} - e^{-i|k|\pi/2}}{2|k|\pi i} \\ &= \begin{cases} \frac{(-1)^j}{(2j+1)\pi}, & \text{if } |k| = 2j+1 \text{ for some } j \in \mathbb{N}_0, \\ 0 & \text{for other } k \in \mathbb{Z} \setminus \{0\}. \end{cases} \end{aligned}$$

Moreover, $a_0 = 1/2$. This implies (4.4).

Next we define the test functions, which will be used in showing the unboundedness of the Toeplitz operator: we set

$$g_m(z) = \frac{r^m e^{im\theta}}{r_m^m v(r_m)}, \quad z = re^{i\theta} \in \mathbb{D}$$

for all $m \in \mathbb{N}_0$, where the definition of the maximum point r_m was given in the beginning of the section so that we obviously have $\|g_m\|_v = 1$. We next show that for all $m \in \mathbb{N}_0$ there holds

$$T_f g_m(z) = \sum_{k=0}^{m} b_{k-m} \frac{\Gamma_{2m}}{\Gamma_{2k}} \frac{z^k}{r_m^m v(r_m)} + \sum_{k=m+1}^{\infty} b_{k-m} \frac{z^k}{r_m^m v(r_m)} \tag{4.5}$$

where $f(z) = \sum_{k=-\infty}^{\infty} b_k r^{|k|} e^{ik\theta}$ and Γ_k is as in (3.1). Indeed, this follows from

$$\begin{aligned} f(z)g(z) &= \sum_{j\in\mathbb{Z}} b_j \frac{r^{m+|j|} e^{i(j+m)\theta}}{r_m^m v(r_m)} \\ &= \sum_{k=m+1}^{\infty} b_{k-m} \frac{r^k e^{ik\theta}}{r_m^m v(r_m)} + \sum_{k=-\infty}^{m} b_{k-m} \frac{r^{2m-k} e^{ik\theta}}{r_m^m v(r_m)} \end{aligned}$$

and (3.8).

Let us now turn to the final proof showing that T_f is unbounded on $H_v^\infty(\mathbb{D})$. We define

$$f_1(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \left(z^{2j+1} + \bar{z}^{2j+1}\right)$$

and note that it suffices to show that T_{f_1} is unbounded since $T_f = T_{1/2} + \pi^{-1} T_{f_1}$ and $T_{1/2}$ (multiplication by constant 1/2) is bounded. Fix a positive integer m, say $m = 4m_0$ for $m_0 \in \mathbb{N}$. Then

$$k-m \text{ is } \begin{cases} \text{odd} & \text{if } k \text{ is odd} \\ \text{even} & \text{if } k \text{ is even} \end{cases} \quad \text{and} \quad j-2m_0 \text{ is } \begin{cases} \text{odd} & \text{if } j \text{ is odd} \\ \text{even} & \text{if } j \text{ is even.} \end{cases}$$

We apply formula (4.5) with $b_k = 0$, if k is even, and with $b_k = (-1)^k/|2k+1|$ if k is odd, to obtain

$$T_{f_1} g_m(z) = \sum_{\substack{k=0,\\ k \text{ odd}}}^{m} b_{k-m} \frac{\Gamma_{2m}}{\Gamma_{2k}} \frac{z^k}{r_m^m v(r_m)} + \sum_{\substack{k=m+1,\\ k \text{ odd}}}^{\infty} b_{k-m} \frac{z^k}{r_m^m v(r_m)}. \tag{4.6}$$

Next, if S is the operator $Sf(z) = (f(z) - if(iz))/2$, we have

$$Sf(z) = \sum_{k=0}^{\infty} f_{4k+1} z^{4k+1} \quad \text{for} \quad f(z) = \sum_{k=0}^{\infty} f_{2k+1} z^{2k+1}, \tag{4.7}$$

since $1 - i \cdot i^{2k+1} = 1 + (-1)^k$. We obtain

$$ST_{f_1} g_m(z) = \sum_{0 \le 4j+1 \le m} b_{4j+1-m} \frac{\Gamma_{2m}}{\Gamma_{8j+2}} \frac{z^{4j+1}}{r_m^m v(r_m)} + \sum_{m+1 \le 4j+1 < \infty} b_{4j+1-m} \frac{z^{4j+1}}{r_m^m v(r_m)}.$$

