A height bound for abelian schemes with real×Q2 multiplication

Arch. Math. 120 (2023), 381–394

0003-889X/23/040381-14
published online February 20, 2023
https://doi.org/10.1007/s00013-023-01833-6

Archiv der Mathematik

A height bound for abelian schemes with real$\times\mathbb{Q}^2$ multiplication

Zachary Youell

Abstract. In this paper, we prove a height bound for points on the base of a family of abelian varieties at which the fibre possesses additional endomorphisms. This complements a result of André in his book (*G*-Functions and Geometry Aspects of Mathematics, E13. Friedrich Vieweg and Sohn, Braunschweig, 1989) as well a result of Daw and Orr (Ann Scuol Norm Super Class Sci 39:1, 2021). The work in this paper will be used to prove a new case of the Zilber-Pink conjecture which will form part of the author's PhD thesis.

Mathematics Subject Classification. Primary 11G18; Secondary 11G50.

Keywords. Shimura varieties, G-functions, Height bound.

1. Introduction. The Zilber-Pink conjecture is a vast open problem in arithmetic geometry concerning Shimura varieties. To give the statement of the conjecture, we first define an atypical subvariety of a Shimura variety $\mathrm{Sh}_K(G, X)$.

Definition 1.1. Let S be a subvariety of $\mathrm{Sh}_K(G, X)$. A subvariety $W \subset S$ is called atypical with respect to S if it is an irreducible component of $S \cap T$ with T a special subvariety of $\mathrm{Sh}_K(G, X)$ and

$$\dim(W) > \dim(S) + \dim(T) - \dim(\mathrm{Sh}_K(G, X)).$$

We write $\mathrm{Atyp}(S)$ for the set of all atypical subvarieties with respect to S.

Conjecture 1.2 (Zilber-Pink conjecture). *Let S be a subvariety of $\mathrm{Sh}_K(G, X)$. Then $\mathrm{Atyp}(S)$ is a finite union of atypical subvarieties with respect to S.*

The author aims to prove in his thesis that, given appropriate arithmetic data, one can combine the methods of Daw and Orr from both [2] and [3] to prove a new case of Conjecture 1.2. These works both employ the Pila-Zannier method in different ways. In [2], Daw and Orr deal with a "acteur" case as they call it, while in [3], the case is "on-facteur". The authors thesis will deal with a

Birkhäuser

combination of facteur and non-facteur cases. The purpose of this paper is to provide a height bound on points on the base of a family of abelian varieties with quarternionic times CM multiplication. This height bound will be used to show that the Galois orbits of such points are sufficiently large, providing the necessary arithmetic data in the authors case.

1.1. Statement of the main theorem. Let V' be a smooth connected algebraic curve over a number field K, and let V denote the complement of a closed point $v_0 \in V'(K)$. Let h denote the Weil height on V'.

Theorem 1.3. *Let $f : X \to V$ be an abelian scheme of relative dimension 4 with multiplicative reduction at v_0. Let $\bar{\eta}$ be a geometric generic point of V. In addition, assume that we have abelian schemes $g_1 : A \to V$ of relative dimension 2 with $\mathrm{End}(A_{\bar{\eta}}) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{d})$ a real quadratic field and $g_2 : E \to V$ of relative dimension 2 with $\mathrm{End}(E_{\bar{\eta}}) \otimes \mathbb{Q} = \mathbb{Q}^2$ such that X is the fibred product $A \times_V E$. Then there exist effective constants C_1 and C_2 such that, for every $v \in V$ with $\mathrm{End}(A_v) \otimes \mathbb{Q}$ a rational quarternion algebra non-split over $\mathbb{Q}$ and $\mathrm{End}(E_v) \otimes \mathbb{Q}$ containing a product of the form $L \times \mathbb{Q}$ with L an imaginary quadratic field, we have*

$$h(v) \leq C_1[K(v) : K]^{C_2}.$$

By multiplicative reduction at v_0 we mean that the fibre X_{v_0} of the Néron model X' of X at v_0 is a torus.

This complements Theorem 1.3 in Chapter X of [1] and Theorem 8.2 in [2]. We shall refer to fibres X_v of the desired form in Theorem 1.3 as exceptional fibres. André deals only with fibres that are simple abelian varieties of odd dimension $g > 1$ over $\mathbb{Q}$ and the endomorphism algebra at exceptional fibres is an extension of the generic endomorphism algebra. Daw and Orr allow their fibres to have even dimension but require that the maximal commutative subalgebra of the generic endomorphism algebra is a totally real field of odd degree. Here our fibres have dimension $g = 4$ and the maximal commutative subalgebra of $\mathrm{End}_V(X) \otimes \mathbb{Q}$ is a product.

1.2. Outline of the paper. In Section 2, we define the generic period matrix and give a loose idea of G-functions and their importance.

In Section 3, we show that, at all exceptional fibres of the desired type, we have an additional relation, then we use André's methods, along with the work of Masser in [7], to show that these relations are in fact non-trivial.

2. Preliminaries.

2.1. Trivial relations. Following André's methods in Chapter X of [1], we construct trivial relations on the fibres of our scheme $f : X \to V$. We may assume that $\mathrm{End}_V(X) \otimes \mathbb{Q} \cong \mathrm{End}(X_{\bar{\eta}}) \otimes \mathbb{Q}$. If this is not the case, then replacing K with a finite extension and V with an étale neighbourhood would achieve this.

Let η be a generic point of V. The relative de Rham cohomology $\mathrm{H}^1_{\mathrm{DR}}(X/V)$ is a locally free sheaf. Further we may assume it is a free sheaf, if not we may replace V by a Zariski open subset. We write

$$W^{\mathrm{DR}} := \Gamma(\mathrm{H}^1_{\mathrm{DR}}(X/V)) \otimes K(V) \cong \mathrm{H}^1_{\mathrm{DR}}(X_\eta).$$

The fibre X_η is isomorphic to $A_\eta \times E_{1,\eta} \times E_{2,\eta}$. This isomorphism, together with the Künneth theorem, establishes

$$\mathrm{H}^1_{\mathrm{DR}}(X_\eta) \cong \mathrm{H}^1_{\mathrm{DR}}(A_\eta) \oplus \mathrm{H}^1_{\mathrm{DR}}(E_{1,\eta}) \oplus \mathrm{H}^1_{\mathrm{DR}}(E_{2,\eta}).$$

Then the action of our generic endomorphism algebra $D := \mathrm{End}_V(X) \otimes \mathbb{Q} \cong \mathrm{End}(X_\eta) \otimes \mathbb{Q} \cong \mathbb{Q}(\sqrt{d}) \times \mathbb{Q} \times \mathbb{Q}$, after possibly extending K to include $\mathbb{Q}(\sqrt{d})$, induces a splitting of this space into $\bigoplus_{i=1}^4 W^{\mathrm{DR}}_{\sigma_i}$. Here the σ_i are the morphisms of algebras $D \to K(V)$ corresponding to the embeddings of each of the simple factors of D into K.

