Vectorial Variational Problems in L^∞ and Applications to Data Assimilation

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Declaration

This thesis describes the work undertaken at the Department of Mathematics and Statistics of the University of Reading, in fulfillment of the requirements for the degree of Doctor of Philosophy.

The results in Chapter 2 comprise joint work with Nikos Kazourakis and Boris Muha. The results in Chapter 3 comprise joint work with Nikos Kazourakis. The results in Chapter 4 comprise joint work with Nikos Kazourakis. Consequently, this thesis provides a modified reproduction of the work produced in [30, 31, 32].

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

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Date: May 2023

Acknowledgments

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Dedication

This thesis is dedicated to my family, friends and everyone else who believed in me.

Abstract

This thesis is a collection of published, submitted and developing papers. Each paper is presented as a chapter of this thesis, in each paper we advance the field of vectorial Calculus of Variations in L^∞. This new progress includes constrained problems, such as the constraint of the Navier-Stokes equations studied in Chapter 2. Additionally the combination of constraints, including a nonlinear operator and a supremal functional, deliberated in Chapter 3. Finally, Chapter 4 presents an alternative supremal constraint, in the contemplation of the second order generalised ∞-eigenvalue problem.

In Chapter 2 we introduce the joint paper with Nikos Katzourakis and Boris Muha. We study a minimisation problem in L^p and L^∞ for certain cost functionals, where the class of admissible mappings is constrained by the Navier-Stokes equations. Problems of this type are motivated by variational data assimilation for atmospheric flows arising in weather forecasting. Herein we establish the existence of PDE-constrained minimisers for all p, and also that L^p minimisers converge to L^∞ minimisers as $p \to \infty$. We further show that L^p minimisers solve an Euler-Lagrange system. Finally, all special L^∞ minimisers constructed via approximation by L^p minimisers are shown to solve a divergence PDE system involving measure coefficients, which is a divergence form counterpart of the corresponding non-divergence Aronsson-Euler system.

In Chapter 3 we present the joint paper with Nikos Katzourakis. We study minimisation problems in L^∞ for general quasiconvex first order functionals, where the class of admissible mappings is constrained by the sublevel sets of another supremal functional and by the zero set of a nonlinear operator. Examples of admissible operators include those expressing pointwise, unilateral, integral isoperimetric, elliptic quasilinear differential, Jacobian and null Lagrangian constraints. Via the method of L^p approximations as $p \to \infty$, we illustrate the existence of a special L^∞ minimiser which solves a divergence PDE system involving certain auxiliary measures as coefficients. This system can be seen as a divergence form counterpart of the Aronsson PDE system which is associated with the constrained L^∞ variational problem.

Chapter 4 provides part of the corresponding developing preprint, joint work with Nikos Katzourakis. We consider the problem of minimising the L^∞ norm of a function of the Hessian over a class of maps, subject to a mass constraint involving the L^∞ norm of a function of the gradient and the map itself. We assume zeroth and first order Dirichlet boundary data, corresponding to the "hinged" and the "clamped" cases. By employing the method of L^p approximations, we establish the existence of a special L^∞ minimiser, which solves a divergence PDE system with measure coefficients as parameters. This is a counterpart of the Aronsson-Euler system corresponding to this constrained variational problem.

Contents

Chapter 1

Background and Motivations

In this chapter we review several background concepts that will be assumed throughout the thesis.

1.1 Sobolev spaces

During the early 20th century, there was a substantial development in the theory of differential equations. Specifically, most partial differential equations (PDEs), either linear or nonlinear, cannot be "solved" in the classical sense of writing an explicit formula representing a solution as differentiable as the equation would suggest. This was the beginning of analytic PDE theory, abandoning to a large extent the search for new calculus techniques to represent formulas of solutions.

A related problem, which arose almost simultaneously, is that in general we have to extend our search for solutions to functions of lower regularity. In fact, for the vast majority of PDEs, it is impossible to prove existence of a solution as differentiable as the terms within the equation. Let alone find an explicit formula to describe the solution in terms of elementary functions.

The modern approach to PDEs consists of searching for appropriately defined generalised solutions. Firstly, we ascertain existence, given a specific domain and certain prescribed boundary/initial conditions. The relevant vector spaces to initiate these questions are the Sobolev spaces. Before we can introduce their definition, we must discuss what it means for a function to have a derivative in the weak sense.

Definition 1.1.1. Let $\Omega \subseteq \mathbb{R}^n$, with $n \in \mathbb{N}$. Suppose $u, v \in L^1_{\text{loc}}(\Omega)$ and $\alpha = (\alpha_1, ..., \alpha_n)$ is a multiindex of order $|\alpha| = \alpha_1 + ... + \alpha_n = k$. We say that $v \in L^1_{\text{loc}}(\Omega)$ is the α^{th}-weak partial derivative of u, written as

$$\partial^\alpha u = v,$$

provided

$$\int_\Omega u \, \partial^\alpha \phi \, \mathrm{d}\mathcal{L}^n = (-1)^{|\alpha|} \int_\Omega v\phi \, \mathrm{d}\mathcal{L}^n,$$

for all test functions $\phi \in C_c^\infty(\Omega)$. Additionally, our integration is with respect to the n-dimensional Lebesgue measure $\mathcal{L}^n$.

Theorem 1.1.2 (Uniqueness of weak derivatives)**.** *A weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

Proof of Theorem 1.1.2. Let $v, w \in L^1_{\text{loc}}(\Omega)$ such that:

$$\int_\Omega u \, \partial^\alpha \phi \, \mathrm{d}\mathcal{L}^n = (-1)^{|\alpha|} \int_\Omega v\phi \, \mathrm{d}\mathcal{L}^n = (-1)^{|\alpha|} \int_\Omega w\phi \, \mathrm{d}\mathcal{L}^n \qquad \forall \phi \in C_c^\infty(\Omega).$$

Then,

$$\int_\Omega v\phi \, \mathrm{d}\mathcal{L}^n = \int_\Omega w\phi \, \mathrm{d}\mathcal{L}^n.$$

Consequently,

$$\int_\Omega (v - w)\phi \, \mathrm{d}\mathcal{L}^n = 0.$$

Thus, $v - w = 0$ a.e and $v = w$ a.e. Hence, we have uniqueness up to a set of measure zero. $\square$

Let us consider some elementary examples of functions possessing weak derivatives.

Example 1.1.3. *If $u \in C^k(\Omega)$ then its classical partial derivatives are indeed weak partial derivatives for $|\alpha| \leq k$.*

Example 1.1.4. Suppose $n = 1$ with $\Omega = (0, 3)$ and

$$u(x) = \begin{cases} 4x - 6 & \text{if } \; 0 < x \leq 2, \\ 2 & \text{if } \; 2 < x < 3. \end{cases}$$

Let

$$v(x) = \begin{cases} 4 & \text{if } \; 0 < x \leq 2, \\ 0 & \text{if } \; 2 < x < 3. \end{cases}$$

We intend to show that $u' = v$ in the weak sense. Choose any $\phi \in C_c^\infty(\Omega)$, we must show that
$$\int_0^3 u\phi' \,\mathrm{d}\mathcal{L} = -\int_0^3 v\phi' \,\mathrm{d}\mathcal{L}.$$
Using additivity and integration by parts, we easily compute
$$\begin{aligned}\int_0^3 u\phi' \,\mathrm{d}\mathcal{L} = \int_0^2 u\phi' \,\mathrm{d}\mathcal{L} + \int_2^3 u\phi' \,\mathrm{d}\mathcal{L} &= \int_0^2 (4x-6)\phi'(x) \,\mathrm{d}\mathcal{L} + \int_2^3 2\phi'(x) \,\mathrm{d}\mathcal{L} \\ &= \Big[\phi(x)(4x-6)\Big]_0^2 - \int_0^2 4\phi(x) \,\mathrm{d}\mathcal{L} + 2\Big[\phi(x)\Big]_2^3 \\ &= 2\phi(2) - \int_0^2 4\phi(x) \,\mathrm{d}\mathcal{L} - 2\phi(2) \\ &= -\int_0^2 4\phi(x) \,\mathrm{d}\mathcal{L} = -\int_0^3 v\phi \,\mathrm{d}\mathcal{L},\end{aligned}$$
as required.

Example 1.1.5. The discontinuous function $f : (0,2) \to \mathbb{R}$
$$f(x) = \begin{cases} 0 & \text{if } \ 0 < x \leq 1, \\ 1 & \text{if } \ 1 < x < 2, \end{cases}$$
is not weakly differentiable. For any $\phi \in C_c^\infty(0,2)$, we compute
$$\int_0^2 f\phi' \,\mathrm{d}\mathcal{L} = \int_0^1 (0)\phi' \,\mathrm{d}\mathcal{L} + \int_1^2 \phi' \,\mathrm{d}\mathcal{L} = \Big[\phi(x)\Big]_1^2 = \phi(2) - \phi(1) = -\phi(1).$$
Consequently, the weak derivative $g = f'$ must satisfy
$$\int_0^2 g\phi \,\mathrm{d}\mathcal{L} = \phi(1),$$
for any $\phi \in C_c^\infty(0,2)$. Suppose for contradiction and assume there exists a $g \in L^1_{\text{loc}}(0,2)$ that satisfies the above. Suppose we have test functions with $\phi(1) = 0$, then $g\phi = 0$ a.e for any $\phi \in C_c^\infty(0,2)$, so $g = 0$ a.e. This must also hold for test functions where $\phi(1) = 1$, but
$$\int_0^2 g\phi \,\mathrm{d}\mathcal{L} = 0 \neq 1,$$
we have a contradiction. As a result, f is not weakly differentiable.

Note that the pointwise derivative of f exists and is zero except at the discontinuity, however the function is not weakly differentiable.

Definition 1.1.6. Let $\Omega \subseteq \mathbb{R}^n$ be open and $p \in [1, \infty]$, then we define the Sobolev spaces as follows:

$$W^{k,p}(\Omega) := \left\{ u \in L^p(\Omega) : \mathrm{D}^\alpha u \in L^p(\Omega), \text{ for } |\alpha| \le k \right\},$$

where the derivatives are taken in the weak sense. If $u \in W^{k,p}(\Omega)$ we define its norm to be:

$$\begin{aligned} \|u\|_{W^{k,p}(\Omega)} &:= \sum_{|\alpha| \le k} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}, \ 1 \le p < \infty, \\ \|u\|_{W^{k,\infty}(\Omega)} &:= \sum_{|\alpha| \le k} \|\mathrm{D}^\alpha u\|_{L^\infty(\Omega)}. \end{aligned}$$

Remark 1.1.7. An alternative choice of norm is given as follows

$$\begin{aligned} \|u\|_{W^{k,p}(\Omega)} &:= \left(\sum_{|\alpha| \le k} \|\mathrm{D}^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, \ 1 \le p < \infty, \\ \|u\|_{W^{k,\infty}(\Omega)} &:= \max_{|\alpha| \le k} \|\mathrm{D}^\alpha u\|_{L^\infty(\Omega)}. \end{aligned}$$

These norms are equivalent to the previous choices, in the sense they generate the same topology. However, throughout this thesis we will employ the norms used in Definition 1.1.6, since they significantly simplify our calculations.

The Sobolev spaces are the correct setting to obtain information for divergence structure PDEs. They also allow us to solve equations in a weaker sense through multiplying by test functions and integrating by parts. Conveniently, the Sobolev spaces inherit functional analytic attributes of the Lebesgue spaces.

Theorem 1.1.8. *The Sobolev spaces are Banach spaces for* $1 \le p \le \infty$. *Additionally, they are separable and reflexive, for* $1 < p < \infty$.

In infinite dimensions, the closed unit ball is not sequentially compact, correspondingly we need some additional properties to establish weak versions of compactness, see [73]. Consequently, the above result is of paramount importance, when constructing bounds to substantiate compactness, to justify limiting processes.

We intend to use Sobolev spaces as means to study PDEs, consequently we must extend the notion of boundary values. A stepping stone to resolving this issue relies on introducing an appropriate closed subspace.

Definition 1.1.9. We denote by

$$W_0^{k,p}(\Omega)$$

the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

Consequently, $u \in W_0^{k,p}(\Omega)$ if and only if there exists functions $u_m \in C_c^\infty(\Omega)$ with $u_m \longrightarrow u$ in $W^{k,p}(\Omega)$. We see the closed subspace $W_0^{k,p}(\Omega)$ as functions within $W^{k,p}(\Omega)$ that exhibit the additional property

$$\mathrm{D}^\alpha u = 0 \text{ on } \partial\Omega \text{ for all } |\alpha| \leq k-1.$$

We must introduce the Trace operator for this expression to make sense, otherwise we have a problem. In the classical setting of $u \in C(\overline{\Omega})$, u has boundary values in the usual sense. However, there is a substantial issue when we encounter functions in a Sobolev space that are not continuous, or only defined a.e. As $\partial\Omega$ is an n-dimensional Lebesgue null set, there is no clear interpretation for the meaning of "u restricted to $\partial\Omega$".

Theorem 1.1.10 (Trace Theorem)**.** *Assume Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

such that
(i) $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$
and
(ii)

$$\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

for each $u \in W^{1,p}(\Omega)$, with the constant C depending only on p and Ω.

Definition 1.1.11. We call Tu the trace of u on $\partial\Omega$.

Theorem 1.1.12. *(Trace-zero functions in $W^{1,p}$). Assume u is bounded and $\partial\Omega$ is C^1. Suppose furthermore that $u \in W^{1,p}(\Omega)$. Then*

$$u \in W_0^{1,p}(\Omega) \textit{ if and only if } Tu = 0 \textit{ on } \partial\Omega.$$

There are some extremely useful inequalities within Sobolev space theory that we use throughout this thesis. The Poincaré and Poincaré Wirtinger inequality are two such examples.

Theorem 1.1.13 (Poincaré's inequality)**.** *Suppose that $1 \leq p < \infty$ and Ω is a bounded open set. Then there exits a constant C (depending on Ω and p) such that*

$$\|u\|_{L^p(\Omega)} \leq C\|\mathrm{D}u\|_{L^p(\Omega)},$$

for any $u \in W_0^{k,p}(\Omega)$.

Theorem 1.1.14 (Poincaré Wirtinger inequality)**.** *Assume that $1 \leq p < \infty$ and Ω is a bounded, connected open set with Lipschitz boundary. Then there exits a constant C, depending only on n, p and Ω, such that*

$$\left\|u - \fint_\Omega u \,\mathrm{d}\mathcal{L}^n\right\|_{L^p(\Omega)} \leq C\|\mathrm{D}u\|_{L^p(\Omega)},$$

for each function $u \in W^{1,p}(\Omega)$.

These results are highly significant, as they allow us to bound the norm of a function, using only the norm of its gradient.

Another useful bound is the Morrey estimate.