Recall that $b_{4j+1-m} = 1/|4(j-m_0)+1|$. So if we take $\theta = 0$ then all summands in the preceding sum are non-negative. Hence

$$\begin{aligned} & \frac{r_m}{5} \log\left(\frac{1}{1-r_m^4}\right) = \frac{r_m}{5}\sum_{j=1}^{\infty}\frac{(r_m^4)^j}{j} \le \sum_{j=0}^{\infty}\frac{r_m^{4j+1}}{4j+1} \\ = & \sum_{m+1\le 4j+1<\infty} b_{4j+1-m}\frac{r_m^{4j+1}v(r_m)}{r_m^m v(r_m)} \le S(T_{f_1}(g_m))(r_m)v(r_m) \\ \le & \ \|S(T_{f_1}(g_m))\|_v \le \|T_{f_1}(g_m)\|_v. \end{aligned}$$

since trivially by the definition of the operator S we have $\sup_{|z|=r}|(Sf)(z)| \le \sup_{|z|=r}|f(z)|$. Since $\lim_{m\to\infty} r_m = 1$, the left-hand side of the preceding estimate grows to the infinity, when $m \to \infty$. Hence T_{f_1} and also T_f cannot be bounded. □

5. General result on multipliers and Toeplitz operators in $H_v^\infty(\mathbb{D})$ with radial symbols

We continue by considering a fixed radial weight v on $\mathbb{D}$ and Toeplitz operators $T_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$, where $T_f = P_v M_f$. A function with radial symmetry on the disk can nearly never be harmonic, and the study of Toeplitz operators with radial symbols requires techniques different from those in Section 4. First we note that if $f \in L^1(\mathbb{D})$ is radial, i.e. $f(z) = f(|z|)$ for almost every $z \in \mathbb{D}$, then T_f is a coefficient multiplier. This is easily seen by expanding the kernel as in (3.4) and a calculation using the usual orthonormality relations of trigonometric polynomials,

$$\begin{aligned} T_f g(z) &= \sum_{n=0}^{\infty}\frac{z^n}{\Gamma_{2n}}\int_0^1\int_0^{2\pi} f(r)g(re^{i\theta})r^{n+1}e^{-in\theta}v(r)\,d\theta dr \\ &= \sum_{n=0}^{\infty}\frac{z^n}{\Gamma_{2n}}\int_0^1 f(r)r^{2n+1}v(r)g_n dr = \sum_{n=0}^{\infty}\gamma_n g_n z^n \qquad (5.1) \end{aligned}$$

where $g = \sum_n g_n z^n$ and

$$\gamma_n = \frac{1}{\Gamma_{2n}}\int_0^1 r^{2n+1}v(r)f(r)dr. \qquad (5.2)$$

We expose here the approach based mainly on the works [17], [19] and [20] dealing with the condition (B), below, which according to Theorem 1.1 of [19] characterizes those radial weights such that the space $H_v^\infty(\mathbb{D})$ is isomorphic to the Banach space ℓ^∞. Examples of weights satisfying (B) are all normal weights (4.2), in particular the standard weights, and the weights of exponential type $v(r) = \exp(-\gamma/(1-r)^\beta)$; see [19].

The very definition of condition (B) is somewhat technical and we cannot quite avoid other technical considerations in this survey either, however, one can follow our presentation without going into the depth of the arguments just by keeping in mind that condition (B) associates to the weight an increasing sequence of indices $(m_n)_{n=1}^{\infty} \subset (0,\infty)$ and radii $\left(r_{m_n}\right)_{n=1}^{\infty} \subset (0,1)$ such that $m_n \to \infty$ and $r_n \to 1$ as $n \to \infty$, and moreover, gives the very useful equivalent representation in Theorem 5.2 for the weighted sup-norm. We recall that the numbers $r_m \in]0,1[$ were defined in the beginning of Section 4.

Definition 5.1. The weight v satisfies the condition (B), if

$$\forall b_1 > 1 \; \exists b_2 > 1 \; \exists c > 0 \; \forall m,n > 0$$
$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \le b_1 \quad \text{and} \quad m,n,|m-n| \ge c \quad \Rightarrow \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \le b_2.$$

Note that here m and n need not be integers. We now fix a number $b > 2$: it is shown in Lemma 5.1. of [19] that it is then possible to choose, by induction, an increasing, unbounded sequence $(m_n)_{n=1}^{\infty} \subset (0,\infty)$ such that

$$b = \min\left(\left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})}\right).$$