If we consider some embedding $K \hookrightarrow \mathbb{C}$, then, along with the relative de Rham cohomology, we have a local system of vector spaces that also splits as

$$R_1 f^{\mathrm{an}}_{\mathbb{C}*}(\mathbb{Q}(\sqrt{d})) =: W = \bigoplus_{i=1}^4 W_{\sigma_i}.$$

Here the sum is over the four algebra morphisms $\sigma_i : D \to \mathbb{C}$.

Looking at the analytification of our curve V over $\mathbb{C}$, we can include an open disc Δ centred at v_0 in $(V')^{\mathrm{an}}_{\mathbb{C}}$. Then we write Δ^* for the punctured open disc centred at v_0 and choose some open dense simply-connected set $\mathcal{V} \subset \Delta^*$. Writing $\mathcal{M}_{\mathcal{V}}$ for the field of meromorphic functions on $\mathcal{V}$, the space $W^{\mathrm{DR}}_\sigma \otimes_{K(V)} \mathcal{M}_{\mathcal{V}}$ is 'dual' to $W_\sigma \otimes_{\mathbb{Q}(\sqrt{d})} \mathcal{M}_{\mathcal{V}}$ via the comparison isomorphism multiplied by $(2\pi i)^{-1}$

$$P^1_{X/V} : \mathrm{H}^1_{\mathrm{DR}}(X/V) \otimes_{\mathcal{O}_V} \mathcal{O}_{V^{\mathrm{an}}_{\mathbb{C}}} \to R^1 f^{\mathrm{an}}_* \mathbb{Q}_{X^{\mathrm{an}}_{\mathbb{C}}} \otimes_{\mathbb{Q}_{V^{\mathrm{an}}_{\mathbb{C}}}} \mathcal{O}_{V^{\mathrm{an}}_{\mathbb{C}}}.$$

The notion of dual we mean here is that the stalk of $W^{\mathrm{DR}}_\sigma \otimes_{K(V)} \mathcal{M}_{\mathcal{V}}$ at a point $v \in V$ is isomorphic to the dual space of the stalk of $W_\sigma \otimes_{\mathbb{Q}(\sqrt{d})} \mathcal{M}_{\mathcal{V}}$ at v. As $\mathcal{V}$ is simply-connected, we can trivialise $R_1 f^{\mathrm{an}}_{\mathbb{C}*}(\mathbb{Q}(\sqrt{d}))|_{\mathcal{V}}$. Then we choose a frame $\{\gamma_{\sigma_i,j}\}$ for $R_1 f^{\mathrm{an}}_{\mathbb{C}*}(\mathbb{Q}(\sqrt{d}))|_{\mathcal{V}}$ adapted to the splitting $W = \bigoplus_{i=1}^4 W_{\sigma_i}$ and, using the freeness of $\mathrm{H}^1_{\mathrm{DR}}(X/V)$, a basis $\{\omega_{\sigma_{k,l}}\}$ for $\Gamma(\mathrm{H}^1_{\mathrm{DR}}(X/V)) \otimes K(V)$ inside $\Gamma(\mathrm{H}^1_{\mathrm{DR}}(X/V))$ adapted to the splitting $W^{\mathrm{DR}} = \bigoplus_{k=1}^4 W^{\mathrm{DR}}_{\sigma_k}$. With this, we get the following relations:

$$\frac{1}{2\pi i} \int\limits_{\gamma_{\sigma_i,j}} \omega_{\sigma_k,l} = 0 \text{ for } i \neq k. \tag{1}$$

Since the generic fibre X_η is an abelian variety, it comes equipped with a skew-symmetric form $< \cdot, \cdot >^{\mathrm{DR}}$ on W^{DR} and $2\pi i < \cdot, \cdot >$ on W taking values in $\mathbb{Q}(\sqrt{d})(1)$. Likewise each of the simple factors $A_\eta, E_{1,\eta}$, and $E_{2,\eta}$ have skew-symmetric forms $< \cdot, \cdot >^{\mathrm{DR}}_{A_\eta}, < \cdot, \cdot >^{\mathrm{DR}}_{E_{1,\eta}}$, and $< \cdot, \cdot >^{\mathrm{DR}}_{E_{2,\eta}}$ on their respective de Rham cohomologies and $2\pi i < \cdot, \cdot >_{A_\eta}, 2\pi i < \cdot, \cdot >_{E_{1,\eta}}$, and $2\pi i < \cdot, \cdot >_{E_{2,\eta}}$ on their respective homology groups.

Proposition 2.1. *The forms* $< \cdot, \cdot >^{\mathrm{DR}}_{A_\eta}$ *and* $2\pi i < \cdot, \cdot >_{A_\eta}$ *split as* $< \cdot, \cdot >^{\mathrm{DR}}_{\sigma_1} \oplus < \cdot, \cdot >^{\mathrm{DR}}_{\sigma_2}$ *and* $2\pi i < \cdot, \cdot >_{\sigma_1} \oplus 2\pi i < \cdot, \cdot >_{\sigma_2}$ *respectively.*