Theorem 1.1.15 (Morrey's inequality)**.** *Assume $n < p \leq \infty$. Then there exists a constant C, depending only on p and n such that*

$$\|u\|_{C^{0,\gamma}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for all $u \in C^1(\Omega)$, where

$$\gamma := 1 - \frac{n}{p}.$$

Thus, if $u \in W^{1,p}(\Omega)$, then u is in fact Hölder continuous of exponent γ. This embedding can actually be made compact. The notion of compact embeddings is used throughout linear and nonlinear functional analysis, it is of the utmost importance within the realm of differential equations.

Definition 1.1.16. Let X and Y be Banach spaces, $X \subset Y$. We say that X is compactly embedded in Y, written

$$X \Subset Y$$

provided
(i) $\|u\|_Y \leq C\|u\|_X (u \in X)$ for some constant C
and
(ii) each bounded sequence in X is precompact in Y.

The second condition means that if $(u_k)_{k=1}^{\infty}$ is a sequence in X with $\sup_k \|u_k\|_X < \infty$, then some subsequence $(u_{k_j})_{j=1}^{\infty} \subseteq (u_k)_{k=1}^{\infty}$ converges in Y to some limit u:

$$\lim_{j\to\infty} \|u_{k_j} - u\|_Y = 0.$$

Theorem 1.1.17 (Rellich-Kondrachov)**.** *Suppose that Ω is bounded with C^1 boundary. Then, for $p > n$, the embedding $W^{1,p}(\Omega) \subset C(\overline{\Omega})$ is compact, i.e $W^{1,p}(\Omega) \Subset C(\overline{\Omega})$.*

This result allows us to prove the existence of a uniformly convergent subsequnce, through a $W^{1,p}(\Omega)$ norm bound.

We refrain from discussing this topic any further, as there is a great deal of accessible literature on Sobolev spaces. The reader should consult [1, 21, 42] for a comprehensive exploration. These references also contain the proofs of the results quoted in this section.

1.2 Variational problems

The study of minimisation problems has been undertaken by a variety of mathematicians for diverse intentions. There has been a substantial focus in understanding the relationship between minimality conditions of a functional and the appreciation of PDEs. As there is no general theory for all PDEs, we must exploit the PDE structure where possible. An important collection of such problems are when we can view minimality through a variational approach, this is a corner stone of Calculus of Variations. For instance, suppose we have some potentially nonlinear PDE with the form

$$A[u] = 0, \tag{1.1}$$

where $A[u]$ is a given differential operator and u is the unknown. Equation (1.1) can be characterised as the minimiser of an appropriate energy functional $E[u]$, such that

$$E'[u] = A[u]. \tag{1.2}$$

The practicality of this method is that now we can prove existence of extrema for the energy functional $E[\cdot]$ and consequently the solution of (1.1). This approach provides a much more tractable method than the direct consideration of problem (1.1).

In this thesis we will not explore the Calculus of Variations as a means to study nonlinear PDEs. Neither will we pursue classical problems from the well established field of minimising integral functionals. However, a strong foundation in the study of integral

Calculus of Variations is necessary to examine the problems we face in this thesis. We will recap some of these fundamental ideas in a subsequent subsection. Our interest lie at the heart of minimising constrained vectorial supremal functionals and finding the necessary conditions these minimisers must satisfy. This is the field of vectorial Calculus of Variations in L^∞ and will be the topic of this thesis.

1.3 Literature review

Due to the extensive nature of this branch of mathematics, it is rather challenging to include and produce a completely comprehensive literature review. A substantial quantity of the appropriate literature is reviewed in the introductions of the papers that are presented in this thesis. However, we will briefly outline the most important previous considerations that have inspired the new progress in this thesis.

1.4 Integral Calculus of Variations

We will now recap some rudimentary details, essentially textbook material of integral Calculus of Variations. See [36, 42, 90] for further details.

Let X be a vector space and $E : X \longrightarrow \mathbb{R}$, a real valued continuously differentiable integral functional. Our first natural question of interest concerns the existence of minimisers, this can be investigated through the well established direct method in the Calculus of Variations.

Theorem 1.4.1 (The Direct Method in the Calculus of Variations)**.** *Suppose X is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subseteq X$ be a weakly closed subset of X. Suppose $E : M \longrightarrow \mathbb{R} \cup \{+\infty\}$ is coercive and sequentially weakly lower semi-continuous on M with respect to X, that is, suppose the following conditions are fulfilled:*

- $E(u) \longrightarrow \infty$ *as* $\|u\| \longrightarrow \infty, u \in X$.
- *For any $u \in M$, any sequence $(u_m) \in M$ such that $u_m \rightharpoonup u$ weakly in X there holds:*

$$E(u) \leq \liminf_{m \to \infty} E(u_m).$$

Then E is bounded from below on M and attains its infimum in M.

Remark 1.4.2. Notice that the direct method is not only restricted to proving the existence of integral functionals.

Once we have established existence of solutions, our next point of inquisition is determining necessary conditions that these minima or maxima must satisfy. These necessary conditions will be in form of PDEs. For vectorial problems these necessary conditions will manifest as a system of PDEs.

If E has local extrema (local minima or maxima) at a point $x_0 \in X$, then

$$E'(x_0) = 0.$$

Under further regularity of E, specifically a C^2 functional, we can deduce that

$$E''(x_0) \geq 0,$$

if x_0 is a local minimum.

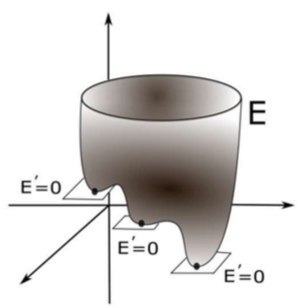

Figure 1.1: Local Extrema

We can intuitively visualise lower dimensional problems like the figure above.

Similarly,

$$E''(x_0) \leq 0.$$

if x_0 is a local maximum.

Now let E be a C^1 real valued functional over the bounded open set $\Omega \subseteq \mathbb{R}^n$. Then for some $u_0 \in \Omega$ and $\phi \in C_c^\infty(\Omega; \mathbb{R}^n)$ and for some sufficiently small $\varepsilon_0 > 0$ the function $E(u_0 + \varepsilon\phi)$ is also continuously differentiable, when $|\varepsilon| < \varepsilon_0$. The first variation is then defined as the derivative of E at point u_0 along the direction of ϕ for $\varepsilon = 0$. When u_0 is a critical point we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} E(u_0 + \varepsilon\phi) = 0. \tag{1.3}$$

We can visualise an elementary situation as follows.

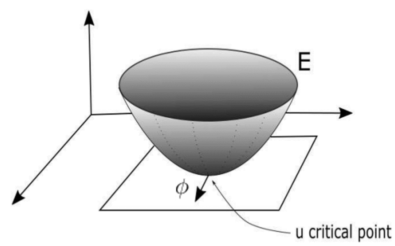

Figure 1.2: Directional Derivative

Consider the functional E defined as above, where $\mathrm{L} \in C^1(\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n})$ is the Lagrangian

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\varepsilon=0} E(u_0 + \varepsilon\phi) &= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} L(\cdot, u + \varepsilon\phi, \mathrm{D}u + \varepsilon\mathrm{D}\phi)\,\mathrm{d}\mathcal{L}^n \\ &= \int_\Omega \Big(L_\eta(\cdot, u, \mathrm{D}u)\cdot\phi + L_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi\Big)\,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

Where $L_\eta := \mathrm{D}_\eta L = \Big(L_{\eta_i}\Big)_{i=1,\dots,N}$ and $L_P := \mathrm{D}_P L = \Big(L_{P_{ij}}\Big)_{i=1,\dots,N}^{j=1,\dots,n}$. Additionally, our integration is with respect to the n-dimensional Lebesgue measure $\mathcal{L}^n$. Finally, for any $A, B \in \mathbb{R}^{N\times n}$ we denote $A : B := \mathrm{tr}(A^T B)$.

If $E(u) = \min\limits_{v\in W^{1,p}(\Omega)} E(v)$ (u is a minimiser of E) then $E(u) \leq E(u + \epsilon\phi)$ and $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} E(u_0 + \varepsilon\phi) = 0$. Thus, if u minimises E, then for all $\phi \in C_c^\infty(\Omega; \mathbb{R}^N)$ we have

$$\int_\Omega \Big(\sum_i L_{\eta_i}(\cdot, u, \mathrm{D}u)\cdot\phi_i + \sum_{ij} L_{P_{ij}}(\cdot, u, \mathrm{D}u) : \mathrm{D}_j\phi_i\Big)\,\mathrm{d}\mathcal{L}^n = 0,$$

as this equality holds for all test functions ϕ we conclude that u solves the Euler-Lagrange PDE system

$$\mathrm{Div}\Big(L_P(\cdot, u, \mathrm{D}u)\Big) = L_\eta(\cdot, u, \mathrm{D}u), \text{ in } \Omega. \tag{1.4}$$

This can be rewritten in the following index notation

$$\sum_i \mathrm{D}_i\Big(L_{P_{ij}}(\cdot, u, \mathrm{D}u)\Big) + L_{\eta_j}(\cdot, u, \mathrm{D}u) = 0, \quad j = 1, ..., N \text{ in } \Omega.$$

For example, consider the p-Dirichlet integral functional

$$E_p(u) := \int_\Omega |\mathrm{D}u|^p \, \mathrm{d}\mathcal{L}^n, \quad u \in W^{1,p}(\Omega; \mathbb{R}^N).$$

The corresponding Euler-Lagrange equations are given by the renowned p-Laplacian

$$\Delta_p u := \mathrm{Div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) = 0 \ \text{ in } \Omega. \tag{1.5}$$

Note that for any $P \in \mathbb{R}^{N\times n}$, the notation $|P|$ denotes its Euclidean (Frobenius) norm:

$$|P| = \left(\sum_{i=1}^{N}\sum_{j=1}^{N}(P_{ij})^2\right)^{\frac{1}{2}}.$$

1.5 Calculus of Variations in L^∞

Calculus of Variations in L^∞ has a reasonably short history, with the first developments being made by Gunnar Arronsson in the 1960s. He considered L^∞ variational problems in the scalar case [4]-[9]. The evolution of vectorial problems did not begin till much later, with Nikos Katzourakis initiating its growth in the 2010s. In this thesis we will study constrained vectorial problems, only a very small quantity of previous literature existed at the commencement of this project [65, 66]. There has already been substantial advancements in the the scalar case, where this interest has arisen from both theoretic and applied settings. Motivations based on applications arise from the following; minimisation of the maximum can provide more realistic models when compared to the case of integral functionals, where instead the average is minimised.

The area of Calculus of Variations in L^∞ is concerned with the study of supremal functionals, alongside their associated PDE systems (as the analogue of Euler-Lagrange equations). Specifically, let $\mathrm{L} \in C^2(\Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n})$ be a Lagrangian, with its arguments denoted as (x, η, P). The vectorial first order case of an L^∞ functional applied to maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ has the form

$$E_\infty(u, \mathcal{O}) := \|\mathrm{L}(\cdot, u, \mathrm{D}u)\|_{L^\infty(\mathcal{O})}, \quad u \in W^{1,\infty}_{loc}(\Omega, \mathbb{R}^N), \ \ \mathcal{O} \Subset \Omega. \tag{1.6}$$

As mentioned, in the classical setting of integral functionals, where

$$E(u)=\int_\Omega \mathrm{L}(\cdot,u,\mathrm{D}u)\,\mathrm{d}\mathcal{L}^n,$$

Euler-Lagrange equations are reasonably well behaved. Specifically, they are of the form

$$\mathrm{Div}\Big(\mathrm{L}_P(\cdot,u,\mathrm{D}u)\Big)=\mathrm{L}_\eta(\cdot,u,\mathrm{D}u),$$

hence exhibiting a divergence form PDE structure, this potentially allows for weak solutions to be characterised via integration by parts. Whilst the PDE system related to (1.6) is degenerate, non-divergence and exhibits discontinuous coefficients, no matter how convex or smooth the Lagrangian might be. Explicitly, the system of associated PDEs generated from (1.6) is given by

$$\mathcal{F}_\infty(\cdot,u,\mathrm{D}u,\mathrm{D}^2u)=0,\quad \text{on } \Omega, \tag{1.7}$$

where $\mathcal{F}_\infty:\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n^2}_s\longrightarrow\mathbb{R}^N$ is the discontinuous Borel measurable map

$$\begin{aligned}\mathcal{F}_\infty(x,\eta,P,X):=&\bigg(\mathrm{L}_P(x,\eta,P)\otimes\mathrm{L}_P(x,\eta,P)+\mathrm{L}(x,\eta,P)\Big[\mathrm{L}_P(x,\eta,P)\Big]^\perp\mathrm{L}_{PP}(x,\eta,P)\bigg):X\\ &+\mathrm{L}(x,\eta,P)\Big[\mathrm{L}_P(x,\eta,P)\Big]^\perp\bigg(\mathrm{L}_{P\eta}(x,\eta,P):P+\mathrm{L}_{Px}(x,\eta,P):I\\ &-\mathrm{L}_\eta(x,\eta,P)\bigg)+\mathrm{L}_P(x,\eta,P)\bigg(\mathrm{L}_\eta(x,\eta,P)^TP+\mathrm{L}_x(x,\eta,P)\bigg).\end{aligned} \tag{1.8}$$

Where $\mathrm{L}_{P_{ij}}\equiv\frac{\partial L}{\partial P_{ij}}$, $\mathrm{L}_{\eta_i}\equiv\frac{\partial L}{\partial\eta_i}$ and $\otimes$ is the tensor product. Additionally, for any matrix $A\in\mathbb{R}^{N\times n}$, $[A]^\perp$ is the orthogonal projection on the orthogonal complement of the Range $R(A)\subseteq\mathbb{R}^N$ of the linear map $A:\mathbb{R}^n\longrightarrow\mathbb{R}^N$:

$$[A]^\perp:=\mathrm{Proj}_{R(A)^\perp}. \tag{1.9}$$

Although, $\mathrm{L}\in C^2(\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n})$ the coefficient $\Big[\mathrm{L}_P(x,\eta,P)\Big]^\perp$ is discontinuous where the rank of $\mathrm{L}_P(x,\eta,P)$ changes. We refer to these types of equations as Aronsson-Euler PDE systems.

An important example of such an equation is given when $\mathrm{L}(x,\eta,P)=|P|^2$ (the Euclidean Matrix norm on $\mathbb{R}^{N\times n}$ squared). Then the relevant associated PDE system is called the ∞-Laplacian:

$$\Delta_\infty:=\Big(\mathrm{D}u\otimes\mathrm{D}u+|\mathrm{D}u|^2[\mathrm{D}u]^\perp\otimes\mathrm{I}\Big):\mathrm{D}^2u=0. \tag{1.10}$$

A standard difficulty, when dealing with these types of problems, is the complexity of the PDE system given in (1.8). As previously mentioned, these systems do not possess a divergence structure, hence weak solutions via Sobolev space methods are inadequate. Furthermore, we can not utilise the notion of Viscosity Solutions, since they are only permitted in the scalar setting [61]. Some recent developments have been made to understand the applicable concept for the vectorial setting, the idea of solutions in the $\mathcal{D}$ sense [63]. However, given the nature of the problems we will study, it will not be necessary to explore this intricate notion of PDE solution. We will bypass these complications, through appropriate rescaling and convergence arguments that will be discussed in the upcoming chapters.