Next, for all $n \in \mathbb{N}$, for the given m_n, we define

$$w_{nk} = \begin{cases} \dfrac{|k| - [m_{n-1}]}{[m_n] - [m_{n-1}]}, & \text{if } m_{n-1} < |k| \le m_n, \quad \text{and} \\[2ex] \dfrac{[m_{n+1}] - |k|}{[m_{n+1}] - [m_n]} & \text{if } m_n < |k| \le m_{n+1}, \end{cases} \tag{5.3}$$

where $k \in \mathbb{Z}$ and $m_0 = 0$. Here $[r]$ is the largest integer not greater than r. With the help of these numbers we define the coefficient multipliers of de la Valleé Poissin type, acting on harmonic functions $f(z) = \sum_{k=-\infty}^{\infty} f_k r^{|k|} e^{ik\theta}$, by

$$W_n : \sum_{k=-\infty}^{\infty} f_k r^{|k|} e^{ik\theta} \mapsto \sum_{k=-\infty}^{\infty} w_{nk} f_k r^{|k|} e^{ik\theta}$$

We will need the following uniform boundedness property of the operators W_n, namely there exists a constant $C > 0$, depending on the weight only, such that

$$M_\infty(W_n g, r) \le C M_\infty(g,r) \tag{5.4}$$

for all $0 \le r \le 1$ and $g \in h_v^\infty(\mathbb{D})$. See (4.1) for the notation. The inequality (5.4) follows e.g. by combining an inequality in Theorem 1 of [20] with Lemma 3.3. of [19].

The operators W_n are important, since they decompose the space $H_v^\infty(\mathbb{D})$ into finite dimensional blocks with a useful representation for the norm. The result is from Theorem 1 of [20], see also Propositions 4.1. and 5.2. of [19].

Theorem 5.2. *Let v satisfy (B). Then there are constants $c_1, c_2 > 0$ such that, for all $g \in h_v^\infty(\mathbb{D})$,*

$$c_1 \sup_{n\in\mathbb{N}} M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \le \|g\|_v \le c_2 \sup_{n\in\mathbb{N}} M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \tag{5.5}$$

and

$$c_1 M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \le \|W_n g\|_v \le c_2 M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \quad (5.6)$$

for all $n \in \mathbb{N}$.

Moreover, it follows from Theorem 5.2 that if the numbers $f_k \in \mathbb{C}$, $k \in \mathbb{Z}$ satisfy

$$\sup_{n\in\mathbb{N}} \sup_{\theta\in[0,2\pi]} \Bigg| \sum_{m_{n-1}<|k|\le m_{n+1}} w_{nk} f_k r_{m_n}^{|k|} e^{ik\theta} \Bigg| v(r_{m_n}) < \infty, \tag{5.7}$$

then the series defining the harmonic function $f(re^{i\theta}) = \sum_{k=-\infty}^{\infty} f_k r^{|k|} e^{ik\theta}$ converges uniformly on compact subsets of $\mathbb{D}$ and f belongs to $h_v^\infty(\mathbb{D})$ and $\|g\|_v$ is bounded by a constant depending on the weight v. For this statement, see Remark 1, (iii) of [20].

Examples. If v is normal then one can take $m_n = 2^{kn}$ for suitable fixed $k > 0$ (see [19], Example 2.4, and [17]). For $v(r) = \exp(-\alpha/(1-r)^\beta)$ one can take $m_n = \beta^2(\beta/\alpha)^{1/\beta} n^{2+2/\beta} - \beta^2 n^2$, see [2].

We now formulate one of the main results of this section, the characterization of boundedness and compactness for coefficient multipliers. The case of Toeplitz operators with radial symbols follows easily from this. The result was already proven for a more restricted class of weights in Theorem 4.1 of [18]. We will assume that a sequence $(\gamma_k)_{k=0}^\infty$ of complex numbers is given, and consider the formal series $f(\theta) = \sum_{k=0}^\infty \gamma_k e^{ik\theta}$, which may or may not converge. The formal series $W_n f$ is then naturally defined as

$$W_n f(\theta) = \sum_{k=0}^{\infty} w_{nk} \gamma_k e^{ik\theta}$$

where the numbers w_{nk} are as in (5.3). We denote by M_f the coefficient multiplier

$$M_f g(z) = \sum_{k=0}^{\infty} \gamma_k g_k r^k e^{ik\theta}, \quad z = re^{i\theta} \tag{5.8}$$

for harmonic functions $g(z) = \sum_{k=-\infty}^{\infty} g_k r^{|k|} e^{ik\theta}$. By definition, $M_f g$ is holomorphic, if the series (5.8) converges.