Proof. We prove only the case for the form $< \cdot, \cdot >^{\mathrm{DR}}_{A_\eta}$ as the other case uses the same argument. Similarly to above, we write $\mathcal{W}^{\mathrm{DR}}$ for the space $\mathrm{H}^1_{\mathrm{DR}}(A_\eta)$. Then $\mathcal{W}^{\mathrm{DR}}$ splits as $\mathcal{W}^{\mathrm{DR}}_{\sigma_1} \oplus \mathcal{W}^{\mathrm{DR}}_{\sigma_2}$ for the two embeddings $\sigma_i : \mathbb{Q}(\sqrt{d}) \to K(V)$. Then pick forms $a_1 \in \mathcal{W}^{\mathrm{DR}}_{\sigma_1}$ and $a_2 \in \mathcal{W}^{\mathrm{DR}}_{\sigma_2}$. For any $\alpha \in \mathbb{Q}(\sqrt{d})$, we have

$$< \alpha a_1, a_2 >^{\mathrm{DR}} = < a_1, \alpha^\dagger a_2 >^{\mathrm{DR}}, \tag{2}$$

where † represents the Rosati involution. As $\mathbb{Q}(\sqrt{d})$ is a real field, it is pointwise invariant under the Rosati involution. By definition, α acts on $\mathcal{W}^{\mathrm{DR}}_{\sigma_i}$ via σ_i and Equation (2) becomes

$$\begin{aligned} < \alpha a_1, a_2 >^{\mathrm{DR}} &= < a_1, \alpha^\dagger a_2 >^{\mathrm{DR}} \\ < \alpha a_1, a_2 >^{\mathrm{DR}} &= < a_1, \alpha a_2 >^{\mathrm{DR}} \\ \sigma_1(\alpha) < a_1, a_2 >^{\mathrm{DR}} &= \sigma_2(\alpha) < a_1, a_2 >^{\mathrm{DR}} . \end{aligned}$$

Choosing $\alpha \in \mathbb{Q}(\sqrt{d})$ such that $\sigma_1(\alpha) \neq \sigma_2(\alpha)$ implies $< a_1, a_2 >^{\mathrm{DR}} = 0$ for all $a_1 \in \mathcal{W}^{\mathrm{DR}}_{\sigma_1}$ and $a_2 \in \mathcal{W}^{\mathrm{DR}}_{\sigma_1}$. Hence the space $\mathcal{W}^{\mathrm{DR}}_{\sigma_1}$ is orthogonal to $\mathcal{W}^{\mathrm{DR}}_{\sigma_2}$ with respect to $< \cdot, \cdot >^{\mathrm{DR}}$ and

$$< \cdot, \cdot >^{\mathrm{DR}} = < \cdot, \cdot >^{\mathrm{DR}}_{\sigma_1} \oplus < \cdot, \cdot >^{\mathrm{DR}}_{\sigma_2} .$$

□

The form $< \cdot, \cdot >^{\mathrm{DR}}$ is dual to the form $2\pi i < \cdot, \cdot >$ via the untwisted comparison isomorphism $Q^1_{X/V} := (2\pi i) P^1_{X/V}$. In other words, the matrix representing $P^1_{X/V}$ satisfies

$$M^{\mathrm{DR}} = (Q^1_{X/V})^{\mathrm{t}} (2\pi i)^{-1} M^B Q^1_{X/V}, \tag{3}$$

where the matrices M^{DR} and M^B represent the form $< \cdot, \cdot >^{\mathrm{DR}}$ and the form defined on the dual of W respectively (by an abuse of notation, we write $P^1_{X/V}$ for both the period isomorphism and the matrix representing it). For further information on the comparison isomorphism, see Chapter IX of [1]. The matrix $P^1_{X/V}$ acts via a right action, hence by ordering our bases $\{\gamma_{\sigma_i, j}\}$ and $\{\omega_{\sigma_k, l}\}$ in a certain way and rescaling where appropriate, we may assume both M^{DR} and M^B are block diagonal with diagonal $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then, this ordering of our bases, combined with Relation (1), gives $P^1_{X/V}$ the following form:

$$P^1_{X/V} = \begin{pmatrix} \Omega_1^{\sigma_1} & \Omega_2^{\sigma_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ N_1^{\sigma_1} & N_2^{\sigma_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_1^{\sigma_2} & \Omega_2^{\sigma_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & N_1^{\sigma_2} & N_2^{\sigma_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega_1^{\sigma_3} & \Omega_2^{\sigma_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1^{\sigma_3} & N_2^{\sigma_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Omega_1^{\sigma_4} & \Omega_2^{\sigma_4} \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1^{\sigma_4} & N_2^{\sigma_4} \end{pmatrix},$$

with the notation used being $\Omega_i^{\sigma_j} = \frac{1}{2\pi i}\int_{\gamma_{\sigma j,i}} \omega_{\sigma_j,1}$ and $N_i^{\sigma_j} = \frac{1}{2\pi i}\int_{\gamma_{\sigma j,i}} \omega_{\sigma_j,2}$. Now that M^{DR} and M^B are as above, Equation (3) gives the classical "Riemann relations", which in our case simplify to

$$N_2^{\sigma_i}\Omega_1^{\sigma_i} - \Omega_2^{\sigma_i}N_1^{\sigma_i} = \frac{1}{2\pi i}. \tag{4}$$

2.2. Locally invariant periods. As $R_1 f_{\mathbb{C}*}^{\mathrm{an}}(\mathbb{Q}(\sqrt{d}))|_{\Delta^*}$ is a local system, it arises from some representation of $\pi_1(\Delta^*)$. At each point $v \in \Delta^*$, the monodromy action gives an automorphism on the vector space $R_1 f_{\mathbb{C}*}^{\mathrm{an}}(\mathbb{Q}(\sqrt{d}))(v)$. We consider the logarithm of this action, which we denote by $2\pi i N$, and write $2\pi i N_v$ for the specific action on the vector space $R_1 f_{\mathbb{C}*}^{\mathrm{an}}(\mathbb{Q}(\sqrt{d}))(v)$. Thanks to Corollary 11.19 in [9], we know that the monodromy action is unipotent and so $2\pi i N$ is nilpotent with degree of nilpotency 2 (see [6] for proof). Looking again at $R_1 f_{\mathbb{C}*}^{\mathrm{an}}(\mathbb{Q}(\sqrt{d}))|_{\Delta^*}$, we shall denote by W^1 the maximal subsystem of $R_1 f_{\mathbb{C}*}^{\mathrm{an}}(\mathbb{Q}(\sqrt{d}))|_{\Delta^*}$ that is invariant under the monodromy action at each point $v \in \Delta^*$. We note that, like W, this maximal constant subsystem also splits as

$$W^1 = \bigoplus_{i=1}^{4} W^1_{\sigma_i},$$

and Chapter IX of [1] tells us that each of the $W^1_{\sigma_i}$ has half the dimension of W_{σ_i}. In our case, since W_{σ_i} has dimension two, this tells us that $W^1_{\sigma_i}$ is a maximal totally isotropic subspace.