1.6 Notation convention

In this area of mathematics, we use compactness arguments throughout. This involves employing modes convergence, through multiple subsequences. We avoid using a multitude of subscripts, instead we simplify our notation. For example, $u_p \longrightarrow u_\infty$ in $W_0^{1,q}(\Omega;\mathbb{R}^n)$ as $p_j \to \infty$, for any fixed $q \in (p, \infty)$ (this means that the convergence holds along the subsequence of indices $(p_j)_1^\infty$, namely $u_{p_j} \longrightarrow u_\infty$ as $j \to \infty$).

1.7 Thesis structure

The aim of this thesis is to advance recent ideas concerning constrained supremal functionals. Our intentions are to further develop the theoretical framework required for establishing PDE conditions for these minimisation problems. However, we can not tackle the L^∞ problem instantaneously. Instead, we employ the standard technique of L^p approximations, this is a well established idea in the theory of vectorial problems. Specifically, we consider the L^p problem for finite p and pass to the limit to deduce information about the L^∞ problem. This approach is motivated by knowing the L^p norm tends to the L^∞ norm (for fixed essentially bounded functions on a domain of finite measure).

We have reached our goal by producing, submitting and publishing papers on a selection of intricate problems within this field of study. Consequently, this thesis is a collection of papers, where each paper is presented as a chapter. A summary of each chapter is outlined below.

In Chapter 2 we provide the joint paper with Nikos Katzourakis and Boris Muha. This paper was published online in the journal Nonlinearity in December 2021. We start with some motivational ideas into why we consider the problem, specifically what is variational data assimilation and how this could support weather prediction. Subsequently, we can pose our research question as a constrained supremal minimisation problem. Once we have established the theoretical foundations, introduced appropriate vector spaces and devised an admissible class of functions, we start to inspect some fundamental questions. The first is clearly existence of minimisers, indeed our initial theorem in this chapter. Once existence has been ascertained, we can pursue PDE conditions that these minimisers satisfy, this is the contents of our second and third theorems. It turns out that our L^∞ minimisers solve a divergence PDE system involving measure coefficients. This is a divergence form counterpart of the corresponding non-divergence Aronsson-Euler systems that have been previously mentioned. Given that measures are present in our equation, we also investigate some of their properties in our third result.

Chapter 3 presents the joint paper with Nikos Katzourakis. This paper was accepted to the journal Advances in Calculus of Variations in March 2023. Here we investigate a more abstract problem: The minimisation of a general quasiconvex first order L^∞ functional that is constrained by two quantities. Specifically, the sublevel set of another supremal functional and the zero set of a nonlinear operator.

The chapter begins as before, by assembling an outline of the problem. Given the anatomy of the research, the same natural questions must be examined. Thus, our first result provides existence of minimisers through utilisation of the direct method, subsequently constructing the connection between minimisers of the L^p and L^∞ problem. Our next step involves exploiting the generalised Kuhn-Tucker theory to discover equations that the constrained minimisers satisfy. The final result is rather challenging to prove, we can not pass to the limit as easily as we did the previous chapter. The issue is we have products that converge in a weak sense and we can not use duality to overcome it. Due to the specificity of the problem, we can bypass the comprehensive machinery of Young measures and employ the theory of Hutchinson's measure function pairs. This allows us to pass to the limit and produce the desired PDE condition. However, this still requires a substantial body of work. Throughout this project, we must impose ever increasing restrictions upon the nonlinear operator Q. The final section illustrates the variety of problems still available to us, despite the initial limitations of assumptions in our previous results. For instance, examples of potential operators include those expressing pointwise, unilateral, integral isoperimetric, elliptic quasilinear differential, Jacobian and null Lagrangian constraints.

In Chapter 4 we illustrate a component of the developing preprint paper, joint work with Nikos Katzourakis. The complete paper was submitted to the journal Proceedings of the Royal Society of Edinburgh, in March 2023. In this final piece of research, we examine an extension of the previously existing first order problem [67]. Specifically, allowing the functional in question to depend on Hessians as opposed to gradients. Additionally, the constraint depends on the gradient and the function itself. Following an analogous line of inquiry, we determine PDE conditions for constrained minimisers, utilising our knowledge of the approximating problems.

In Chapter 5 we discuss the conclusions and future work.

Appendix A provides the derivation of a bound stated in Chapter 2.

Appendix B contains a simple computational proof of the modified Hölder inequality utilised in Chapter 2.

Chapter 2

Vectorial Variational Problems in L^∞ Constrained by the Navier Stokes Equations

2.1 Introduction and main results

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set and let also $n \geq 2$ and $\nu, T > 0$. Consider the Navier-Stokes equations

$$\left\{ \begin{array}{ll} \partial_t u \; - \nu \Delta u \; + (u \cdot \mathrm{D})u \; + \mathrm{Dp} \; - f \; = \; y, & \text{in } \Omega \times (0,T), \\ \operatorname{div} u \; = 0, & \text{in } \Omega \times (0,T), \\ u(\cdot, 0) = u_0, & \text{on } \Omega, \\ u = \; 0, & \text{on } \partial\Omega \times (0,T), \end{array} \right. \tag{2.1}$$

and for brevity let us henceforth symbolise $\nabla u := (\mathrm{D}u, \partial_t u)$ and $\Omega_T := \Omega \times (0,T)$, where $\mathrm{D}u = (\partial_{x_1} u, ..., \partial_{x_n} u) \in \mathbb{R}^{n \times n}$ symbolises the spatial gradient. The system of PDEs (2.1) describes the velocity $u : \Omega_T \longrightarrow \mathbb{R}^n$ and the pressure $\mathrm{p} : \Omega_T \longrightarrow \mathbb{R}$ of a flow, for some given initial data $u_0 : \Omega \longrightarrow \mathbb{R}^n$ with source $f : \Omega_T \longrightarrow \mathbb{R}^n$. Here the map $y : \Omega_T \longrightarrow \mathbb{R}^n$ is a parameter and should be understood as a (deterministic) noise or error. Let also $N \in \mathbb{N}$ and suppose we are given a mapping noise or error. Let also $N \in \mathbb{N}$ and suppose we are given a mapping

$$\mathrm{Q} \; : \; \Omega_T \times \Big(\mathbb{R}^n \times \mathbb{R}^{(n+1) \times n} \times \mathbb{R} \Big) \longrightarrow \mathbb{R}^N. \tag{2.2}$$

A problem of interest in the geosciences, in particular in data assimilation for atmospheric flows in relation to weather forecasting (see e.g. [22, 23, 24]), can be formulated as follows: find solutions (u, p) to (2.1) such that, in an appropriate sense,

$$\left\{ \begin{aligned} y \,&\approx\, 0, \\ \mathrm{Q}(\cdot, \cdot, u, \nabla u, \mathrm{p}) - q \,&\approx\, 0, \end{aligned} \right. \tag{2.3}$$

where $q : \Omega_T \longrightarrow \mathbb{R}^N$ is a vector of given measurable "data" arising from some specific measurements, taken through the "observation operator" Q of (2.2). In (2.1) and (2.3), y represents an error in the measurements which forces the Navier-Stokes equations to be satisfied only approximately for solenoidal (divergence-free) vector fields. Namely, we are looking for solutions to (2.1) such that simultaneously the error y vanishes, and also the measurements q match the prediction of the solution (u, p) through the observation operator (2.2). In application, Q is typically some component (e.g. linear projection or nonlinear submersion) of the atmospheric flow that we can observe. Unfortunately, the data fitting problem (2.3) is severely ill-posed; an exact solution may well not exist, and even if it does, it may not be unique.

In this paper, inspired by the methodology of data assimilation, especially variational data assimilation in continuous time (for relevant works we refer e.g. to [18, 25, 39, 47, 48, 75, 77, 86]), we seek to minimise the misfit functional

$$(u, \mathrm{p}, y) \,\mapsto\, (1 - \lambda)\Big\|\mathrm{Q}(\cdot, \cdot, u, \nabla u, \mathrm{p}) - q\Big\| \,+\, \lambda\Big\|y\Big\|$$

over all admissible triplets (u, p, y) which satisfy (2.1), for a fixed weight $\lambda \in (0, 1)$. The role of this weight is to obtain essentially a Pareto family of extremals, one for each value λ, even though in this paper we do not pursue further this viewpoint of vector-valued minimisation (the interested reader may e.g. consult [29]). The standard approach to data assimilation is to use Hilbert space methods (or least squares in the discrete case), hence minimising in L^2. The novelty of our approach, which is also justified from the viewpoint of applications, is to consider instead *minimisation in* L^∞, namely by interpreting the norms above as L^∞ ones (or maxima in the discrete case). There is a significant advantage of considering a min-max problem instead of minimising the squared averages: the misfit becomes uniformly small throughout the space-time domain Ω_T and not just on average, hence large "spikes" of deviations from zero are at the outset excluded.

When one passes from a variational problem for an integral norm to one for the supremum norm, even though this is justified from the viewpoint of desired outputs, it poses some serious challenges. The L^∞ norm is neither differentiable nor strictly convex, and the space L^∞ is neither reflexive nor separable. Additionally, with respect to the domain argument, the L^∞ norm is not additive but only sub-additive. Further, one would also need estimates for (2.1) in appropriate subspaces of L^∞ for weakly differentiable functions, which, to the best of our knowledge, do not exist even for linear strongly elliptic systems (see e.g. [52]). Even then, if one somehow solves the L^∞ minimisation problem (by using, for instance, the direct method of the Calculus of Variations as in [36], under the appropriate quasiconvexity assumptions for $|\mathrm{Q}-q|+|y|$ as in [17]), the analogue of the Euler-Lagrange equations for the L^∞ problem cannot be derived directly by perturbation/sensitivity methods due to the lack of smoothness of the L^∞ norm.

In this paper, to overcome the difficulties described above, we follow the methodology of the relatively new field of Calculus of Variations in L^∞ (see e.g. [34, 61] for a general introduction to the scalar-valued theory), and in particular the ideas from [64, 65, 66, 68] involving higher order and vectorial problems, as well as problems involving PDE-constraints, which have only recently started being investigated. To this end, we follow the approach of solving the desired L^∞ variational problem by solving respective approximating L^p variational problems for all p, and obtain appropriate compactness estimates which allow to pass to the limit as $p \to \infty$. The case of finite $p > 2$ studied herein is also of independent interest, especially for numerical discretisation schemes in L^∞ (see e.g. [70, 71]), but in this paper we treat it mostly as an approximation device to solve efficiently the L^∞ problem. The idea of this approach is based on the observation that, for a fixed essentially bounded function on a domain of finite measure, the L^p norm tends to the L^∞ norm of the function as $p \to \infty$.

In order to state our hypotheses and main results, let us set

$$\mathrm{K}\Big(x,t,\eta,A,a,r\Big) \,:=\, \mathrm{Q}\Big(x,t,\eta,A,a,r\Big) \,-\, q(x,t) \tag{2.4}$$

(note that in (2.4) $(x,t) \in \Omega_T$ is treated as two arguments and the two arguments (A,a) are for $\nabla u = (\mathrm{D}u, \partial_t u)$, which we conveniently display abbreviated as one) and, by considering the (strong) divergence operator $\mathrm{div} : W^{1,1}(\Omega;\mathbb{R}^n) \longrightarrow L^1(\Omega)$, we henceforth assume that

$$\left\{ \begin{array}{ll} (a) & \Omega \text{ is bounded and has } C^2 \text{ boundary } \partial\Omega, \\ (b) & u_0 \in \Big(W^{2,\infty} \cap W^{1,\infty}_0\Big)(\Omega;\mathbb{R}^n) \cap \ker(\mathrm{div}), \\ (c) & f \in L^\infty(\Omega_T;\mathbb{R}^n) \quad \& \quad q \in L^\infty(\Omega_T;\mathbb{R}^N), \\ (d) & \mathrm{K}(x,t,\cdot,\cdot,\cdot,\cdot) \text{ is } C^1 \text{ for almost every } (x,t), \\ (e) & \mathrm{K}(\cdot,\cdot,\eta,A,a,r) \text{ is } L^\infty \text{ for all } (\eta,A,a,r), \\ (f) & |\mathrm{K}(x,t,\eta,A,\cdot,\cdot)|^2 \text{ is convex for all } (x,t,\eta,A). \end{array} \right. \tag{2.5}$$

Then, for any $p \in (1,\infty)$, we define the L^p misfit $\mathrm{E}_p : \mathfrak{X}^p(\Omega_T) \longrightarrow \mathbb{R}$ by setting

$$\mathrm{E}_p\Big(u,\mathrm{p},y\Big) \;:=\; (1-\lambda)\Big\|\mathrm{K}(\cdot,u,\nabla u,\mathrm{p})\Big\|_{\dot{L}^p(\Omega_T)} +\, \lambda\|y\|_{\dot{L}^p(\Omega_T)}. \tag{2.6}$$

We note that in (2.6) and subsequently, the dotted $\dot{L}^p$ quantities are regularisations of the respective norms at the origin, obtained by regularising the Euclidean norm in the respective target space:

$$\|h\|_{\dot{L}^p(\Omega_T)} := \Big\||h|_{(p)}\Big\|_{L^p(\Omega_T)}, \qquad |\cdot|_{(p)} \;:=\; \sqrt{|\cdot|^2+p^{-2}}. \tag{2.7}$$

Further, since we will only be dealing with finite measures, we will always be using the normalised L^p norms in which we replace the integral over the domain with the respective average, for example for $L^p(\Omega_T)$ with the $(n+1)$-Lebesgue measure, the norm will be

$$\|h\|_{L^p(\Omega_T)} := \bigg(\fint_{\Omega_T} |h|^p \,\mathrm{d}\mathcal{L}^{n+1}\bigg)^{1/p}.$$

The admissible minimisation class $\mathfrak{X}^p(\Omega_T)$ over which E_p is considered, is defined as follows:

$$\mathfrak{X}^p(\Omega_T) \;:=\; \Big\{(u,\mathrm{p},y) \in \mathcal{W}^p(\Omega_T) \;:\; (u,\mathrm{p},y) \text{ satisfies weakly (2.1)}\Big\}, \tag{2.8}$$

where

$$\mathcal{W}^p(\Omega_T) \;:=\; W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n) \times W^{1,0;p}_{\sharp}(\Omega_T) \times L^p(\Omega_T;\mathbb{R}^n). \tag{2.9}$$

The rather complicated functional spaces appearing in (2.9) are defined as follows. The space $W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n)$ consists of solenoidal maps which are $W^{2,p}$ in space and $W^{1,p}$ in time, and also laterally vanishing on $\partial\Omega \times (0,T)$:

$$\left\{\begin{aligned} W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n) \;&:=\; L^p\Big((0,T);W^{2,p}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big) \bigcap W^{1,p}\Big((0,T);L^p(\Omega;\mathbb{R}^n)\Big),\\ W^{2,p}_{0,\sigma}(\Omega;\mathbb{R}^n) \;&:=\; \Big(W^{2,p}\cap W^{1,p}_0\Big)(\Omega;\mathbb{R}^n) \cap \ker(\mathrm{div}). \end{aligned}\right. \tag{2.10}$$