Theorem 5.3. *Let the weight v satisfy condition (B). Then M_f maps $h_v^\infty(\mathbb{D})$ into $H_v^\infty(\mathbb{D})$ and is bounded, if and only if*

$$\sup_{n\in\mathbb{N}} \int_0^{2\pi} |(W_n f)(\theta)| d\theta < \infty. \tag{5.9}$$

Moreover, assume (5.9) *holds. Then* $M_f : h_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ *is compact, if and only if*

$$\int_0^{2\pi} |(W_n f)(\theta)| d\theta \to 0 \quad \text{as } n \to \infty. \tag{5.10}$$

We present here the proof of the boundedness-statement, comment on the compact case only briefly and refer to [4] for the details. Let us first prove that (5.9) implies the boundedness of the operator. By (5.3), for every n there are only finitely many non-zero w_{nk}, hence, we can write $M_{W_n f}$, cf. (5.8), as a convolution

$$M_{W_n f} g(z) = \frac{1}{2\pi} \int_0^{2\pi} W_n f(\theta - \psi) g(re^{i\psi}) d\psi, \qquad z = re^{i\theta} \in \mathbb{D}.$$

We obtain the estimate

$$\left| M_{W_n f} g(z) \right| v(r) \leq \frac{1}{2\pi} \int_0^{2\pi} |(W_n f)(\theta)| d\theta \, \|g\|_v \tag{5.11}$$

for all $g \in h_v^\infty(\mathbb{D})$, Hence,

$$M_\infty(M_{W_n f} g, r) v(r) \leq C \|g\|_v$$

for all n and r, where the constant $C > 0$ is the supremum on the left-hand side of (5.9). According to the remark concerning (5.7) the series on the right-hand side of (5.8) converges uniformly on compact subsets of $\mathbb{D}$, defines an element of $H_v^\infty(\mathbb{D})$ and is bounded by $\|g\|_v$. This means that M_f maps $h_v^\infty(\mathbb{D})$ continuously into $H_v^\infty(\mathbb{D})$.

As for the compactness of the operator M_f under the assumption (5.10), one takes a sequence $(g_j)_{j=1}^\infty$ contained in the closed unit ball of $h_v^\infty(\mathbb{D})$ and converging to 0 uniformly on compact subsets of $\mathbb{D}$. One needs to show that M_f maps such a sequence into a one converging to 0 with respect to the norm; see for example [26], Section 2.4. Roughly speaking, one can improve the boundedness proof to get this, by using the assumption (5.10) together with the assumption on the convergence in the compact-open topology. One needs a more sophisticated use of Theorem 5.2.

As usual, the proof for the necessity of the condition (5.9) for the boundedness requires a careful enough choice of appropriate test functions. To this end we fix an arbitrary $0 < \varepsilon < 1$ as well as $n \in \mathbb{N}$ and $\varphi \in [0, 2\pi]$. Using the Fejer approximation theorem we find a trigonometric polynomial $g(z) = \sum_{k \in \mathbb{Z}} g_k r^{|k|} e^{ik\theta}$, depending on n, φ and ε, such that

$$\left| g(r_{m_n} e^{i\theta}) - \frac{\overline{W_n f(\varphi - \theta)}}{|W_n(\varphi - \theta)| v(r_{m_n})} \right| < \frac{\varepsilon}{v(r_{m_n})} \tag{5.12}$$

for all $\theta \in [0, 2\pi|$, in particular

$$M_\infty(g, r_{m_n}) v(r_{m_n}) \leq 2. \tag{5.13}$$

As a consequence,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}|(W_nf)(\theta)|d\theta &= \frac{1}{2\pi}\int_0^{2\pi}|(W_nf)(\varphi-\theta)|d\theta \\ &\leq \frac{1}{2\pi}\left|\int_0^{2\pi}(W_nf)(\varphi-\theta)g(r_{m_n}e^{i\theta})d\theta\right| v(r_{m_n})+\varepsilon \\ &= \frac{1}{2\pi}\left|\int_0^{2\pi}f(\varphi-\theta)(W_ng)(r_{m_n}e^{i\theta})d\theta\right| v(r_{m_n})+\varepsilon \\ &= |M_fW_ng(r_{m_n}e^{i\varphi})|\, v(r_{m_n})+\varepsilon \leq \|M_f\|\cdot\|W_ng\|_v+\varepsilon. \end{aligned} \quad (5.14)$$

Using Theorem 5.2 and (5.4), (5.13) we find a constant $C > 0$ such that

$$\|W_ng\|_v \leq c_2M_\infty(W_ng, r_{m_n})v(r_{m_n}) \leq c_2dM_\infty(g, r_{m_n})v(r_{m_n}) \leq 2Cc_2.$$

Hence $\sup_n \int_0^{2\pi} |(W_nf)(\theta)|d\theta < \infty$.