The isotropy of $W^1_{\sigma_i}$ allows us to choose the frame $\{\gamma_{\sigma_i,j}\}$ for $R_1 f_{\mathbb{C}*}^{\mathrm{an}}(\mathbb{Q}(\sqrt{d}))|_{\Delta^*}$ in such a way that, for each σ_i, we have $\gamma_{\sigma_i,1} \in W^1_{\sigma_i}$. Doing so may change the values of the non-zero entries of $P^1_{X/V}$, but does not change the form of the matrix $P^1_{X/V}$ or the relation given in Equation (4).

Definition 2.2. A locally invariant period is one of the form

$$\frac{1}{2\pi i}\int_{\gamma} \omega,$$

with $\gamma \in W^1$.

2.3. *G*-functions. For a place v of a field K, we write K_v for the completion of K with respect to v and let $i_v : K \hookrightarrow K_v$ denote the associated embedding.

Definition 2.3. Let F be a fixed number field. A G-function over F is a formal power series of the form

$$f(z) := \sum_{n\geq 0} a_n z^n, \; a_n \in F,$$

satisfying the following properties:

(1) f is the solution to a linear differential equation with coefficients that are polynomials in z over F,
(2) there exists a sequence of natural numbers $\{d_n\}_{n\in\mathbb{N}}$ that grows at most geometrically such that $d_n a_m \in \mathcal{O}_F$ for $m = 0, \dots, n$,

(3) for every infinite place v of F, the series $\sum_{n\geq 0} i_v(a_n)z^n \in F_v[[z]]$ defines an analytic function around zero.

Theorem 2.4. *Consider the periods to be functions on the disc Δ. Let x be a local parameter at v_0 for V' as in André* [1]*. Then there exists a basis of sections for $\mathrm{H}^1_{\mathrm{DR}}(X/V)$ over $\mathcal{V}$ such that the Taylor expansions in x of the locally invariant relative periods are G-functions.*

Proof. See Chapter IX, Section 4 of [1]. □

The next two sections show that at exceptional fibres we can construct a non-trivial global relation between the locally invariant periods. Then using the following theorem of Bombieri together with Theorem 2.4, this proves Theorem 1.3.

Theorem 2.5 (Theorem E from [1], Introduction). *Let Υ_δ denote the set of points $\xi \in \bar{\mathbb{Q}}$ where there exists some global non-trivial relation of degree δ at ξ between given G-functions $y_1, \ldots, y_\mu$. Then Υ_δ has bounded height (at most a power of $\delta + 1$).*

3. Relations at a fixed Archimedean place.

3.1. Additional relations. Let ν be an infinite place of the field K. To this place, we may associate an embedding $\iota_\nu : K \hookrightarrow \mathbb{C}$. We fix this embedding and construct additional relations present only at exceptional fibres. The method for doing this varies from that of André in [1]. As our exceptional fibre is a semi-simple abelian variety, we construct an additional relation at two of its simple factors. The fact that we have two additional relations allows us to eliminate any factor of $2\pi i$, yielding a relation with coefficients in $\bar{\mathbb{Q}}$.

3.2. Additonal relation for the abelian surface. Let X_v be an exceptional fibre. This gives a fibre A_v from $g_1 : A \to V$ with a non-split rational quarternion algebra B as its endomorphism algebra. The algebra B not only contains $\mathbb{Q}(\sqrt{d})$ but also, by [2, Lemma 8.7], an imaginary quadratic field $\mathbb{Q}(\sqrt{-c})$, stable under the Rosati involution on B. We denote by E and $\hat{E}$ the fields $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{-c})$ respectively and let F denote their compositum. We let $\hat{K}$ denote the compositum of $K(v)$ and F. We construct an additional non-trivial relation at A_v with the desired property via André's method on the scheme $g_1 : A \to V$.

We set up the problem in the same way as for our larger abelian scheme. We again let V' be a smooth connected algebraic curve over a number field K, with V denoting the complement of the closed point $v_0 \in V'$. Then $g_1 : A \to V$ is an abelian scheme of relative dimension 2 with multiplicative reduction at v_0 and $\mathrm{End}(A_\eta) \otimes \mathbb{Q} = E$. Once again $\bar{\eta}$ is a geometric generic point of V. This setup, after possibly extending K to include E, allows us to construct a relative period matrix $P^1_{A/V}$ as we did before. This matrix has the form

$$P^1_{A/V} = \begin{pmatrix} \Omega_1^{\sigma_1} & \Omega_2^{\sigma_1} & 0 & 0 \\ N_1^{\sigma_1} & N_2^{\sigma_1} & 0 & 0 \\ 0 & 0 & \Omega_1^{\sigma_2} & \Omega_2^{\sigma_2} \\ 0 & 0 & N_1^{\sigma_2} & N_2^{\sigma_2} \end{pmatrix}.$$

Lemma 3.1. *There exists a linear or quadratic relation with coefficients in $\hat{K}(2\pi i)$, among the values at v of the locally invariant entries of the period matrix $P^1_{A/V}$.*

We write $\mathcal{W}_v$ for the space $R_1 g^{\mathrm{an}}_{1,\mathbb{C}^*}(E)(v) \cong \mathrm{H}_1(A^{\mathrm{an}}_{v,\mathbb{C}}, E)$ that splits as $\mathcal{W}_{\sigma_1,v} \oplus \mathcal{W}_{\sigma_2,v}$, where the $\sigma_i : E \to \mathbb{C}$ are algebra morphisms. The space $\mathcal{W}_v$ also has a subspace arising from the maximal constant subsystem $\mathcal{W}^1$ of the local system that we shall call $\mathcal{W}^1_v$, which also splits as

$$\mathcal{W}^1_v = \mathcal{W}^1_{\sigma_1} \oplus \mathcal{W}^1_{\sigma_2}.$$

As with our main case, we may replace V with an open neighbourhood to ensure that the relative de Rham cohomology $\mathrm{H}^1_{\mathrm{DR}}(A/V)$ is free and in particular

$$\mathcal{W}^{\mathrm{DR}} := \Gamma(\mathrm{H}^1_{\mathrm{DR}}(A/V)) \otimes K(V) \cong \mathrm{H}^1_{\mathrm{DR}}(A_\eta),$$

where η is a generic point of V. This splits as

$$\mathcal{W}^{\mathrm{DR}} = \mathcal{W}^{\mathrm{DR}}_{\sigma_1} \oplus \mathcal{W}^{\mathrm{DR}}_{\sigma_2}.$$