The space $W^{1,0;p}_{\sharp}(\Omega_T)$ consists of scalar-valued functions which are $W^{1,p}$ in space with zero average, and L^p in time:

$$\left\{\begin{aligned} W^{1,0;p}_{\sharp}(\Omega_T) \;&:=\; L^p\Big((0,T);W^{1,p}_{\sharp}(\Omega)\Big),\\ W^{1,p}_{\sharp}(\Omega) \;&:=\; \Big\{g \in W^{1,p}(\Omega) \;:\; \int_\Omega g \,\mathrm{d}\mathcal{L}^n = 0\Big\}. \end{aligned}\right. \tag{2.11}$$

The associated norms in these spaces are the expected ones, namely

$$\begin{cases} \|v\|_{W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T)} := \|v\|_{L^p(\Omega_T)} + \|\nabla v\|_{L^p(\Omega_T)} + \|\mathrm{D}^2 v\|_{L^p(\Omega_T)}, \\ \|g\|_{W^{1,0;p}_{\sharp}(\Omega_T)} := \|g\|_{L^p(\Omega_T)} + \|\mathrm{D}g\|_{L^p(\Omega_T)}. \end{cases} \tag{2.12}$$

Note also that the divergence-free condition for u in (2.1) has now been incorporated in the functional space $W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T)$. Finally, the L^∞ misfit $\mathrm{E}_\infty : \mathfrak{X}^\infty(\Omega_T) \longrightarrow \mathbb{R}$ is defined by setting

$$\mathrm{E}_\infty\big(u, \mathrm{p}, y\big) := (1-\lambda)\Big\|\mathrm{K}(\cdot,\cdot,u,\nabla u,\mathrm{p})\Big\|_{L^\infty(\Omega_T)} + \lambda\|y\|_{L^\infty(\Omega_T)}, \tag{2.13}$$

where the admissible class $\mathfrak{X}^\infty(\Omega_T)$ is given by

$$\mathfrak{X}^\infty(\Omega_T) := \bigcap_{1<p<\infty} \mathfrak{X}^p(\Omega_T). \tag{2.14}$$

Note that the natural topology of $\mathfrak{X}^\infty(\Omega_T)$ is not induced by a complete norm in a Banach space, but instead its topology is defined as the locally convex topology induced from the ambient Frechét space $\bigcap_{1<p<\infty} \mathcal{W}^p(\Omega_T)$. Notwithstanding, no difficulties will arise from this fact, which is a necessity that stems from the lack of $W^{2,\infty}$-$W^{1,\infty}$ estimates for (2.1). In particular, $\mathfrak{X}^\infty(\Omega_T)$ is *not* obtained by considering the strictly smaller Cartesian product space

$$\mathcal{W}^\infty(\Omega_T) = W^{2,1;\infty}_{\mathrm{L},\sigma}(\Omega_T) \times W^{1,0;\infty}_{\sharp}(\Omega_T) \times L^\infty(\Omega_T;\mathbb{R}^n).$$

However, we will assume that the solution (u, p) to (2.1) is strong and satisfies $W^{2,p}$-$W^{1,p}$ estimates for any finite p. This is deducible under assumption (2.5) in the case of $n = 2$ (see e.g. [50, 87]), and also under smallness conditions on u_0 in any dimension $n \geq 3$ (see e.g. [3, 53, 88]). Hence, our additional hypothesis is

$$\begin{cases} \text{Either} \\ \qquad\qquad n = 2, \\ \text{or, } n \geq 3 \text{ but for any } p \in (1,\infty) \text{, exists } C > 0 \text{ depending only on } p \text{ and} \\ \text{on } \partial\Omega,\ T,\ \|u_0\|_{L^2(\Omega)}, \|f\|_{L^2(\Omega_T)} \text{, such that} \\ \|u\|_{W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T)} + \|\mathrm{p}\|_{W^{1,0;p}_{\sharp}(\Omega_T)} \leq C\Big(\|u_0\|_{W^{2-\frac{2}{p},p}(\Omega)} + \|f\|_{L^p(\Omega_T)}\Big), \\ \text{when } (u,\mathrm{p}) \text{ solves weakly (2.1) with } y \equiv 0. \end{cases} \tag{2.15}$$

Assumption (2.15), albeit restrictive, is compatible with situations of interest in weather forecasting (see e.g. [22, 23, 24]). Our first main result concerns the existence of E_p-minimisers in $\mathfrak{X}^p(\Omega_T)$, the existence of E_∞-minimisers in $\mathfrak{X}^\infty(\Omega_T)$ and the approximability of the latter by the former as $p \to \infty$.

Theorem 2.1.1 (E_∞-minimisers, E_p-minimisers & convergence as $p \to \infty$)**.** *Suppose that* (2.5) *and* (2.15) *hold true. Then, for any $p \in (n+2, \infty]$, the functional E_p (given by* (2.6) *for $p < \infty$ and by* (2.13) *for $p = \infty$) has a constrained minimiser (u_p, p_p, y_p) in the admissible class $\mathfrak{X}^p(\Omega_T)$:*

$$\mathrm{E}_p\Big(u_p, \mathrm{p}_p, y_p\Big) \ = \ \inf\Big\{\mathrm{E}_p\Big(u, \mathrm{p}, y\Big) \ : \ \Big(u, \mathrm{p}, y\Big) \in \mathfrak{X}^p(\Omega_T)\Big\}. \tag{2.16}$$

Additionally, there exists a subsequence of indices $(p_j)_1^\infty$ such that the sequence of respective E_{p_j}-minimisers $(u_{p_j}, \mathrm{p}_{p_j}, y_{p_j})$ satisfies $(u_p, \mathrm{p}_p, y_p) \rightharpoonup (u_\infty, \mathrm{p}_\infty, y_\infty)$ in $\mathcal{W}^q(\Omega_T)$ for any $q \in (1, \infty)$, as $p_j \to \infty$. Additionally,

$$\left\{ \begin{array}{lll} u_p \rightharpoonup u_\infty, & & \text{in } W^{2,1;q}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n), \\ u_p \longrightarrow u_\infty, & & \text{in } C\Big(\overline{\Omega_T};\mathbb{R}^n\Big), \\ \mathrm{D}u_p \longrightarrow \mathrm{D}u_\infty, & & \text{in } C\Big(\overline{\Omega_T};\mathbb{R}^{n\times n}\Big), \\ \mathrm{p}_p \rightharpoonup \mathrm{p}_\infty, & & \text{in } W^{1,0;q}_{\#}(\Omega_T;\mathbb{R}^n), \\ y_p \rightharpoonup y_\infty, & & \text{in } L^q(\Omega_T), \end{array} \right. \tag{2.17}$$

for any $q \in (1, \infty)$, and also

$$\mathrm{E}_p(u_p, \mathrm{p}_p, y_p) \longrightarrow \mathrm{E}_\infty(u_\infty, \mathrm{p}_\infty, y_\infty) \tag{2.18}$$

as $p_j \to \infty$.

Given the existence of constrained minimisers established by Theorem 2.1.1 above, the next natural question concerns the existence of necessary conditions in the form of PDEs governing the constrained minimisers. We first consider the case of $p < \infty$. Unsurprisingly, the PDE constraint of (2.1) used in defining (2.8) gives rise to a generalised Lagrange multiplier in the Euler-Lagrange equations, obtained by utilising well-known results on the Kuhn-Tucker theory from [94]. Interestingly, however, the incorporation of the solenoidality constraint into the functional space (recall (2.10)), allows us to have only one generalised multiplier corresponding only to the parabolic system in (2.1), instead of two.

To state our second main result, we first need to introduce some notation. For any $M \in \mathbb{N}$ and $p \in (1,\infty)$, we define the operator

$$\mathfrak{M}_p \ : \quad L^p(\Omega_T;\mathbb{R}^M) \longrightarrow L^{p'}(\Omega_T;\mathbb{R}^M),$$

where $p' := p/(p-1)$, by setting

$$\mathfrak{M}_p(V) \ := \ \frac{|V|_{(p)}^{p-2}\, V}{\Big(\|V\|_{\dot{L}^p(\Omega_T)}\Big)^{p-1}}. \tag{2.19}$$

Here $|\cdot|_{(p)}$ is the regularisation of the Euclidean norm of $\mathbb{R}^M$, as defined in (2.7). By Hölder's inequality it is immediate to verify that (for the normalised $L^{p'}$ norm) we actually have

$$\Big\|\mathfrak{M}_p(V)\Big\|_{L^{p'}(\Omega_T)} \ \leq \ 1,$$

and therefore $\mathfrak{M}_p$ is valued in the unit ball of $L^{p'}(\Omega_T;\mathbb{R}^M)$. Further, for brevity we will use the notation

$$\left\{ \begin{array}{rl} \mathrm{K}[u,\mathrm{p}] & := \mathrm{K}\Big(\cdot,\cdot,u,\nabla u,\mathrm{p}\Big), \\ \mathrm{K}_\eta[u,\mathrm{p}] & := \mathrm{K}_\eta\Big(\cdot,\cdot,u,\nabla u,\mathrm{p}\Big), \\ \mathrm{K}_{(A,a)}[u,\mathrm{p}] & := \mathrm{K}_{(A,a)}\Big(\cdot,\cdot,u,\nabla u,\mathrm{p}\Big), \\ \mathrm{K}_r[u,\mathrm{p}] & := \mathrm{K}_r\Big(\cdot,\cdot,u,\nabla u,\mathrm{p}\Big), \end{array} \right. \tag{2.20}$$

for K and its partial derivatives $\mathrm{K}_\eta, \mathrm{K}_{(A,a)}, \mathrm{K}_r$ with respect to the arguments for $u, \nabla u$ and p respectively.

Theorem 2.1.2 (Variational Equations in L^p)**.** *Suppose that* (2.5) *and* (2.15) *hold true. Then, for any $p \in (n+2,\infty)$, there exists a Lagrange multiplier*

$$\Psi_p \ \in \ \Big(W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big)^* \tag{2.21}$$

associated with the constrained minimisation problem (2.16)*, such that the minimising triplet $(u_p,\mathrm{p}_p,y_p) \in \mathfrak{X}^p(\Omega_T)$ satisfies the relations*

$$\left\{ \begin{array}{l} (1-\lambda)\displaystyle\int_{\Omega_T}\Big(\mathrm{K}_\eta[u_p,\mathrm{p}_p]\cdot u \ + \ \mathrm{K}_{(A,a)}[u_p,\mathrm{p}_p] : \nabla u\Big)\cdot \mathfrak{M}_p\Big(\mathrm{K}[u_p,\mathrm{p}_p]\Big)\,\mathrm{d}\mathcal{L}^{n+1} \\ \ = -\lambda\displaystyle\int_{\Omega_T}\Big(\partial_t u - \nu\Delta u + (u\cdot \mathrm{D})u_p + (u_p\cdot \mathrm{D})u\Big)\cdot\mathfrak{M}_p(y_p)\,\mathrm{d}\mathcal{L}^{n+1} \ + \ \Big\langle \Psi_p, u(\cdot,0)\Big\rangle, \end{array} \right. \tag{2.22}$$

$$(1-\lambda)\int_{\Omega_T} \mathrm{K}_r[u_p,\mathrm{p}_p]\,\mathrm{p}\cdot \mathfrak{M}_p\Big(\mathrm{K}[u_p,\mathrm{p}_p]\Big)\,\mathrm{d}\mathcal{L}^{n+1} \,=-\lambda\int_{\Omega_T}\mathrm{D}\mathrm{p}\cdot\mathfrak{M}_p(y_p)\,\mathrm{d}\mathcal{L}^{n+1} \tag{2.23}$$

for all test mappings

$$(u,\mathrm{p})\,\in\, W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n)\times W^{1,0;p}_{\sharp}(\Omega_T),$$

where the operators $\mathrm{K},\mathrm{K}_\eta,\mathrm{K}_{(A,a)},\mathrm{K}_r$ *are given by* (2.20).

Now we consider the case of $p=\infty$. For this extreme case, which is obtained by an appropriate passage to limits as $p\to\infty$ in Theorem 2.1.2, we need to assume additionally that the operator $\mathrm{K}[u,\mathrm{p}]$ does not depend on $(\partial_t u,\mathrm{p})$, hence in this case we will symbolise

$$\left\{\begin{aligned} \mathrm{K}[u] \,&:= \mathrm{K}\Big(\cdot,\cdot,u,\mathrm{D}u\Big),\\ \mathrm{K}_\eta[u] \,&:= \mathrm{K}_\eta\Big(\cdot,\cdot,u,\mathrm{D}u\Big),\\ \mathrm{K}_A[u] \,&:= \mathrm{K}_A\Big(\cdot,\cdot,u,\mathrm{D}u\Big),\end{aligned}\right. \tag{2.24}$$

for K and its partial derivatives $\mathrm{K}_\eta,\mathrm{K}_A$ with respect to the arguments for $u,\mathrm{D}u$ respectively, all of which will also need to be assumed to be continuous. We note that, when $p=\infty$, there is no direct analogue of the divergence structure Euler-Lagrange equations. Instead, one of the central points of Calculus of Variations in L^∞ is that Aronsson-Euler PDE systems may be derived, under appropriate (stringent) assumptions. Even in the unconstrained case, these PDE systems are always non-divergence and even fully nonlinear and with discontinuous coefficients (see e.g. [12, 13, 35, 63, 70]). The case of L^∞ problems involving only first order derivative of scalar-valued functions is nowadays a well established field which originated from the work of Aronsson in the 1960s [4, 5], today largely interconnected to the theory of Viscosity Solutions to nonlinear elliptic PDE (for a general pedagogical introduction see e.g. [34, 61]). However, vectorial and higher L^∞ variational problems involving constraints, have only recently been explored (see [65, 66], but also the relevant earlier contributions [10, 11, 15]). For several interesting developments on L^∞ variational problems we refer the interested reader to [14, 16, 19, 20, 27, 39, 49, 76, 80, 81, 84].

In this paper, motivated by recent progress on higher order and on constrained L^∞ variational problems made in [68] by the second author jointly with Moser and by the second author in [65, 66] (inspired by earlier contributions by Moser and Schwetlick deployed in a geometric setting in [79]), we follow a slightly different approach which does not lead an Aronsson-Euler type system; instead, it leads to a *divergence structure* PDE system. However, there is a toll to be paid, as the divergence PDEs arising as necessary conditions involve measures as auxiliary parameters whose determination becomes part of the problem. Notwithstanding, the central point of this idea is to use a scaling in the

Euler-Lagrange equations before letting $p \to \infty$, which is different from the scaling used to (formally) derive the Aronsson-Euler equations as $p \to \infty$.