The proof for the necessity of the condition (5.10) for the compactness of M_f needs a number of additional technical details. □

Since Riesz projection P, (4.3), is bounded by the assumptions of Theorem 5.3, it follows that the boundedness and compactness of $M_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ are also equivalent to (5.9) and (5.10), respectively.

Let us turn back to Toeplitz operators. Let T_a be a Toeplitz operator on $H_v^\infty(\mathbb{D})$ with a given radial symbol $a \in L^1(\mathbb{D})$, i.e. $a(z) = a(|z|)$ for almost every z. Then, defining

$$\gamma_k = \frac{1}{\Gamma_{2k}}\int_0^1 r^{2k+1}v(r)a(r)dr,\ k\in\mathbb{N}_0 \quad \text{and} \quad f_a(\theta)=\sum_{k=0}^\infty \gamma_k e^{ik\theta}, \quad (5.15)$$

it was shown in (5.1)–(5.2) that T_a coincides with the Taylor multiplier with coefficients $(\gamma_k)_{k=0}^\infty$. The previous theorem thus yields the main result on the boundedness and compactness.

Theorem 5.4. *Let the weight satisfy (B). If $a \in L^1$ is radial then T_a is bounded as an operator $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ if and only if*

$$\sup_n \int_0^{2\pi} |(W_nf_a)(\theta)|d\theta < \infty, \quad (5.16)$$

and T_a is a compact operator $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$, if and only if

$$\lim_{n\to\infty}\int_0^{2\pi} |(W_nf_a)(\theta)|d\theta = 0. \quad (5.17)$$

We finally recall that Theorems 1.1 and 3.3 of the article [21] contain necessary and sufficient conditions for the boundedness and compactness of $T_a : A^p_v(\mathbb{D}) \to A^p_v(\mathbb{D})$ for $1 < p < \infty$, with minimal assumptions on the radial weights v. However, the characterization is in terms of the boundedness of coefficient multipliers in Hardy spaces, which is another open problem.

6. Supplementary results on Toeplitz operators with radial symbols

According to Theorem 4.1, the boundedness of the symbol does not suffice to imply the boundedness of the Toeplitz operator of $T_f : H^\infty_v(\mathbb{D}) \to H^\infty_v(\mathbb{D})$. In this section we continue working with radial symbols and present results, where additional regularity or decay of the symbol at the boundary of the disk $\mathbb{D}$ implies the boundedness of T_a. The proofs are based on Theorem 5.4, although here we will only sketch some ideas of them.

In Theorem 5.4, the conditions for the boundedness and compactness of the Toeplitz operator may not be easy to verify for concrete weights and symbols, but the results of this section also serve the purpose of presenting some sufficient conditions that are quite easy to formulate and control. The setting for the spaces and symbols is the same as in the previous section, but in addition to condition (B) we also assume that, for some $\epsilon > 0$, v satisfies the following technical condition

$$\sup_{n \in \mathbb{N}} \frac{\int_0^1 r^{n-n^\epsilon} v(r) dr}{\int_0^1 r^n v(r) dr} < \infty. \tag{6.1}$$

It is not difficult to see that (6.1) holds for example for the important classes of standard, normal and exponential weights. For normal weights, condition (6.1) with $\epsilon = 1/2$ follows from Lemma 4.5. of [3]. In the case $v(r) = \exp(-1/(1-r))$ it is known that $\int_0^1 r^m v(r) dr$, $m > 1$, is proportional to the quantity $m^{-3/4} \exp(-Bm^{1/2})$ for some constant $B > 0$ independent of m (see e.g. Lemma 2.2. in [7] or Lemma 4.28 in [1]). Hence, assuming $\epsilon < 1/2$ we obtain

$$\int_0^1 r^{n-n^\epsilon} v(r) dr \le C(n - n^\epsilon)^{3/4} \exp(-B(n-n^\epsilon)^{1/2})$$