We have another splitting of both $\mathcal{W}_v$ and $\mathcal{W}^{\mathrm{DR}}_v$,

$$\mathcal{W}_v \otimes_E F = \hat{\mathcal{W}}_{\hat{\sigma}_1} \oplus \hat{\mathcal{W}}_{\hat{\sigma}_2},$$
$$\mathcal{W}^{\mathrm{DR}}_v \otimes_{K(v)} \hat{K} = \hat{\mathcal{W}}^{\mathrm{DR}}_{\hat{\sigma}_1} \oplus \hat{\mathcal{W}}^{\mathrm{DR}}_{\hat{\sigma}_2},$$

where $\hat{\sigma}_{1,2}$ denote the embeddings of $\hat{E}$ into $\mathbb{C}$ instead. Then, if

$$\hat{\mathcal{W}}_{\hat{\sigma}_2} \cap [\mathcal{W}^1_v \otimes_E F] \neq \{0\},$$

as in Case 2 of Construction 2.4.1 from [1], we can choose a non-zero cycle γ that belongs to this space. By Relation (1) (after replacing σ_i with $\hat{\sigma}_i$), for any $\hat{\omega} \in \hat{\mathcal{W}}^{\mathrm{DR}}_{\hat{\sigma}_1}$, we have

$$\int_\gamma \hat{\omega} = 0. \tag{5}$$

Writing $\hat{\omega}$ in terms of our basis for $\mathcal{W}^{\mathrm{DR}}_v$ and γ in terms of our basis for $\mathcal{W}^1_v$ gives us a linear relation of locally invariant periods.

If the intersection $\hat{\mathcal{W}}_{\hat{\sigma}_2} \cap [\mathcal{W}^1_v \otimes_E F]$ is trivial, then the situation is similar to what André calls Case 3.

Before showing how to construct an additional relation in this case, we first establish that the splittings of $\mathcal{W}_v$ into $\mathcal{W}_{\sigma_1} \oplus \mathcal{W}_{\sigma_2}$ and $\hat{\mathcal{W}}_{\hat{\sigma}_1} \oplus \hat{\mathcal{W}}_{\hat{\sigma}_2}$ are not identical.

Proposition 3.2. *We have $\mathcal{W}_{\sigma_i,v} \otimes_E F \neq \hat{\mathcal{W}}_{\hat{\sigma}_j}$ for any pair i, j.*

Proof. We know $\mathcal{W}_v$ is a four dimensional E-vector space and that it inherits an action of $\hat{E}$ from the action of B on A_v. Let us extend scalars so that we work with the space $\mathcal{W}_v \otimes_E F$, then this space splits as both $\mathcal{W}_{\sigma_1,v} \otimes_E F \oplus \mathcal{W}_{\sigma_2,v} \otimes_E F$ and $\hat{\mathcal{W}}_{\hat{\sigma}_1} \otimes_{\hat{E}} F \oplus \hat{\mathcal{W}}_{\hat{\sigma}_2} \otimes_{\hat{E}} F$. Then the action of B is a representation $\rho : B \to \mathrm{GL}(\mathcal{W}_v)$, and we may represent the action of elements of $\hat{E} \subset B$ by matrices with entries in E. To study the action of this representation, we need only look at the action of $j = \sqrt{-c} \in \hat{E}$ on $\mathcal{W}_v \otimes_E F$, where we view j as an endomorphism via ρ. We know that the action of $\rho(j)$ splits $\mathcal{W}_v \otimes_E F$ into two eigenspaces $\hat{\mathcal{W}}_{\hat{\sigma}_1} \otimes_{\hat{E}} F$ and $\hat{\mathcal{W}}_{\hat{\sigma}_2} \otimes_{\hat{E}} F$, with eigenvalues j and $-j$ respectively. Pick $w \in \hat{\mathcal{W}}_{\hat{\sigma}_1} \otimes_{\hat{E}} F$, then by definition,

$$\rho(j)w = jw,$$

where on the right j is acting by standard scalar multiplication. Now we can take an element $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ that acts trivially on E with $\sigma(j) = -j$. Then

$$\sigma(\rho(j)w) = \sigma(jw),$$

which is equivalent to

$$\begin{aligned}\sigma(\rho(j))\sigma(w) &= \sigma(j)\sigma(w)\\ &= -j\sigma(w).\end{aligned}$$

But the matrix representing $\rho(j)$ has entries in E, therefore $\sigma(\rho(j)) = \rho(j)$, $\rho(j)\sigma(w) = -j\sigma(w)$, and $\sigma(w) \in \hat{\mathcal{W}}_{\hat{\sigma}_2}$. Thus for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as above, we have that

$$\sigma(\mathcal{W}_{\sigma_1,v} \otimes_E F) = \mathcal{W}_{\sigma_1,v} \otimes_E F,\ \sigma(\hat{\mathcal{W}}_{\hat{\sigma}_1} \otimes_{\hat{E}} F) = \hat{\mathcal{W}}_{\hat{\sigma}_2} \otimes_{\hat{E}} F.$$

Hence $\mathcal{W}_{\sigma_i,v} \otimes_E F \neq \hat{\mathcal{W}}_{\hat{\sigma}_j} \otimes_{\hat{E}} F$ for any pair i, j. □

As we saw earlier, there is a symplectic form $2\pi i < \cdot, \cdot >$ on $\mathcal{W}_v$.

Lemma 3.3. *The subspaces $\hat{\mathcal{W}}_{\hat{\sigma}_2} \otimes F$ and $\mathcal{W}_v^1 \otimes F \subset \mathcal{W}_v \otimes F$ are both Lagrangian.*

Proof. First take $\mathcal{W}_v^1 \otimes F$. Since $E \subset F$, it has a decomposition into $\mathcal{W}_{\sigma_1}^1 \otimes F \oplus \mathcal{W}_{\sigma_2}^1 \otimes F$. Both of these subspaces are maximal isotropic subspaces of $\mathcal{W}_{\sigma_i} \otimes F$ respectively (since they are both one dimensional). We can then use that $\mathcal{W}_{\sigma_1}$ is symplectic with orthogonal complement $\mathcal{W}_{\sigma_2}$ to establish that for any two vectors $v, w \in \mathcal{W}_v^1 \otimes F$, we have

$$2\pi i < v, w >= 2\pi i < w, v >= 0,$$

and so $\mathcal{W}_v^1 \otimes F$ is isotropic.