In the light of the above comments, our final main result concerns the satisfaction of necessary PDE conditions for the PDE-constrained minimisers in L^∞ constructed in Theorem 2.1.1, and reads as follows.

Theorem 2.1.3 (Variational Equations in L^∞)**.** *Suppose that* (2.5) *and* (2.15) *hold true, and that additionally* K *does not depend on* $(\partial_t u, \mathrm{p})$ *with* $\mathrm{K}, \mathrm{K}_\eta, \mathrm{K}_A$ *in* (2.24) *being continuous on* $\overline{\Omega_T} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$*. Then, there exists a linear functional*

$$\Psi_\infty \in \bigcap_{r>n+2} \Big(W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega;\mathbb{R}^n) \Big)^* \tag{2.25}$$

which is a Lagrange multiplier associated with the constrained minimisation problem (2.16) *for* $p=\infty$*. There also exist vector measures*

$$\Sigma_\infty \in \mathcal{M}\Big(\overline{\Omega_T};\mathbb{R}^N\Big), \quad \sigma_\infty \in \mathcal{M}\Big(\overline{\Omega_T};\mathbb{R}^n\Big) \tag{2.26}$$

such that the minimising triplet $(u_\infty, \mathrm{p}_\infty, y_\infty) \in \mathfrak{X}^\infty(\Omega_T)$ *satisfies the relations*

$$\left\{ \begin{aligned} & (1-\lambda)\int_{\overline{\Omega_T}} \Big(\mathrm{K}_\eta[u_\infty]\cdot u \,+\, \mathrm{K}_A[u_\infty] : \mathrm{D}u\Big)\cdot \mathrm{d}\Sigma_\infty \\ & = -\lambda \int_{\overline{\Omega_T}} \Big(\partial_t u - \nu\Delta u + (u\cdot \mathrm{D})u_\infty + (u_\infty\cdot\mathrm{D})u\Big)\cdot \mathrm{d}\sigma_\infty \,+\, \Big\langle \Psi_\infty, u(\cdot,0)\Big\rangle, \end{aligned} \right. \tag{2.27}$$

$$\int_{\overline{\Omega_T}} \mathrm{Dp}\cdot \mathrm{d}\sigma_\infty \,=\, 0, \tag{2.28}$$

for all test mappings

$$(u,\mathrm{p}) \,\in\, \Big(W^{2,1;\infty}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n)\cap C^2\Big(\overline{\Omega_T};\mathbb{R}^n\Big)\Big) \times \Big(W^{1,0;\infty}_{\sharp}(\Omega_T)\cap C^1\Big(\overline{\Omega_T}\Big)\Big).$$

Further, the multiplier Ψ_∞ *and the measures* $\Sigma_\infty, \sigma_\infty$ *can be approximated as follows:*

$$\left\{ \begin{aligned} & \Psi_p \xrightharpoonup{*} \Psi_\infty, \quad \text{in } \Big(W^{2-2/r,r}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big)^*, \text{ for all } r>n+2, \\ & \Sigma_p \xrightharpoonup{*} \Sigma_\infty, \quad \text{in } \mathcal{M}\Big(\overline{\Omega_T};\mathbb{R}^N\Big), \\ & \sigma_p \xrightharpoonup{*} \sigma_\infty, \quad \text{in } \mathcal{M}\Big(\overline{\Omega_T};\mathbb{R}^n\Big), \end{aligned} \right. \tag{2.29}$$

along a subsequence $p_j \to \infty$*, where*

$$\left\{ \begin{aligned} & \Sigma_p \,:=\, \mathfrak{M}_p\Big(\mathrm{K}[u_p]\Big)\mathcal{L}^{n+1}\llcorner_{\Omega_T}, \\ & \sigma_p \,:=\, \mathfrak{M}_p(y_p)\mathcal{L}^{n+1}\llcorner_{\Omega_T}. \end{aligned} \right. \tag{2.30}$$

Finally, Σ_∞ concentrates on the set whereon $|\mathrm{K}[u_\infty]|$ is maximised over $\overline{\Omega_T}$

$$\Sigma_\infty\left(\left\{\left|\mathrm{K}[u_\infty]\right| < \left\|\mathrm{K}[u_\infty]\right\|_{L^\infty(\Omega_T)}\right\}\right) = 0, \tag{2.31}$$

and σ_∞ asymptotically concentrates on the set whereon $|y_\infty|$ is approximately maximised over $\overline{\Omega_T}$, in the sense that for any $\varepsilon > 0$ small,

$$\lim_{p\to\infty} \sigma_p\left(\left\{|y_p| < \|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\right\}\right) = 0. \tag{2.32}$$

Even though the weak interpretation of the equations (2.22)-(2.23) is relatively obvious, this is not the case for (2.27)-(2.28) despite having a simpler form. The reason is that the limiting measures $(\Sigma_\infty, \sigma_\infty)$ are not product measures on $\overline{\Omega_T} = \overline{\Omega} \times [0, T]$ in order to use the Fubini theorem, therefore due to the temporal dependence, (2.28) cannot be simply interpreted as "$\mathrm{div}(\sigma_\infty) = 0$". Similar arguments can be made for (2.27) as well. Since this point is not utilised any further in this paper, we only provide a brief discussion in the next section.

We conclude this introduction with some remarks regarding the organisation of this paper. This introduction is followed by Section 2.2, in which we discuss some preliminaries and also establish some basic estimates which are utilised subsequently to establish our main results. In Section 2.3 we prove Theorem 2.1.1 by establishing the existence of constrained minimisers for all p including $p = \infty$, as well as the convergence of minimiser of the former problems to those of the latter. In Section 2.4 we prove Theorem 2.1.2, deriving the necessary PDE conditions which constrained minimisers in L^p satisfy. Finally, in Section 2.5 prove Theorem 2.1.3, deriving the necessary PDE conditions that constrained minimisers in L^∞ satisfy, as well as the additional properties that the measures arising in these PDEs satisfy. A key ingredient here is that we establish appropriate weak* compactness for the Lagrange multipliers arising in the L^p problems in order to pass to the limit as $p \to \infty$.

2.2 Preliminaries and the main estimates

We begin by recording for later use the following modified Hölder inequality for the dotted $\dot{L}^p$ regularised "norms" defined in (2.7): for any $1 \leq q \leq p < \infty$ and $h \in L^p(\Omega_T)$, we have the inequality

$$\|h\|_{\dot{L}^q(\Omega_T)} \leq \|h\|_{\dot{L}^p(\Omega_T)} + \sqrt{q^{-2} - p^{-2}},$$

which can be very easily confirmed by a direct computation. Next, we continue with a brief discussion regarding the weak interpretation of the equations (2.27)-(2.28). As already noted in the introduction, since $(\Sigma_\infty, \sigma_\infty)$ are generally neither product measures or absolutely continuous with respect to the $(n+1)$-Lebesgue measure on $\overline{\Omega_T} = \overline{\Omega} \times [0,T]$, one needs to use the disintegration "slicing" theorem for Young measures in order to express them appropriately, as follows. Since σ_∞ is a vector measure in $\mathcal{M}(\overline{\Omega_T};\mathbb{R}^n)$, by the Radon-Nikodym theorem, we may decompose

$$\sigma_\infty \,=\, \frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}\,\|\sigma_\infty\|,$$

where $\|\sigma_\infty\| \in \mathcal{M}(\overline{\Omega_T})$ is the scalar total variation measure and $\mathrm{d}\sigma_\infty/\mathrm{d}\|\sigma_\infty\|$ is the vector-valued Radon-Nikodym derivative of σ_∞ with respect to $\|\sigma_\infty\|$. Fix any $h \in L^1(\overline{\Omega_T}, \|\sigma_\infty\|)$. By the disintegration "slicing" theorem for Young measures (see se.g. [44, Theorem 3.2, p. 179]), we have the representation formula

$$\int_{\overline{\Omega_T}} h\,\mathrm{d}\|\sigma_\infty\| \,=\, \int_{[0,T]}\left(\int_{\overline{\Omega}} h(x,t)\,\mathrm{d}\|\sigma_\infty\|_t(x)\right)\mathrm{d}\|\sigma_\infty\|^o(t)$$

where the measure $\|\sigma_\infty\|^o \in \mathcal{M}([0,T])$ and the family of measures $(\|\sigma_\infty\|_t)_{t\in[0,T]} \subseteq \mathcal{M}(\overline{\Omega})$ are defined as follows:

$$\|\sigma_\infty\|^o \,:=\, \|\sigma_\infty\|\Big(\overline{\Omega}\times\cdot\Big), \qquad \|\sigma_\infty\|_t(A) \,:=\, \frac{\mathrm{d}\|\sigma_\infty\|\Big(A\times\cdot\Big)}{\mathrm{d}\|\sigma_\infty\|\Big(\overline{\Omega}\times\cdot\Big)}(t), \ \text{ for } A \subseteq \overline{\Omega} \text{ Borel}.$$

Namely, $\|\sigma_\infty\|^o$ is one of the marginals of σ_∞ and for $\|\sigma_\infty\|^o$-a.e. $t \in [0,T]$, the measure $\|\sigma_\infty\|_t$ evaluated at A is defined as the Radon-Nikodym derivative of the measure $\|\sigma_\infty\|\Big(A\times\cdot\Big)$ with respect to $\|\sigma_\infty\|\Big(\overline{\Omega}\times\cdot\Big)$ at the point $t \in [0,T]$. Then, in view of (2.28), by choosing p in the form $\mathrm{p}(x,t) = \pi(x)\tau(t)$, we have

$$\begin{aligned}
0 \,&=\, \int_{\overline{\Omega_T}} \mathrm{Dp}\cdot\mathrm{d}\sigma_\infty \\
&=\, \int_{\overline{\Omega_T}} \left(\mathrm{Dp}\cdot\frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}\right)\mathrm{d}\|\sigma_\infty\| \\
&=\, \int_{[0,T]}\left(\int_{\overline{\Omega}}\left(\mathrm{Dp}\cdot\frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}\right)(x,t)\,\mathrm{d}\|\sigma_\infty\|_t(x)\right)\mathrm{d}\|\sigma_\infty\|^o(t) \\
&=\, \int_{[0,T]}\left(\int_{\overline{\Omega}}\left(\mathrm{D}\pi(x)\cdot\frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}(x,t)\right)\mathrm{d}\|\sigma_\infty\|_t(x)\right)\tau(t)\,\mathrm{d}\|\sigma_\infty\|^o(t).
\end{aligned}$$

The arbitrariness of τ implies that for $\|\sigma_\infty\|^o$-a.e. $t \in [0,T]$, we have

$$\int_{\overline{\Omega}} \left(\mathrm{D}\pi(x) \cdot \frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}(x,t) \right) \mathrm{d}\|\sigma_\infty\|_t(x) \;=\; 0.$$

When restricting our attention to those test function for which $\pi|_{\partial\Omega} \equiv 0$, we obtain the next weak interpretation of (2.28):

$$\operatorname{div}\left(\frac{\mathrm{d}\sigma_\infty}{\mathrm{d}\|\sigma_\infty\|}(\cdot,t)\, \|\sigma_\infty\|_t \right) \;=\; 0, \qquad \text{in } \Omega,$$

for $\|\sigma_\infty\|^o$-a.e. $t \in [0,T]$. Similar considerations apply also to equation (2.27), but the arguments are considerably more complicated.

Next we prove a general compact embedding lemma by means of interpolation theory.

Lemma 2.2.1. *Suppose that $p > n+2$. Then, there exists $\alpha \in (0,1)$ such that the space*

$$W^{2,1;p}(\Omega_T) \;:=\; L^p\Big((0,T); W^{2,p}(\Omega)\Big) \bigcap W^{1,p}\Big((0,T); L^p(\Omega)\Big)$$

is compactly embedded in the space $C^{0,\alpha}\Big([0,T]; C^{1,\alpha}(\overline{\Omega})\Big)$.

Proof of Lemma 2.2.1. Let us use the abbreviated space notation

$$\mathrm{X}_1 \;:=\; W^{2,p}(\Omega), \quad \mathrm{X}_0 \;:=\; L^p(\Omega)$$

and select θ such that

$$\frac{p+n}{2p} \;<\; \theta \;<\; \frac{p-1}{p},$$

which is possible since

$$\frac{p-1}{p} - \frac{p+n}{2p} \;=\; \frac{p-(n+2)}{2p} \;>\; 0.$$

Since $1-\theta > 1/p$, direct application of the interpolation result in [2, Theorem 5.2] for the exponents $s_0 := 1$, $s_1 := 0$ and $p_0 \equiv p_1 := p$ yields that space $W^{2,1;p}(\Omega_T)$ is compactly embedded in the space $C^{0,\alpha}\Big([0,T]; \mathrm{X}\Big)$, where $0 < \alpha < 1-\theta-1/p$ and $\mathrm{X} = (\mathrm{X}_0, \mathrm{X}_1)_{\theta,p}$ symbolises the real interpolation between the Banach spaces X_0 and X_1. Now it remains to identify the space X. By using standard results in interpolation theory (see e.g. [92, Theorem 4.3.1.1 and formula (2.4.2/9)] or [93] for Lipschitz domains) we get:

$$\Big(L^p(\Omega), W^{2,p}(\Omega)\Big)_{\theta,p} \;=\; \mathrm{B}^{2\theta}_{pp}(\Omega) \;=\; W^{2\theta,p}(\Omega).$$

Since $2\theta > 1+n/p$, by the standard Sobolev embedding theorem for fractional spaces (e.g. [38, Theorem 8.2], we have that $W^{2\theta,p}(\Omega)$ is continuously embedded in the space $C^{1,\alpha}(\overline{\Omega})$, where $0<\alpha\leq 2\theta-1-n/p$. The conclusion ensues. $\square$

Remark 2.2.2. Let us now record for later use the following simple inclusion of space (which is in fact a continuous embedding):

$$C^{0,\alpha}\Big([0,T];C^{0,\alpha}(\overline{\Omega})\Big)\ \subseteq\ C^{0,\alpha}\Big(\overline{\Omega_T}\Big).$$

Indeed, for any $h\in C^{0,\alpha}\Big([0,T];C^{0,\alpha}(\overline{\Omega})\Big)$, we compute

$$\begin{aligned}\Big|h(t_1,x_1)-h(t_2,x_2)\Big|\ &\leq\ |h(t_1,x_1)-h(t_2,x_1)|+|h(t_2,x_1)-h(t_2,x_2)|\\ &\leq\ \|h(t_1,\cdot)-h(t_2,\cdot)\|_{C(\overline{\Omega})}\ +\ \|h(t_2,\cdot)\|_{C^{0,\alpha}(\overline{\Omega})}|x_1-x_2|^\alpha\\ &\leq\ \Big(|t_1-t_2|^\alpha+|x_1-x_2|^\alpha\Big)\|h\|_{C^{0,\alpha}([0,T];C^{0,\alpha}(\overline{\Omega}))}\end{aligned}$$

which establishes the claim.