$$\le \quad C' n^{3/4} \exp(-Bn^{1/2} + C'') \le C''' \int_0^1 r^n v(r) dr$$

for some positive constants C, C' etc., since

$$(n - n^\epsilon)^{1/2} = n^{1/2}(1 - n^{\epsilon-1})^{1/2} = n^{1/2}\Big(1 - \frac{1}{2} n^{\epsilon-1} + O(n^{2\epsilon-2})\Big)$$

$$= \quad n^{1/2} - \frac{1}{2} n^{\epsilon-1/2} + O(n^{2\epsilon-3/2}) \ge n^{1/2} - C''$$

for all n. Thus, (6.1) holds. The same argument works for the more general weights $v(r) = \exp(-\alpha/(1-r)^\beta)$, $\alpha, \beta > 0$.

It was proven in [19] that normal and exponential weights satisfy (B).

Theorem 6.1. *Let v satisfy (B) and (6.1) and assume that the symbol $a \in L^1$ is real valued and radial. The operator T_a is a bounded operator $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ in any of the following cases:*
(i) The restriction $a|_{[\delta,1[}$ is differentiable (with respect to r) for some $\delta \in]0,1[$ and there holds

$$\limsup_{r\to 1} a'(r) < \infty \qquad or \qquad \liminf_{r\to 1} a'(r) > -\infty, \tag{6.2}$$

and, in addition,

$$\limsup_{r\to 1} |a(r)\log(1-r)| < \infty \tag{6.3}$$

(ii) The restriction $a|_{[\delta,1[}$ is differentiable for some $\delta \in]0,1[$, a' satisfies (6.2) and, for some constant $C > 0$, there holds the bound

$$|a'(r)| \le \frac{C}{(1-r)\big(\log(1-r)\big)^2} \quad for\ r \in]\delta, 1[. \tag{6.4}$$

(iii) The symbol a is continuously differentiable on $[0,1]$.

Theorem 6.1 holds also in the case of complex valued symbols a, namely, the assumptions need to be satisfied by both $\mathrm{Re}\, a$ and $\mathrm{Im}\, a$.

The symbol $a(r) = 1/(1-\log(1-r))$ satisfies the second condition (6.2) and, of course, (6.3) so that $T_a : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is bounded. The same is true for $a(r) = (1-r)^\delta$ with any $\delta > 0$. The latter symbol even induces a compact operator, as can be seen by the next result.

Theorem 6.2. *Let v satisfy (B) and (6.1) and assume that the symbol $a \in L^1$ is real valued and radial.*
(i) If the restriction $a|_{[\delta,1[}$ is differentiable for some $\delta \in]0,1[$, satisfies (6.2) and, in addition,

$$\limsup_{r\to 1} |a(r)\log(1-r)| = 0 \tag{6.5}$$

then the operator $T_a : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is compact.
(ii) Assume that the restriction $a|_{[\delta,1[}$ is differentiable for some $\delta \in]0,1[$, satisfies (6.2), and there holds

$$\lim_{r\to 1} |a'(r)|(1-r)\big(\log(1-r)\big)^2 = 0. \tag{6.6}$$

Then T_a is compact, if and only if $\lim_{r\to 1} a(r) = 0$.

Here, the case of complex valued symbols can be treated in the same way as in the previous theorem.

The item (i) in both Theorems 6.1 and 6.2 follows from Theorem 5.4. We do not present the proof but only refer to [5]. Recall that the coefficients of the series f_a in (5.16), (5.17) are given in (5.15), which involves integrals

$\int_0^1 r^n a(r)v(r)dr$: the proofs of (i) of Theorems 6.1 and 6.2 are based on quite technical estimates and calculations with these expressions.

However, it is not so difficult to see that the sufficient condition (ii) essentially implies (i) in Theorem 6.1. Assume a is real-valued and that (6.4) holds. For all $r \in]\delta, 1[$ we get by the change of the integration variable $\log(1-s) =: x$ and $dx/ds = -1/(1-s)$ that

$$\int_r^1 |a'(s)|ds \le C \int_r^1 \frac{1}{(1-s)\big(\log(1-s)\big)^2} ds = C \int_{-\infty}^{\log(1-r)} \frac{1}{x^2} dx = \frac{C}{|\log(1-r)|}. \tag{6.7}$$

This implies that we can extend a as a continuous function to $]\delta, 1]$ by setting

$$a(1) = \int_\delta^1 a'(s)ds + a(\delta) \ \big(= \lim_{r\to 1} a(r) \ \big).$$

Now, (6.7) yields for all $r \in]\delta, 1[$

$$|a(r) - a(1)| = \Big| \int_r^1 a'(s)ds \Big| \le \frac{C}{|\log(1-r)|}, \tag{6.8}$$

which means that the function $a - a(1)$ satisfies (6.3). Note that the Toeplitz operator with the constant symbol $a(1)$ is bounded as it is just a constant multiplier.