For $\hat{\mathcal{W}}_{\hat{\sigma}_2} \otimes F$, we recall that $\hat{E} \subset F$ is an imaginary quadractic field stable under the Rosati involution on B, our quarternion algebra over $\mathbb{Q}$. If we pick $a \in \hat{E}$ with a totally imaginary (so that the complex conjugate $\bar{a} = -a$) and non-zero vectors $v, w \in \hat{\mathcal{W}}_{\hat{\sigma}_2} \otimes F$, then

$$\begin{aligned}\hat{\sigma}_2(a)(2\pi i < v, w >) &= 2\pi i < \hat{\sigma}_2(a)v, w >\\ &= 2\pi i < -a \cdot v, w >\\ &= 2\pi i < v, -\bar{a}w >\\ &= 2\pi i < v, a \cdot w >\\ &= 2\pi i < v, -\hat{\sigma}_2(a)w >\\ &= -\hat{\sigma}_2(a)(2\pi i < v, w >),\end{aligned}$$

here we have assumed that $\hat{\sigma}_1 : \hat{E} \to \mathbb{C}$ acts as the identity and that the Rosati involution restricts to complex conjugation on $\hat{E}$. This implies that

$2\pi i < v, w >= 0$, hence, $\hat{\mathcal{W}}_{\hat{\sigma}_2}$ is isotropic as well. As both subspaces have half the dimension of $W_v \otimes F$ and are isotropic, this makes both Lagrangian. $\square$

Now that we have two Lagrangian subspaces we can construct an additional relation when $\hat{\mathcal{W}}_{\hat{\sigma}_2} \cap [\mathcal{W}_v^1 \otimes_E F]$ is trivial.

Using Lemma 1.4.35 of [8], Lemma 3.3, and the assumption that $\hat{\mathcal{W}}_{\hat{\sigma}_2} \cap [\mathcal{W}_v^1 \otimes_E F] = \{0\}$, we may write

$$\mathcal{W}_v \otimes F = \mathcal{W}_v^1 \otimes F \oplus \hat{\mathcal{W}}_{\hat{\sigma}_2} \otimes F$$

with a symplectic basis $\{\gamma_{\sigma_{1,1}}, \gamma_{\sigma_{2,1}}, \alpha, \beta\}$ for $\mathcal{W}_v$ with respect to $< \cdot, \cdot >$. This basis is chosen such that $\gamma_{\sigma_{1,1}}, \gamma_{\sigma_{2,1}} \in \mathcal{W}_v^1 \otimes F$ and $\alpha, \beta \in \hat{\mathcal{W}}_{\hat{\sigma}_2} \otimes F$. Then, because $\hat{E}$ is an imaginary quadratic field and stable under the Rosati involution, we may write $\{\omega_{\hat{\sigma}_1,1}, \omega_{\hat{\sigma}_1,2}, \omega_{\hat{\sigma}_2,1}, \omega_{\hat{\sigma}_2,2}\}$ for a symplectic basis for $\mathcal{W}_v^{\mathrm{DR}} \otimes F$ with respect to the form $< \cdot, \cdot >^{\mathrm{DR}}$ as both $\hat{\mathcal{W}}_{\hat{\sigma}_1}^{\mathrm{DR}}$ and $\hat{\mathcal{W}}_{\hat{\sigma}_2}^{\mathrm{DR}}$ are Lagrangian by similar arguments to Lemma 3.3.

Now consider the "Period Matrix" P defined as

$$P := \frac{1}{2\pi i}\begin{pmatrix} \int_{\gamma_{\sigma_1,1}} \omega_{\hat{\sigma}_1,1} & \int_{\gamma_{\sigma_2,1}} \omega_{\hat{\sigma}_1,1} & \int_\alpha \omega_{\hat{\sigma}_1,1} & \int_\beta \omega_{\hat{\sigma}_1,1} \\ \int_{\gamma_{\sigma_1,1}} \omega_{\hat{\sigma}_1,2} & \int_{\gamma_{\sigma_2,1}} \omega_{\hat{\sigma}_1,2} & \int_\alpha \omega_{\hat{\sigma}_1,2} & \int_\beta \omega_{\hat{\sigma}_1,2} \\ \int_{\gamma_{\sigma_1,1}} \omega_{\hat{\sigma}_2,1} & \int_{\gamma_{\sigma_2,1}} \omega_{\hat{\sigma}_2,1} & \int_\alpha \omega_{\hat{\sigma}_2,1} & \int_\beta \omega_{\hat{\sigma}_2,1} \\ \int_{\gamma_{\sigma_1,1}} \omega_{\hat{\sigma}_2,2} & \int_{\gamma_{\sigma_2,1}} \omega_{\hat{\sigma}_2,2} & \int_\alpha \omega_{\hat{\sigma}_2,2} & \int_\beta \omega_{\hat{\sigma}_2,2} \end{pmatrix}.$$

For simplicity, we shall name the four quadrants of this matrix,

$$P = \begin{pmatrix} \Omega_1 & \Omega_2 \\ N_1 & N_2 \end{pmatrix}.$$

Since $\alpha, \beta \in \hat{\mathcal{W}}_{\hat{\sigma}_2} \otimes F$, we have

$$\Omega_2 = 0 \tag{6}$$

by Relation (1) (again replacing σ_i with $\hat{\sigma}_i$). As the bases above have been chosen to be symplectic, both $2\pi i < \cdot, \cdot >$ and $< \cdot, \cdot >^{\mathrm{DR}}$ are represented by $\left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. Since both bases are symplectic and the two forms are dual, we have that P multiplies the matrix representing the symplectic form by $(2\pi i)^{-1}$, the justification of this is the same as that of Equation (3). Thus we can establish additional relations between its entries via

$$P \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right) P^{\mathrm{t}} = \frac{1}{2\pi i}\left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right).$$