Lemma 2.2.3. *Suppose that assumptions (2.5) and (2.15) are satisfied. We have that*

$$\mathfrak{X}^\infty(\Omega_T)\ \neq\ \emptyset$$

(and consequently we have $\mathfrak{X}^p(\Omega_T)\neq\emptyset$ for all $p>1$). Further, if $(u,\mathrm{p},y)\in\mathfrak{X}^p(\Omega_T)$ for some $p>1$ which satisfies

$$\mathrm{E}_p(u,\mathrm{p},y)\ \leq\ M$$

for some $M>0$, then for any $q\leq p$ there exists $C(q,M)>0$ such that

$$\|u\|_{W^{2,1;q}_{\mathrm{L},\sigma}(\Omega_T)}+\ \|\mathrm{p}\|_{W^{1,0;q}_{\sharp}(\Omega_T)}+\ \|y\|_{L^q(\Omega_T)}\ \leq\ C(q,M).$$

Further, if $p>n+2$ and $q\in(n+2,p]$, then there exists $\alpha\in(0,1)$ and a constant $C(M,q)>0$ such that additionally

$$\|u\|_{C^{0,\alpha}(\Omega_T)}+\ \|\mathrm{D}u\|_{C^{0,\alpha}(\Omega_T)}\ \leq\ C(q,M).$$

We note that the constants above also depend on $n,\partial\Omega,T,f,u_0,\lambda$, but as all these are fixed throughout this paper, we suppress denoting the explicit dependence on them.

Proof of Lemma 2.2.3. By assumptions (2.5)(b)-(2.5)(c), we have that the triplet $(u_0,0,y_0)$, where

$$y_0\ :=\ -\nu\Delta u_0\ +\ (u_0\cdot\mathrm{D})u_0\ -\ f$$

satisfies that $(u_0, 0, y_0) \in \mathfrak{X}^\infty(\Omega_T)$, and in fact lies also in the smaller space

$$W^{2,1;\infty}_{\mathrm{L},\sigma}(\Omega_T) \times W^{1,0;\infty}_{\sharp}(\Omega_T) \times L^\infty(\Omega_T; \mathbb{R}^n).$$

Next, if $(u, \mathrm{p}, y) \in \mathfrak{X}^p(\Omega_T)$ with $\mathrm{E}_p(u, \mathrm{p}, y) \leq M$, then we readily have that

$$\|y\|_{L^q(\Omega_T)} \leq \|y\|_{\dot{L}^p(\Omega_T)} \leq \frac{M}{\lambda},$$

whilst by assumptions (2.15) and (2.5)(c) we have that

$$\|u\|_{W^{2,1;q}_{\mathrm{L},\sigma}(\Omega_T)} + \|\mathrm{p}\|_{W^{1,0;q}_{\sharp}(\Omega_T)} \leq C(q)\Big(1 + \frac{M}{\lambda}\Big).$$

for some q-dependent constant $C(q)$, for any $q \in (n, p]$. Further, suppose $p > n + 2$ and $n + 2 < q \leq p$. Then, the above estimate in particular implies

$$\|u\|_{L^q(\Omega_T)} + \|\nabla u\|_{L^q(\Omega_T)} \leq C(q, M),$$

whereat application of the Morrey imbedding theorem yields

$$\|u\|_{C^{0,\alpha'}(\Omega_T)} \leq C(q, M),$$

for a new constant $C(q, M)$ and some $\alpha' \in (0, 1)$. Next, by Lemma 2.2.1, Remark 2.2.2 and the established estimate for $q > n + 2$, we have

$$C(q)\|\mathrm{D}u\|_{C^{0,\alpha''}(\Omega_T)} \leq \|u\|_{W^{2,1;q}(\Omega_T)} \leq C(q, M),$$

for some $\alpha'' \in (0, 1)$ and some constant $C(q) > 0$. By choosing $\alpha := \min\{\alpha', \alpha''\}$, the conclusion ensues. $\square$

2.3 Minimisers of L^p problems and convergence as $p \to \infty$

In this section we establish Theorem 2.1.1, by utilising the results of Section 2.2.

Proof of Theorem 2.1.1. Fix $p \in (n + 2, \infty)$. By Lemma 2.2.1, $\mathfrak{X}^p(\Omega_T) \neq \emptyset$, therefore $0 \leq \inf_{\mathfrak{X}^p(\Omega_T)} \mathrm{E}_p < \infty$. By Lemma 2.2.3, it follows that $\mathfrak{X}^p(\Omega_T)$ is sequentially weakly compact. Note now that $y \mapsto \|y\|^p_{\dot{L}^p(\Omega_T)}$ is trivially convex, and by the identity

$$\Big\|\mathrm{K}\big(\cdot, u, \mathrm{D}u, \partial_t u, \mathrm{p}\big)\Big\|^p_{\dot{L}^p(\Omega_T)} = \int_{\Omega_T} \Big(\big|\mathrm{K}\big(\cdot, u, \mathrm{D}u, \partial_t u, \mathrm{p}\big)\big|^2 + p^{-2}\Big)^{\frac{p}{2}} \mathrm{d}\mathcal{L}^{n+1},$$

assumption (2.5)(f) yields that

$$(\partial_t u, \mathrm{p}) \,\mapsto\, \Big\|\mathrm{K}\Big(\cdot, u, \mathrm{D}u, \partial_t u, \mathrm{p}\Big)\Big\|^p_{\dot{L}^p(\Omega_T)}$$

is also convex. By standard results in the Calculus of Variations (see e.g. [36]) it follows that E_p is weakly lower semicontinuous in $\mathcal{W}^p(\Omega_T)$. Since the convex combination of p-th roots of two weakly lower semicontinuous functionals is indeed a weakly lower semicontinuous functional. By the bounds obtained in Lemma 2.2.3, it follows that $\mathfrak{X}^p(\Omega_T)$ is weakly closed in $\mathcal{W}^p(\Omega_T)$. Furthermore, E_p is weakly lower semicontinuous in $\mathfrak{X}^p(\Omega_T)$. Hence, E_p attains its infimum at some $(u_p, \mathrm{p}_p, y_p) \in \mathfrak{X}^p(\Omega_T)$.

Consider now the family of minimisers $(u_p, \mathrm{p}_p, y_p)_{p>n+2}$. For any $(u, \mathrm{p}, y) \in \mathfrak{X}^\infty(\Omega_T)$ and any $q \leq p$, minimality and the Hölder inequality for the dotted $\dot{L}^p$ functionals yield

$$\mathrm{E}_p(u_p, \mathrm{p}_p, y_p) \,\leq\, \mathrm{E}_p(u, \mathrm{p}, y) \,\leq\, \mathrm{E}_\infty(u, \mathrm{p}, y) \,+\, p^{-1}.$$

By choosing $(u, \mathrm{p}, y) = (u_0, 0, y_0)$, by Lemma 2.2.3 and a standard diagonal argument, we have that the family of minimisers is weakly precompact in $\mathcal{W}^q(\Omega_T)$ for all $q \in (n+2, \infty)$. Further, by Lemma 2.2.1 and Remark 2.2.2, $W^{2,1;q}_{\mathrm{L},\sigma}(\Omega_T; \mathbb{R}^n)$ is compactly embedded in $C^{0,\alpha}\Big([0,T]; C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^n)\Big)$. Hence, for any sequence of indices $p_j \to \infty$, there exists $(u_\infty, \mathrm{p}_\infty, y_\infty) \in \cap_{q\in(n+2,\infty)} \mathcal{W}^q(\Omega_T)$ and a subsequence denoted again as $(p_j)_1^\infty$ such that (2.17) holds true. Additionally, due to these modes of convergence, it follows that $(u_\infty, \mathrm{p}_\infty, y_\infty)$ solves (2.1), therefore in fact $(u_\infty, \mathrm{p}_\infty, y_\infty) \in \mathfrak{X}^\infty(\Omega_T)$. Again now by minimality and the Hölder inequality for the dotted $\dot{L}^p$ functionals, for any $(u, \mathrm{p}, y) \in \mathfrak{X}^\infty(\Omega_T)$ we have

$$\mathrm{E}_q(u_p, \mathrm{p}_p, y_p) - \sqrt{q^{-2} - p^{-2}} \,\leq\, \mathrm{E}_p(u_p, \mathrm{p}_p, y_p) \,\leq\, \mathrm{E}_\infty(u, \mathrm{p}, y) \,+\, p^{-1}.$$

Since as we have already shown, E_q is weakly lower semicontinuous in $\mathfrak{X}^q(\Omega_T)$, by letting $p \to \infty$ along the subsequence in the above inequality yields

$$\begin{aligned}
\mathrm{E}_q(u_\infty, \mathrm{p}_\infty, y_\infty) - \sqrt{q^{-2}} \,&\leq\, \liminf_{p_j \to \infty} \mathrm{E}_p(u_p, \mathrm{p}_p, y_p) \\
&\leq\, \limsup_{p_j \to \infty} \mathrm{E}_p(u_p, \mathrm{p}_p, y_p) \\
&\leq\, \mathrm{E}_\infty(u, \mathrm{p}, y).
\end{aligned}$$

By further letting $q \to \infty$, we obtain

$$\begin{aligned}
\mathrm{E}_\infty(u_\infty, \mathrm{p}_\infty, y_\infty) \,&\leq\, \liminf_{p_j \to \infty} \mathrm{E}_p(u_p, \mathrm{p}_p, y_p) \\
&\leq\, \limsup_{p_j \to \infty} \mathrm{E}_p(u_p, \mathrm{p}_p, y_p) \\
&\leq\, \mathrm{E}_\infty(u, \mathrm{p}, y),
\end{aligned}$$

for any $(u,\mathrm{p},y) \in \mathfrak{X}^\infty(\Omega_T)$. The above inequality establishes on the one hand that $(u_\infty,\mathrm{p}_\infty,y_\infty)$ minimises E_∞ over $\mathfrak{X}^\infty(\Omega_T)$, and on the other hand by choosing $(u,\mathrm{p},y) := (u_\infty,\mathrm{p}_\infty,y_\infty)$ that (2.18) holds true. Hence, Theorem 2.1.1 has been established. $\square$

2.4 The equations for L^p PDE-constrained minimisers

In this section we establish the proof of Theorem 2.1.2. We begin with some preparation. Firstly, it will be convenient to consider the functional E_p of (2.6) as being defined in the wider Banach space $\mathcal{W}^p(\Omega_T)$ defined in (2.9):

$$\mathrm{E}_p \ : \ \mathcal{W}^p(\Omega_T) \longrightarrow \mathbb{R}.$$

Next, we introduce a mapping on $\mathcal{W}^p(\Omega_T)$ which incorporates the PDE constraint (2.1) appearing in (2.8) as follows. We define

$$\mathrm{G} = \left[\begin{array}{c} \mathrm{G}_1 \\ \mathrm{G}_2 \end{array}\right] \ : \ \mathcal{W}^p(\Omega_T) \longrightarrow L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)$$

by setting

$$\left\{ \begin{array}{l} \mathrm{G}_1(u,\mathrm{p},y) \ := \ \partial_t u \ - \nu\Delta u \ + (u\cdot \mathrm{D})u \ + \mathrm{Dp} \ - (y+f), \\ \mathrm{G}_2(u,\mathrm{p},y) \ := \ u(\cdot,0) - u_0. \end{array}\right.$$

Then, we may express (2.8) as

$$\mathfrak{X}^p(\Omega_T) \ = \ \mathcal{W}^p(\Omega_T) \cap \{\mathrm{G}=0\}.$$

We are now ready to prove our second main result.

Proof of Theorem 2.1.2. By assumption (2.5), for any $p \in (n+2,\infty)$ the functional $\mathrm{E}_p : \mathcal{W}^p(\Omega_T) \longrightarrow \mathbb{R}$ is Frechét differentiable and its derivative

$$\begin{aligned} \mathrm{dE}_p \ &: \ \mathcal{W}^p(\Omega_T) \longrightarrow \Big(\mathcal{W}^p(\Omega_T)\Big)^*, \\ (\mathrm{dE}_p)_{(\bar u,\bar{\mathrm{p}},\bar y)}(u,\mathrm{p},y) \ &= \ \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathrm{E}_p\Big(\bar u + \varepsilon u, \bar{\mathrm{p}}+\varepsilon \mathrm{p}, \bar y + \varepsilon y\Big) \end{aligned}$$

can be easily computed and is given by the formula

$$\begin{aligned} (\mathrm{dE}_p)_{(\bar u,\bar{\mathrm{p}},\bar y)}(u,\mathrm{p},y) \ = \ p(1-\lambda) & -\!\!\!\!\!\!\int_{\Omega_T}\Big(\mathrm{K}_\eta[\bar u,\bar{\mathrm{p}}]\cdot u \ + \ \mathrm{K}_{(A,a)}[\bar u,\bar{\mathrm{p}}] : \nabla u + \ \mathrm{K}_r[\bar u,\bar{\mathrm{p}}]\,\mathrm{p}\Big)\cdot \\ & \cdot \mathfrak{M}_p\Big(\mathrm{K}[\bar u,\bar{\mathrm{p}}]\Big)\,\mathrm{d}\mathcal{L}^{n+1} \ + \ p\lambda -\!\!\!\!\!\!\int_{\Omega_T} \mathfrak{M}_p(\bar y)\cdot y\,\mathrm{d}\mathcal{L}^{n+1}, \end{aligned}$$

where the operator $\mathfrak{M}_p : L^p(\Omega_T;\mathbb{R}^M) \longrightarrow L^{p'}(\Omega_T;\mathbb{R}^M)$ (for $M \in \{N, n\}$) is given by (2.19) and we have used the notation introduced in (2.20). Next, we note that the mapping G which incorporates the PDE constraint is also Fréchet differentiable and it can be easily confirmed that its derivative

$$\begin{aligned} \mathrm{dG} \ &: \ \mathcal{W}^p(\Omega_T) \longrightarrow \mathcal{B}\Big(\mathcal{W}^p(\Omega_T), L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big), \\ (\mathrm{dG})_{(\bar u,\bar{\mathrm{p}},\bar y)}(u,\mathrm{p},y) \ &= \ \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathrm{G}\Big(\bar u + \varepsilon u, \bar{\mathrm{p}} + \varepsilon \mathrm{p}, \bar y + \varepsilon y\Big) \end{aligned}$$

is given by the formula

$$\Big(\mathrm{dG}\Big)_{(\bar u,\bar{\mathrm{p}},\bar y)}(u,\mathrm{p},y) \ = \ \left[\begin{array}{c} \partial_t u - \nu\Delta u + (u\cdot \mathrm{D})\bar u + (\bar u \cdot \mathrm{D})u + \mathrm{Dp} - y \\ u(\cdot,0) \end{array} \right].$$