It is plain that (iii) implies (ii) in Theorem 6.1.

Also, as regards to Theorem 6.2, the assumptions in (ii) imply those of (i). Namely, if (6.6) holds, then we can repeat the calculation (6.7)–(6.8) so that the constant C is replaced by a positive function $C(r)$ with $C(r) \to 0$ as $r \to 1$. Then, we see from the analogue of (6.8) that the function $a - a(1)$ even satisfies (6.5). If in addition $a(1) = 0$ then also a satisfies (6.5). Note that if $\lim_{r\to 1} a(r) = a(1) \ne 0$, then T_a is a compact perturbation of a non-zero multiple of the identity which is not compact, and thus it cannot be a compact operator.

In [5] it is shown that if v is a normal weight, the assumptions on a in the previous theorems can be relaxed, namely the boundedness of $T_a : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ follows just from (6.3) and the compactness from (6.5) without any smoothness assumptions on the symbol. Also, in the case of exponential weights $v(r) = \exp\big(-\alpha/(1-r)^\beta\big)$, $\alpha, \beta > 0$, the smoothness requirements on a can be dropped, namely, if

$$\limsup_{r\to 1} |a(r)|(1-r)^{-1/2-\beta/4} < \infty, \tag{6.9}$$

then $T_a : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is bounded, and if

$$\limsup_{r\to 1} |a(r)|(1-r)^{-1/2-\beta/4} = 0, \tag{6.10}$$

then T_a is compact on $H_v^\infty(\mathbb{D})$.

Let us finally consider reflexive weighted Bergman spaces $A^p_v(\mathbb{D})$. For radial symbols, the boundedness of T_a as an operator from the Bergman-Hilbert space $A^2_v(\mathbb{D})$ into itself is characterized by the condition

$$\sup_{n\in\mathbb{N}} |\gamma_n| < \infty, \tag{6.11}$$

where the numbers γ_n are as in (5.2). The idea of trying to characterize the boundedness and compactness of $T_a : A^p_v(\mathbb{D}) \to A^p_v(\mathbb{D})$ for $2 < p < \infty$ (or $1 < p < 2$) by interpolating does not seem to work, but one can derive a sufficient condition similar to (5.9) for the boundedness of T_a in $A^p_v(\mathbb{D})$.

To formulate and sketch the proof of the result we need some modifications of the notions that were used in the case of weighted sup-norms. We again assume that the weight v satisfies condition (B). First, instead of the de la Valleé Poissin operators it is enough just to use the Dirichlet projections $Q_n g(z) = \sum_{k=0}^n g_k z^k$ for holomorphic $g(z) = \sum_{k=0}^\infty g_k z^k$. It is known that there are constants $c_p > 0$ with $M_p(Q_n g, r) \le c_p M_p(g, r)$ for all $0 < r < 1$, $1 < p < \infty$, where c_p does not depend on g, n or r and we write $M_p(g, r)^p = (2\pi)^{-1} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta$.

Analogously with the case of weighted sup-norms one picks up suitable increasing numerical sequences $(\ell_n)_{n=1}^\infty$ with $\ell_1 = 0$ and $\lim_{n\to\infty} \ell_n = \infty$ and $(s_n)_{n=1}^\infty \subset (0,1)$ with $\lim_{n\to\infty} s_n = 1$ and then defines the operators

$$Z_n = Q_{[\ell_{n+1}]} - Q_{[\ell_n]}, \quad n \in \mathbb{N}.$$

These are used to derive an equivalent form of the weighted L^p-norm: for some constants $c_2 \ge c_1 > 0$, for every $f \in A^p_v(\mathbb{D})$, there holds

$$c_1 \|f\|_{p,v} \le \Big(\sum_{n=1}^\infty \omega_n^p M_p^p\,(Z_n f, s_n) \Big)^{1/p} \le c_2 \|f\|_{p,v}, \tag{6.12}$$