This, combined with (6), gives the following two relations:

$$N_1^{\mathrm{t}}\Omega_1 - \Omega_1^{\mathrm{t}} N_1 = 0, \tag{7}$$

$$N_2^{\mathrm{t}}\Omega_1 = \frac{1}{2\pi i} I. \tag{8}$$

Using the fact that we can also write

$$\mathcal{W}_v \otimes F = \hat{\mathcal{W}}_{\hat{\sigma}_1} \otimes F \oplus \hat{\mathcal{W}}_{\hat{\sigma}_2} \otimes F,$$

we observe that $\gamma_{\sigma_j,1}$ can be written as $v + a_j\alpha + b_j\beta$ with $v \in \hat{\mathcal{W}}_{\hat{\sigma}_1} \otimes F$. This observation allows us to express N_1 in terms of the entries of N_2. Observe

$$\begin{aligned} N_1 &= \begin{pmatrix} a_1 N_2^{11} + b_1 N_2^{12} & a_2 N_2^{11} + b_2 N_2^{12} \\ a_1 N_2^{21} + b_1 N_2^{22} & a_2 N_2^{21} + b_2 N_2^{22} \end{pmatrix} \\ &= N_2 T, \end{aligned}$$

where $T = \left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right) \in \mathrm{M}_2(F)$ and N_2^{ij} denotes the entry of N_2 in the ith row and jth column. Here, all terms involving cycles from $\hat{\mathcal{W}}_{\hat{\sigma}_1}$ automatically become zero as we are integrating forms from $\hat{\mathcal{W}}^{\mathrm{DR}}_{\hat{\sigma}_2}$ over them.

Thanks to Proposition 3.2, we know that

$$\mathcal{W}_{\sigma_1} \otimes F \not\subset \hat{\mathcal{W}}_{\hat{\sigma}_1} \otimes F,$$

to establish that $T \neq 0$. Now, if we re-examine Equation (8), we may use that $N_1^{\mathrm{t}} = T^{\mathrm{t}} N_2^{\mathrm{t}}$ to obtain

$$T^{\mathrm{t}} N_2^{\mathrm{t}} \Omega_1 = \frac{1}{2\pi i} T^{\mathrm{t}}, \tag{9}$$

$$N_1^{\mathrm{t}} \Omega_1 = \frac{1}{2\pi i} T^{\mathrm{t}}. \tag{10}$$

Now we are guaranteed at least one (possibly more) inhomogenous quadratic relation between our locally invariant periods with coefficients in the field $F(2\pi i)$ because $T \neq 0$ implies that Equation (9) is not just $0 = 0$.

3.3. Additional relation on CM elliptic curve. The field L in our exceptional endomorphism algebra $B \times L \times \mathbb{Q}$ arises due to the elliptic curve E_1 having complex multiplication. To construct a relation in this case, we use a result proved by Masser in [7].

For an elliptic curve E, we denote a basis for $\mathrm{H}_1(E(\mathbb{C}), \mathbb{Q})$ by $\{\gamma_1, \gamma_2\}$. Likewise we denote a basis for $\mathrm{H}^1_{\mathrm{DR}}(E(\mathbb{C}), \mathbb{Q})$ by $\{\omega_1, \omega_2\}$. Then the periods τ_i and pseudo-periods η_i of the elliptic curve are given by

$$\tau_i = \frac{1}{2\pi i} \int\limits_{\gamma_i} \omega_1, \ \ \eta_i = \frac{1}{2\pi i} \int\limits_{\gamma_i} \omega_2,$$

we note that this definition has an extra factor of $1/2\pi i$ compared to the one given in [7] to bring it in line with Definition 2.2. While this is not the relative case, by writing $\gamma_i = \gamma_{\sigma_3,i}(v)$ and $\omega_j = \omega_{\sigma_3,j}(v)$, we can think of τ_1 and η_1 to be the so called "locally invariant" periods between which we wish to find an additional relation.

Theorem 3.4 (Theorem III from [7]). *Let E be an elliptic curve defined over $\bar{\mathbb{Q}}$. Let U be the $\bar{\mathbb{Q}}$-linear span of the set $\{1, \pi, \tau_1, \tau_2, \eta_1, \eta_2\}$. If E has complex multiplication, then* $\dim_{\bar{\mathbb{Q}}} U = 4$.

Masser gives two relations to reduce the dimension to 4 when the curve E has complex multiplication. Let E be defined by the equation $y^2 = 4x^3 -$

$g_2x - g_3$. Then the first relation is $\tau_2 = \alpha\tau_1$, where α has minimal polynomial $ax^2 + bx + c$ over $\mathbb{Z}$. The second relation is

$$a^2\alpha\eta_2 - ac\eta_1 + \gamma\tau_1 = 0,$$

where $\gamma \in \mathbb{Q}(g_2, g_3, \alpha)$. Combining both, along with the relation given in Equation 4, gives us a relation in terms of the "locally invariant" periods

$$-\frac{\gamma}{a^2\alpha}\tau_1^2 + (\frac{c}{a\alpha} - \alpha)\tau_1\eta_1 = \frac{1}{2\pi i}. \tag{11}$$

The numbers g_2, g_3, and α all belong to the field $\hat{K}$ due to choices we have made. So the left hand side of this relation is defined over $\hat{K}$. Any fibre of $(E \times E)$ which has an endomorphism algebra containing $L \times \mathbb{Q}$ is isomorphic to the product of a CM elliptic curve with another elliptic curve so we may apply Theorem 3.4 to obtain a relation of the form found in Equation (11). We note that Case 3 in Chapter X of [1] may also be used to construct a relation of the form found in Equation (11).

3.4. Relation on an exceptional fibre. At the exceptional fibre X_v, we now have a relation between the periods on A_v and on one of the elliptic factors. The relation on the abelian surface can be expressed as $F_1 = t/2\pi i$ for some algebraic number t and F_1 is a linear or quadratic combination of periods. Likewise the relation on the elliptic curve can be expressed as $F_2 = 1/2\pi i$ with F_2 of the same form as the left hand side of Equation (11). After rescaling F_2 by t, we gain a relation for the infinite place ν of K, the relation is

$$F_1 - tF_2 = 0. \tag{12}$$

4. Relation at all Archimedean places. To construct a relation at every Archimedean place of the field K, we follow [1], Chapter X, Section 3. Let ν be an infinite place of K and $\iota : K \hookrightarrow \mathbb{C}$ the corresponding complex embedding. Following Section 2 above, we obtain a period matrix and the locally invariant part of this matrix, after making the same choice of local parameter as we did in Theorem 2.4, yields a matrix of G-functions. These G-functions are Taylor expansions in the local parameter x around v_0 via the embedding $\iota : K \hookrightarrow \mathbb{C}$. We write $y_1, \ldots, y_8 \in K[[x]]$ for these G-functions over K and the third point of Definition 2.3 guarantees $\iota(y_i)$ is also an analytic function (here ι acts coefficient wise).