We now claim that the differential

$$(\mathrm{dG})_{(\bar u,\bar{\mathrm{p}},\bar y)} \ : \ \mathcal{W}^p(\Omega_T) \longrightarrow L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n)$$

is a surjective map, for any $(\bar u, \bar{\mathrm{p}}, \bar y) \in \mathcal{W}^p(\Omega)$. This is equivalent to the statement that for any $p > n+2$, the linearised Navier-Stokes problem

$$\left\{ \begin{array}{rll} \partial_t u - \nu\Delta u + (u\cdot \mathrm{D})\bar u + (\bar u \cdot \mathrm{D})u + \mathrm{Dp} \ = & F, & \text{in } \Omega_T, \\ \operatorname{div} u \ = & 0, & \text{in } \Omega_T, \\ u(\cdot,0) \ = & v, & \text{on } \Omega, \\ u \ = & 0, & \text{on } \partial\Omega \times (0,T), \end{array} \right.$$

has a solution $(u,\mathrm{p}) \in W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n) \times W^{1,0;p}_{\sharp}(\Omega_T)$, for any $\bar u \in W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T;\mathbb{R}^n)$ and any data

$$(F,v) \ \in \ L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n).$$

This is indeed the case, and it is a consequence of a classical result of Solonnikov [87, Th. 4.2] for $n = 3$ and of Giga-Sohr [54, Th. 2.8] for $n > 3$, as a perturbation of the Stokes problem. As a consequence, the assumptions of the generalised Kuhn-Tucker theorem hold true (see e.g. Zeidler [94, Cor. 48.10 & Th. 48B]). Hence, there exists a Lagrange multiplier

$$\Lambda_p \ \in \ \Big(L^p(\Omega_T;\mathbb{R}^n) \times W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega_T;\mathbb{R}^n)\Big)^*$$

such that

$$\Big(\mathrm{dE}_p\Big)_{(u_p,\mathrm{p}_p,y_p)}(u,\mathrm{p},y) \ = \ \Big\langle \Big(\mathrm{dG}\Big)_{(u_p,\mathrm{p}_p,y_p)}(u,\mathrm{p},y),\, \Lambda_p \Big\rangle,$$

for any $(u, \mathrm{p}, y) \in \mathcal{W}^p(\Omega)$. By standard duality arguments, the Riesz representation theorem and by taking into account the form of the differentials dE_p and dG, we may identify Λ_p with a pair of Lagrange multipliers

$$(\phi_p, \Psi_p) \,\in\, L^{p'}(\Omega_T;\mathbb{R}^n) \times \Big(W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega_T;\mathbb{R}^n)\Big)^*$$

such that, the constrained minimiser $\big(u_p, p_p, y_p\big) \in \mathfrak{X}^p(\Omega_T)$ satisfies the equation

$$\begin{aligned}
(1-\lambda)\int_{\Omega_T}&\Big(\mathrm{K}_\eta[u_p,\mathrm{p}_p]\cdot u \,+\, \mathrm{K}_{(A,a)}[u_p,\mathrm{p}_p] : \nabla u + \,\mathrm{K}_r[u_p,\mathrm{p}_p]\,\mathrm{p}\Big)\\
&\qquad\cdot\,\mathfrak{M}_p\Big(\mathrm{K}[u_p,\mathrm{p}_p]\Big)\,\mathrm{d}\mathcal{L}^{n+1} \;+\; \lambda\int_{\Omega_T}\mathfrak{M}_p(y_p)\cdot y\,\mathrm{d}\mathcal{L}^{n+1}\\
=\int_{\Omega_T}&\Big(\partial_t u - \nu\Delta u + (u\cdot\mathrm{D})u_p + (u_p\cdot\mathrm{D})u + \mathrm{Dp} - y\Big)\cdot\phi_p\,\mathrm{d}\mathcal{L}^{n+1} \;+\; \langle\Psi_p, u(\cdot,0)\rangle,
\end{aligned}$$

for any $(u, \mathrm{p}, y) \in \mathcal{W}^p(\Omega_T)$. We note that here we have tacitly rescaled (ϕ_p, Ψ_p) by multiplying them with the factor $p(\mathcal{L}^{n+1}(\Omega_T))^{-1}$, in order to remove the averages arising from E_p on the left hand side and to be able to obtain non-trivial limits as $p\to\infty$ of the multipliers themselves later on. By using linear independence, the above equation actually decouples to the triplet of relations

$$\left\{\begin{aligned}
&(1-\lambda)\int_{\Omega_T}\Big(\mathrm{K}_\eta[u_p,\mathrm{p}_p]\cdot u \,+\, \mathrm{K}_{(A,a)}[u_p,\mathrm{p}_p] : \nabla u\Big)\cdot\mathfrak{M}_p\Big(\mathrm{K}[u_p,\mathrm{p}_p]\Big)\,\mathrm{d}\mathcal{L}^{n+1}\\
&\ =\int_{\Omega_T}\Big(\partial_t u - \nu\Delta u + (u\cdot\mathrm{D})u_p + (u_p\cdot\mathrm{D})u\Big)\cdot\phi_p\,\mathrm{d}\mathcal{L}^{n+1} \;+\; \langle\Psi_p, u(\cdot,0)\rangle,
\end{aligned}\right.$$

$$(1-\lambda)\int_{\Omega_T}\Big(\mathrm{K}_r[u_p,\mathrm{p}_p]\,\mathrm{p}\Big)\cdot\mathfrak{M}_p\Big(\mathrm{K}[u_p,\mathrm{p}_p]\Big)\,\mathrm{d}\mathcal{L}^{n+1} \;=\; \int_{\Omega_T}\mathrm{Dp}\cdot\phi_p\,\mathrm{d}\mathcal{L}^{n+1},$$

$$\lambda\int_{\Omega_T}\mathfrak{M}_p(y_p)\cdot y\,\mathrm{d}\mathcal{L}^{n+1} \;=\; -\int_{\Omega_T} y\cdot\phi_p\,\mathrm{d}\mathcal{L}^{n+1}.$$

The arbitrariness of $y \in L^p(\Omega_T;\mathbb{R}^n)$ in the third equation readily yields that the multiplier ϕ_p equals

$$\phi_p \;=\; -\lambda\,\mathfrak{M}_p(y_p).$$

By substituting this into the first two equations, we see that the theorem has been established. $\square$

2.5 The equations for L^∞ PDE-constrained minimisers

Proof of Theorem 2.1.3. By Theorem 2.1.2 it follows that for any $p \in (n+2, \infty)$, the minimising triplet $(u_p, \mathrm{p}_p, y_p) \in \mathfrak{X}^p(\Omega_T)$ satisfies

$$\left\{ \begin{array}{l} (1-\lambda) \displaystyle\int_{\Omega_T} \Big(\mathrm{K}_\eta[u_p] \cdot u \, + \, \mathrm{K}_A[u_p] : \mathrm{D}u \Big) \cdot \mathfrak{M}_p\Big(\mathrm{K}[u_p]\Big) \, \mathrm{d}\mathcal{L}^{n+1} \\ \; = -\lambda \displaystyle\int_{\Omega_T} \Big(\partial_t u - \nu \Delta u + (u \cdot \mathrm{D}) u_p + (u_p \cdot \mathrm{D}) u \Big) \cdot \mathfrak{M}_p(y_p) \, \mathrm{d}\mathcal{L}^{n+1} \, + \, \Big\langle \Psi_p, u(\cdot, 0) \Big\rangle, \end{array} \right.$$

and also

$$\int_{\Omega_T} \mathrm{D}\mathrm{p} \cdot \mathfrak{M}_p(y_p) \, \mathrm{d}\mathcal{L}^{n+1} \, = \, 0,$$

for all test mappings $(u, \mathrm{p}) \in W^{2,1;p}_{\mathrm{L},\sigma}(\Omega_T; \mathbb{R}^n) \times W^{1,0;p}_{\sharp}(\Omega_T)$. The first goal is to pass to the limit as $p \to \infty$ in these equations in order to obtain (2.27)-(2.28). Since by (2.19) we readily have that $\mathfrak{M}_p(y_p)$ and $\mathfrak{M}_p\Big(\mathrm{K}[u_p]\Big)$ are valued in the unit balls of $L^{p'}(\Omega_T; \mathbb{R}^n)$ and of $L^{p'}(\Omega_T; \mathbb{R}^N)$ respectively, by defining Σ_p and σ_p as in (2.30), the existence of limiting measures $\Sigma_\infty, \sigma_\infty$ is guaranteed along perhaps a further subsequence such that

$$\Sigma_p \xrightharpoonup{\;*\;} \Sigma_\infty \text{ in } \mathcal{M}\Big(\overline{\Omega_T}; \mathbb{R}^N\Big) \quad \text{and} \quad \sigma_p \xrightharpoonup{\;*\;} \sigma_\infty \text{ in } \mathcal{M}\Big(\overline{\Omega_T}; \mathbb{R}^n\Big),$$

as $p_j \to \infty$. Further, by Lemma 2.2.3 we have that $u_p \longrightarrow u_\infty$ and $\mathrm{D}u_p \longrightarrow \mathrm{D}u_\infty$, both uniformly on $\overline{\Omega_T}$ as $p_j \to \infty$. Also, by the continuity assumption on $\mathrm{K}, \mathrm{K}_\eta, \mathrm{K}_A$ on $\overline{\Omega_T} \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$ and again Lemma 2.2.3, it follows that

$$\mathrm{K}[u_p] \longrightarrow \mathrm{K}[u_\infty], \;\; \mathrm{K}_\eta[u_p] \longrightarrow \mathrm{K}_\eta[u_\infty] \;\; \text{and} \;\; \mathrm{K}_A[u_p] \longrightarrow \mathrm{K}_A[u_\infty],$$

all uniformly on $\overline{\Omega_T}$ as $p_j \to \infty$. Putting all this together, we see that the remaining main point is to obtain a uniform estimate on the family of Lagrange multipliers $(\Psi_p)_{p > n+2}$ in order to deduce that

$$\Psi_p \xrightharpoonup{\;*\;} \Psi_\infty \;\; \text{in } \Big(W^{2-2/r, r}_{0,\sigma}(\Omega; \mathbb{R}^n) \Big)^*, \;\; \text{for all } r > n+2,$$

which would allow to pass to the limit as $p_j \to \infty$. Once this has been achieved, passing to the limit in the equations follows by standard duality pairing arguments, which are made possible by restricting the class of test functions (u, p) to those which are continuous together with those derivatives appearing in the relations.

In order to derive the desired estimate on $(\Psi_p)_{p>n+2}$, we argue as follows. Consider (2.22) for $\mathrm{K}_a \equiv 0$ (the first equation appearing in this proof) and let us fix the initial value on $\Omega \times \{0\}$

$$u(\cdot,0) \,\equiv\, \hat{u} \,\in W^{2,\infty}_{0,\sigma}(\Omega;\mathbb{R}^n)$$

of the arbitrary test function u, but we will select u on Ω_T such that the term in the bracket in the integral on the right-hand-side becomes a gradient. Then, this term will vanish identically as a consequence of (2.23) when $\mathrm{K}_r \equiv 0$ (the second equation appearing in this proof). Indeed, let $p > n+2$ and let also $(\tilde{u}, \tilde{\mathrm{p}})$ be the (unique) solution to

$$\left\{\begin{array}{rcll} \partial_t \tilde{u} - \nu\Delta\tilde{u} + (\tilde{u}\cdot \mathrm{D})u_p + (u_p \cdot \mathrm{D})\tilde{u} + \mathrm{D}\tilde{\mathrm{p}} &=& 0, & \text{in } \Omega_T, \\ \operatorname{div}\tilde{u} &=& 0, & \text{in } \Omega_T, \\ \tilde{u}(\cdot,0) &=& \hat{u}, & \text{on } \Omega, \\ \tilde{u} &=& 0, & \text{on } \partial\Omega\times(0,T), \end{array}\right.$$

The solvability of the above problem is a consequence of the classical result of Solonnikov [87, Th. 4.2] for $n = 3$ and of Giga-Sohr [54, Th. 2.8] for $n > 3$, as a perturbation of the Stokes problem: by choosing $q > n+2$ in Solonnikov's assumption (4.14), a solution as claimed does exist. Further, since $\hat{u}$ is in $W^{2,\infty}_{0,\sigma}(\Omega;\mathbb{R}^n)$, by [87, Cor. 2, p. 489] we have the uniform estimate

$$\|\tilde{u}\|_{W^{2,1;r}_{\mathrm{L},\sigma}(\Omega_T)} + \|\tilde{\mathrm{p}}\|_{W^{1,0;r}_{\sharp}(\Omega_T)} \,\le\, C(r)\|\hat{u}\|_{W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega)},$$

for any $r \in (1,\infty)$. By Lemmas 2.2.1 and 2.2.3 and Remark 2.2.2, if we restrict our attention to $r \in (n+2,\infty)$, we additionally have the bound

$$\|\tilde{u}\|_{L^\infty(\Omega_T)} + \|\mathrm{D}\tilde{u}\|_{L^\infty(\Omega_T)} \le\, C(r)\|\hat{u}\|_{W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega)},$$

for some new constant $C(r)$ (which is unbounded as $r \searrow n+2$). By setting

$$\left\{\begin{array}{l} K_\infty \,:=\, \sup\Big\{|\mathrm{K}_\eta| + |\mathrm{K}_A| :\ \Omega_T \times \mathbb{B}^n_{R_\infty}(0) \times \mathbb{B}^{n\times n}_{R_\infty}(0)\Big\}, \\ R_\infty \,:=\, \sup\limits_{p>n+2}\Big(\|u_p\|_{L^\infty(\Omega_T)} + \|\mathrm{D}u_p\|_{L^\infty(\Omega_T)}\Big), \end{array}\right.$$

where $\mathbb{B}^n_{R_\infty}(0)$ and $\mathbb{B}^{n\times n}_{R_\infty}(0)$ denote the balls of radius R_∞ centred at the origin of $\mathbb{R}^n$ and of $\mathbb{R}^{n\times n}$ respectively, we estimate by using (2.22)-(2.23) (for $\mathrm{K}_a \equiv 0$, $\mathrm{K}_r \equiv 0$) and that by

(2.19) we have $\left\|\mathfrak{M}_p\Big(\mathrm{K}[u_p]\Big)\right\|_{L^1(\Omega_T)} \leq 1$ (for the normalised L^1 norm):

$$
\begin{aligned}
\Big|\Big\langle \Psi_p, \hat{u}\Big\rangle\Big| \,&\leq \lambda \left| \int_{\Omega_T} \mathrm{D}\tilde{\mathrm{p}} \cdot \mathfrak{M}_p(y_p)\, \mathrm{d}\mathcal{L}^{n+1} \right| \\
&\qquad + (1-\lambda) \left| \int_{\Omega_T} \Big(\mathrm{K}_\eta[u_p]\cdot \tilde{u} \,+\, \mathrm{K}_A[u_p] : \mathrm{D}\tilde{u} \Big) \cdot \mathfrak{M}_p\Big(\mathrm{K}[u_p]\Big)\, \mathrm{d}\mathcal{L}^{n+1} \right| \\
&= (1-\lambda) T\mathcal{L}^n(\Omega) \left| \fint_{\Omega_T} \Big(\mathrm{K}_\eta[u_p]\cdot \tilde{u} \,+\, \mathrm{K}_A[u_p] : \mathrm{D}\tilde{u} \Big) \cdot \mathfrak{M}_p\Big(\mathrm{K}[u_p]\Big)\, \mathrm{d}\mathcal{L}^{n+1} \right| \\
&\leq (1-\lambda) T\mathcal{L}^n(\Omega) K_\infty \Big(\|\tilde{u}\|_{L^\infty(\Omega_T)} + \|\mathrm{D}\tilde{u}\|_{L^\infty(\Omega_T)} \Big) \\
&\leq (1-\lambda) T\mathcal{L}^n(\Omega) K_\infty C(r) \|\hat{u}\|_{W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega)},
\end{aligned}
$$

for any r fixed. Therefore, for any $r \in (n+2,\infty)$ and any $\hat{u} \in W^{2,\infty}_{0,\sigma}(\Omega;\mathbb{R}^n)$, we have the estimate

$$
\Big|\Big\langle \Psi_p, \hat{u}\Big\rangle\Big| \,\leq \Big(T\mathcal{L}^n(\Omega)K_\infty\Big) C(r) \|\hat{u}\|_{W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega)}.
$$