where the numbers ω_n are determined by the weight. The details of the definitions of the various parameters and proof of (6.12) can be found in [13] for $p = 1$ and in [20] for $1 < p < \infty$. Examples and calculations in concrete cases can be found in the paper [3]: there it is shown that one can obtain (6.12) for the exponential weights $v(r) = \exp\big(-\alpha/(1-r)^\beta\big)$, $\alpha, \beta > 0$ by using

$$\ell_n = \beta^{1+1/\beta} \alpha^{-1/\beta} n^{2+2/\beta} - \beta n^2, \quad s_n = 1 - \Big(\frac{\alpha}{\beta}\Big)^{1/\beta} \frac{1}{n^{2/\beta}}. \tag{6.13}$$

Proposition 6.3. *Let the weight satisfy (B), let $a \in L^1$ be a radial function and let $f_a(\theta) = \sum_{k=0}^\infty \gamma_k e^{ik\theta}$ be as in (5.2). Then the Toeplitz operator $T_a : A^p_v(\mathbb{D}) \to A^p_v(\mathbb{D})$ is a well-defined, bounded operator, if*

$$\sup_{n\in\mathbb{N}} \int_0^{2\pi} |(Z_n f_a)(\theta)| d\theta < \infty, \tag{6.14}$$

and $T_a : A^p_v(\mathbb{D}) \to A^p_v(\mathbb{D})$ *is compact, if*

$$\int_0^{2\pi} |(Z_n f_a)(\theta)| d\theta \to 0 \quad as\ n \to \infty. \tag{6.15}$$

Proof. Let M_f be the convolution operator, or the sequence space multiplier, corresponding to T_a, see (5.2). For all $g \in A^p_v(\mathbb{D})$ and $z = re^{i\theta} \in \mathbb{D}$ we get

$$(Z_n M_f g)(z) = (M_{Z_n f} g)(z) = \int_0^{2\pi} Z_n f(\theta - \psi) Z_n g(re^{id\psi}) d\psi,$$

where we replaced g by $Z_n g$ by the usual orthogonality relations of trigonometric monomials. The Young inequality $\|a * b\|_{L^p(\partial\mathbb{D})} \le \|a\|_{L^1(\partial\mathbb{D})} \|b\|_{L^p(\partial\mathbb{D})}$ yields

$$M_p(Z_n M_f g, r) \le \int_0^{2\pi} |(Z_n f)(\theta)| d\theta \, M_p(Z_n g, r) \tag{6.16}$$

The inequality $\|M_f g\|_{p,v} \le C \|g\|_{p,v}$ thus follows by applying (6.14) and (6.12) to both $\|M_f g\|_{p,v}$ and $\|g\|_{p,v}$, and this implies the boundedness of T_a.

Assume next (6.15) holds, and let $(g_j)_{j=1}^\infty$ be a sequence which is contained in the unit ball of $A^p_v(\mathbb{D})$ and which converges to 0 uniformly on compact subsets of $\mathbb{D}$, and assume $\varepsilon > 0$ is given. We choose $N \in \mathbb{N}$ such that $\int_0^{2\pi} |(Z_n f)(\theta)| d\theta < \varepsilon$. The convergence of the sequence in the compact-open topology can be used to find a large enough $J \in \mathbb{N}$ such that

$$\sup_{|z| \le r_{m_n}} |Z_n M_f g_j(z)| v(z) < \frac{\varepsilon}{2\pi N \omega_n} \quad \Rightarrow \quad M_p(Z_n M_f g_j, r_{m_n}) < \frac{\varepsilon}{N \omega_n}$$

for all $n \le N$, all $j \ge J$. This, (6.16) and (6.12) imply

$$\begin{aligned} \|M_f g_j\|^p_{p,v} &\le \sum_{n=1}^N \omega_n^p M_p(Z_n M_f g_j, r_{m_n})^p + \sum_{n=N+1}^\infty \omega_n^p M_p(Z_n M_f g_j, r_{m_n})^p \\ &\le \varepsilon + \varepsilon \sum_{n=N+1}^\infty \omega_n^p M_p(Z_n g_j, r_{m_n})^p \le 2\varepsilon \|g_j\|^p_{p,v} \le 2\varepsilon. \end{aligned}$$

We infer that the sequence $(g_j)_{j=1}^\infty$ converges to 0 in the norm of $A^p_v(\mathbb{D})$, which proves the compactness of the operator. □

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