Lemma 4.1. *For any other complex embedding $\iota' : K \to \mathbb{C}$, the complex Taylor series $\iota'(y_1), \ldots, \iota'(y_8)$ (where ι' acts coefficient wise) are again expansions in x of the locally invariant entries of a period matrix attatched to the same basis of local sections of $\mathrm{H}^1_{\mathrm{DR}}(X/V)$ and some local frame in $(R_1 f^{\mathrm{an}}_{\mathbb{C}\ via\ \iota'})_*(\mathbb{Q})$.*

Proof. See Chapter X, Section 3 of [1]. □

For each infinite place ν of $\hat{K}$, apply the construction of Section 3 above to the exceptional fibre $X_v \times_{\hat{K}} \iota_\nu(\hat{K})$. By Lemma 4.1, we obtain a linear or quadratic polynomial relation $q_\nu(y_1,\ldots,y_8)$ over $\hat{K}$ such that, for $\xi = x(v)$, $q_\nu(y_1,\ldots,y_8)(\xi) = 0$ holds ν-adically if

$$|\xi|_\nu < \mathrm{R}_\nu(y_1,\ldots,y_8),$$

where R_ν denotes the ν-adic radius of convergence.

We define a polynomial

$$q = \prod q_\nu, \tag{13}$$

where the product is taken over all infinite places ν of $\hat{K}$ such that $|\xi|_\nu < \mathrm{R}_\nu(y_1,\ldots,y_8)$. The polynomial q is then a homogeneous polynomial with coefficients in $\hat{K}$ of degree at most $2[\hat{K} : \mathbb{Q}]$. We now show that this is a non-trivial relation of G-functions.

4.1. Non-triviality of the relation. The proof of this is once again very similar to the one that André gives, as it boils down to checking that the relations we constructed in 3.1 cannot be generated from the relations (2.3.1), (2.3.2) or (2.3.3) in Chapter X of [1]. We note that in [1], the generic endomorphism algebra is a totally real field of odd degree. Here it is $E\times\mathbb{Q}^2$, but in Subsection 2.1, we established that the same relations are present in our case, namely Equation (4) and the non-diagonal blocks of the period matrix being zero. Further to this, we note that the sublemma of Tankeev in Chapter X of [1] does not cover our case, but does hold for the simple factors of a non-exceptional fibre X_v, then the generic special Mumford-Tate group is $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{Sp}_{2,E}\times\mathrm{Sp}_{2,\mathbb{Q}}^2$, or $\mathrm{Sp}_{2,E}^4$ after extending scalars to E. Then the results in the last two sections of [4] ensure that the ideal vanishing on the coefficients of the period matrix is the same as the one given by André in [1].

Let the ideal Θ be as in André [1, Lemma 3.3] and $V(\Theta)$ the vanishing locus of this ideal in $\mathbb{A}^{16}$ (where we think of the coordinates as $\Omega_1^{\sigma_i}, \Omega_2^{\sigma_i}, N_1^{\sigma_i}$, and $N_2^{\sigma_i}$ for $1 \le i \le 4$). In [1], André states that showing non-triviality is equivalent to showing that the variety given by our relation is a proper subvariety of the image of $V(\Theta)$ after projecting onto the space spanned by the locally invariant periods ($\Omega_1^{\sigma_i}$ and $N_1^{\sigma_i}$ for each i). In André's work, the projection of $V(\Theta)$ onto this space is defined by the relation saying $(N_1^{\sigma_i})^t\Omega_1^{\sigma_i}$ is symmetric for each i. However in our case, this relation is trivial as each of $N_1^{\sigma_i}$ and $\Omega_1^{\sigma_i}$ are just complex numbers and hence commute with one another, so the projection of $V(\Theta)$ onto the locally invariant subspace is just the whole of the subspace. Hence the ideal defined by the relation given in Equation (13) defines a proper subvariety of this space and the relation is non-trivial.

4.2. The relation is global. The proof of globality of the relation is unchanged from Lemma 3.4 that André gives in [1]. The key condition is that $\mathrm{End}\, X_v \not\hookrightarrow M_4(\mathbb{Q})$, this holds in our case. The maximal commutative subalgebra of $\mathrm{End}\, X_v$ is five dimensional over $\mathbb{Q}$ and is isomorphic to $E\times L\times\mathbb{Q}$, which has no nilpotent elements. By [5], the maximal commutative subalgebra of $M_4(\mathbb{Q})$ has dimension five over $\mathbb{Q}$ but has nilpotent elements, hence there is no isomorphism between the two and $\mathrm{End}\, X_v$ does not embed into $M_4(\mathbb{Q})$.

Acknowledgements. The author would like to thank the EPSRC for its support via grant EP/R513301/1. He would also like to thank the University of Reading and his supervisor Christopher Daw for guidance while writing.

References

[1] André, Y.: G-Functions and Geometry Aspects of Mathematics, E13. Friedrich Vieweg & Sohn, Braunschweig (1989)

[2] Daw, C., Orr, M.: Unlikely intersections with $E \times$ CM curves in $\mathcal{A}_2$. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **22**(4), 1705–1745 (2021)

[3] Daw, C., Orr, M.: Quantitative reduction theory and unlikely intersections. IMRN **2022**(20), 16138–16195 (2022)

[4] Ichikawa, T.: Algebraic groups associated with abelian varieties. Math. Ann. **289**(1), 133–142 (1991)

[5] Jacobson, N.: Schur's theorems on commutative matrices. Bull. Amer. Math. Soc. **50**(6), 431–436 (1944)

[6] Landman, A.: On the Picard-Lefschetz transformation for algebraic manifolds acquiring general singularities. Trans. Amer. Math. Soc. **181**, 89–126 (1973)

[7] Masser, D.W.: Elliptic Functions and Transcendence. Lecture Notes in Mathematics, Vol. 437. Springer, Berlin, New York (1975)

[8] Piccione, P., Tausk, D.V.: A Student's Guide to Symplectic Spaces. Grassmannians and Maslov index. Publicações Matemáticas do IMPA. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro (2008)

[9] Steenbrink, G.: Mixed Hodge Structures. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 52. Springer, Berlin (2008)

Zachary Youell
University of Reading
Reading
UK
e-mail: z.youell@pgr.reading.ac.uk

Received: 9 August 2022

Revised: 22 December 2022

Accepted: 17 January 2023