Since $W^{2,\infty}_{0,\sigma}(\Omega;\mathbb{R}^n)$ is dense in $W^{2-2/r,r}_{0,\sigma}(\Omega;\mathbb{R}^n)$, by the Hahn-Banach theorem, the above estimate implies that for any fixed $p > n+2$, the bounded linear functional

$$
\Psi_p \;\;:\;\; W^{2-\frac{2}{p},p}_{0,\sigma}(\Omega;\mathbb{R}^n) \longrightarrow \mathbb{R}
$$

can be (uniquely) extended to a functional $\Psi_p : W^{2-2/r,r}_{0,\sigma}(\Omega;\mathbb{R}^n) \longrightarrow \mathbb{R}$ for all $r \in (n+2,p]$, whose extension we denote again by Ψ_p. Therefore, Ψ_p can be extended to a unique continuous linear functional

$$
\Psi_p \;\;:\;\; \bigcup_{r>n+2} W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega;\mathbb{R}^n) \longrightarrow \mathbb{R}
$$

on the above Fréchet space, whose topology can be defined in the standard locally convex sense by the family of seminorms

$$
\Big\{ \|\cdot\|_{W^{2-2/r,r}(\Omega)} :\; r > n+2 \Big\}.
$$

Additionally, the uniformity of the estimate with respect to p implies that

$$
(\Psi_p)_{p>n+2} \text{ is bounded in } \left(\bigcup_{r>n+2} W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega;\mathbb{R}^n) \right)^*
$$

(in the locally convex sense). Hence, as it can be seen by a customary diagonal argument in the scale of Banach spaces $\Big\{W^{2-2/r,r}_{0,\sigma}(\Omega_T;\mathbb{R}^n) : r > n+2\Big\}$ comprising the Fréchet space, there exists a continuous linear functional

$$\Psi_\infty \ : \ \bigcup_{r>n+2} W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega;\mathbb{R}^n) \longrightarrow \mathbb{R}$$

and a further subsequence as $p \to \infty$ such that along which we have $\Psi_p \xrightharpoonup{*} \Psi_\infty$ in the locally convex sense. Additionally, since

$$\Psi_\infty \in \bigcap_{r>n+2} \Big(W^{2-\frac{2}{r},r}_{0,\sigma}(\Omega;\mathbb{R}^n)\Big)^*$$

the convergence $\Psi_p \xrightharpoonup{*} \Psi_\infty$ is equivalent to weak* convergence in the Banach space $W^{2-2/r,r}_{0,\sigma}(\Omega;\mathbb{R}^n)$ for any fixed $r > n+2$. In conclusion, we see that (2.27)-(2.28) have now been established.

Now we complete the proof of Theorem 2.1.3 by establishing (2.31)-(2.32). Since $\mathrm{K}[u_p] \longrightarrow \mathrm{K}[u_\infty]$ in $C\Big(\overline{\Omega_T};\mathbb{R}^N\Big)$, by applying [64, Prop. 10], we immediately obtain that Σ_∞ concentrates on the set whereon $|\mathrm{K}[u_\infty]|$ is maximised over $\overline{\Omega_T}$:

$$\Sigma_\infty\bigg(\Big\{\Big|\mathrm{K}[u_\infty]\Big| < \max_{\overline{\Omega_T}}\Big|\mathrm{K}[u_\infty]\Big|\Big\}\bigg) \,=\, 0.$$

This proves (2.31). For (2.32), we argue as follows. We first note that

$$\|y_p\|_{\dot{L}^p(\Omega_T)} \longrightarrow \|y_\infty\|_{L^\infty(\Omega_T)}$$

as $p \to \infty$, along a subsequence. In view of (2.6) and (2.13), this is a consequence of (2.18) and the fact that $\mathrm{K}[u_p] \longrightarrow \mathrm{K}[u_\infty]$ uniformly on $\overline{\Omega_T}$, which implies

$$\Big\|\mathrm{K}[u_p]\Big\|_{\dot{L}^p(\Omega_T)} \longrightarrow \Big\|\mathrm{K}[u_\infty]\Big\|_{L^\infty(\Omega_T)}.$$

As a consequence of the convergence of $\|y_p\|_{\dot{L}^p(\Omega_T)}$ to $\|y_\infty\|_{L^\infty(\Omega_T)}$, for any $\varepsilon > 0$ we may choose p large so that

$$\|y_p\|_{\dot{L}^p(\Omega_T)} \geq \|y_\infty\|_{L^\infty(\Omega_T)} - \frac{\varepsilon}{2}.$$

Let us define now the following subset of Ω_T, which without loss of generality we may assume it has positive $\mathcal{L}^{n+1}$-measure:

$$A_{p,\varepsilon} := \Big\{|y_p| \leq \|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\Big\}.$$

In particular, if $\mathcal{L}^{n+1}(A_{p,\varepsilon}) > 0$, then necessarily $\|y_\infty\|_{L^\infty(\Omega_T)} > 0$. For any Borel set $B \subseteq \Omega_T$ such that $\mathcal{L}^{n+1}(\Omega_T \cap B) > 0$, we estimate by using (2.30), (2.19), (2.7) and the above:

$$\begin{aligned}
\sigma_p(A_{p,\varepsilon} \cap B) &\leq \frac{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)}{\|y_p\|^{p-1}_{\dot{L}^p(\Omega_T)}} ⨍_{A_{p,\varepsilon}\cap B} \Big(|y_p|_{(p)}\Big)^{p-1} \,\mathrm{d}\mathcal{L}^{n+1} \\
&\leq \frac{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)}{\|y_p\|^{p-1}_{\dot{L}^p(\Omega_T)}} ⨍_{A_{p,\varepsilon}\cap B} \Big(\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\Big)^{p-1} \,\mathrm{d}\mathcal{L}^{n+1} \\
&\leq \frac{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)}{\|y_p\|^{p-1}_{\dot{L}^p(\Omega_T)}} \Big(\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon\Big)^{p-1} \\
&\leq \mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B) \left(\frac{\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon}{\|y_\infty\|_{L^\infty(\Omega_T)} - \frac{\varepsilon}{2}}\right)^{p-1}.
\end{aligned}$$

As a result, for any $\varepsilon > 0$ small, any p large enough and any Borel set $B \subseteq \Omega_T$ with $\mathcal{L}^{n+1}(\Omega_T \cap B) > 0$, we have obtained the density estimate

$$\frac{\sigma_p(A_{p,\varepsilon} \cap B)}{\mathcal{L}^{n+1}(A_{p,\varepsilon} \cap B)} \leq \left(1 - \frac{\varepsilon}{2\|y_\infty\|_{L^\infty(\Omega_T)} - \varepsilon}\right)^{p-1}.$$

The above estimate in particular implies that $\sigma_p(A_{p,\varepsilon}) \longrightarrow 0$ as $p \to \infty$ for any $\varepsilon > 0$ fixed, therefore establishing (2.32). The proof of Theorem 2.1.3 is now complete. □

Remark 2.5.1. It is perhaps worth noting (in relation to the preceding arguments in the proof of (2.32)) that the modes of convergence

$$\|y_p\|_{L^p(\Omega_T)} \longrightarrow \|y_\infty\|_{L^\infty(\Omega_T)} \quad \text{and} \quad y_p \overset{*}{\rightharpoonup} y_\infty \ \text{ in } L^\infty(\Omega_T; \mathbb{R}^n)$$

as $p \to \infty$, in general by themselves do not suffice to obtain $y_p \longrightarrow y_\infty$ in any strong sense, hence precluding the derivation of a stronger property than (2.32), along the lines of (2.31). A simple counter-example, even in one dimension, is the following: let $p \in 2\mathbb{N}$ and set

$$y_p := \sum_{j=0}^{(p-2)/2} \left[\chi_{\left(\frac{2j}{p}, \frac{2j+1}{p}\right)} - \chi_{\left(\frac{2j+1}{p}, \frac{2j+2}{p}\right)}\right] + \chi_{(1,2)},$$

and also $y_\infty := \chi_{(1,2)}$. Then, we have $|y_p| = 1$ $\mathcal{L}^1$-a.e. on $(0,2)$ for all p, hence we deduce that $\|y_p\|_{L^p(0,2)} \longrightarrow \|y_\infty\|_{L^\infty(0,2)}$, whilst we also have $y_p \overset{*}{\rightharpoonup} y_\infty$ in $L^\infty(0,2)$ as $p \to \infty$, but $y_p \not\longrightarrow y_\infty$ neither a.e., nor in L^1 or in measure.

Chapter 3

On the Isosupremic L^∞ Vectorial Minimisation Problem with PDE Constraints

3.1 Introduction and main results

Let $n, N \in \mathbb{N}$ and let also $\Omega \Subset \mathbb{R}^n$ be a bounded open set with Lipschitz continuous boundary. Consider two functions

$$f,\, g \ : \ \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}, \tag{3.1}$$

which will be assumed to satisfy certain natural structural assumptions. Additionally, let $\bar{p} > n$ be fixed and consider a given nonlinear operator

$$\mathrm{Q} \ : \ W^{1,\bar{p}}_0\Big(\Omega;\mathbb{R}^N\Big) \longrightarrow \mathbf{E}, \tag{3.2}$$

where $(\mathbf{E}, \|\cdot\|)$ is an arbitrary Banach space. In this paper we are interested in the following variational problem: given $G \geq 0$ and the supremal functionals

$$\mathrm{F}_\infty, \mathrm{G}_\infty \ : \ W^{1,\infty}_0(\Omega;\mathbb{R}^N) \longrightarrow \mathbb{R},$$

defined by

$$\left\{ \begin{aligned} \mathrm{F}_\infty(u) :=& \ \operatorname*{ess\,sup}_\Omega f(\cdot, u, \mathrm{D}u), \\ \mathrm{G}_\infty(u) :=& \ \operatorname*{ess\,sup}_\Omega g(\cdot, u, \mathrm{D}u), \end{aligned} \right. \tag{3.3}$$

find $u_\infty \in W^{1,\infty}_0(\Omega;\mathbb{R}^N)$ such that

$$\mathrm{F}_\infty(u_\infty) \,=\, \inf\Big\{\mathrm{F}_\infty(u) \,:\, u \in W^{1,\infty}_0(\Omega;\mathbb{R}^N),\ \mathrm{G}_\infty(u) \le G \ \&\ \mathrm{Q}(u) = 0\Big\}. \tag{3.4}$$

We are also interested in deriving appropriate differential equations as necessary conditions that such constrained minimisers u_∞ might satisfy. Hence, the problem under consideration is that of minimising the supremal functional F_∞ within the G-sublevel set of the supremal functional G_∞, constrained also by the zero level set of the mapping Q. Minimisation problems in L^∞ with an L^∞ sublevel-set constraints are called *isosupremic*, a terminology introduced by Aronsson-Barron in [11], by analogy to the terminology of *isoperimetric* constraints utilised for integral functionals.

The assumptions required for our mapping Q expressing the nonlinear level-set constraint are very general, allowing for the solvability of a very large class of problems. In Section 3.4 we provide a fairly comprehensive list of explicit examples of operators Q to which our results apply. Indeed, Q may manifest itself as a pointwise, unilateral, or inclusion constraint (Subsection 3.4.1), as an integral isoperimetric constraint (Subsection 3.4.2), or as a nonlinear PDE constraint, including second order quasilinear divergence systems (Subsection 3.4.3), Jacobian equations and other PDEs driven by null Lagrangians (Subsection 3.4.4).

Minimisation of supremal functionals poses several important challenges. Firstly, the L^∞ norm is neither Gateaux differentiable nor strictly convex, and the space L^∞ is neither reflexive nor separable. A related additional complication is that the L^∞ norm is not additive but only sub-additive with respect to the domain. As a result, if one solves the L^∞ minimisation existence problem via the direct method, then the analogue of the Euler-Lagrange equations for the L^∞ problem cannot be derived directly by considering variations, due to the lack of smoothness of the L^∞ norm.

In this paper, we transcend the difficulties illustrated above, by following the relatively standard strategy of using appropriate approximations by L^p functionals as $p \to \infty$. We solve the desired L^∞ variational problem by solving approximating L^p variational problems for all p, and obtaining the necessary compactness estimates that allow us to pass to the limit as $p \to \infty$. The case of finite p, which is of independent interest and a byproduct of our analysis, for us is just the mechanism to solve the desired L^∞ problem. The intuition behind the use of L^p approximations is based on the fact that for a fixed essentially bounded function on a set of finite measure, the L^p norm tends to the L^∞ norm as $p \to \infty$.

Problems involving constraints are relatively new in the Calculus of Variations in L^∞ and previous work has been relatively sparse, even more so in the vectorial case. To the best of our knowledge, the only work which directly studies isosupremic problems is [11] by Aronsson-Barron. Among other questions answered therein, it considers some aspects of the one-dimensional case for $n = 1$, but with no additional constraints of any type (which amounts to $\mathrm{Q} \equiv 0$ in our setting).

More broadly, very few previous works involve vectorial problems with general constraints in L^∞. Certain vectorial and higher order problems involving eigenvalues in L^∞ have been considered in [65, 69]. Examples of problems with PDE and other constraints are considered in [30, 63, 64, 66]. In the paper [15] of Barron-Jensen, a scalar L^∞ constrained problem was considered, but the constraint was integral. With the exception of the paper [11], it appears that vectorial variational problems in L^∞ involving isosupremic constraints have not been studied before, especially including additional nonlinear constraints which cover numerous different cases, as in this work. For assorted interesting works within the wider area of Calculus of Variations in L^∞ we refer to [10, 14, 16, 17, 19, 20, 26, 27, 68, 76, 80, 81, 84].

Let us note that, in this work, we refrain from discussing the question of defining and studying localised versions of L^∞ minimisers on subdomains. Such minimisers are commonly called *absolute* in the Calculus of Variations in L^∞, a concept first introduced and studied by Aronsson in his seminal papers in the 1960s. As already noted in [11], the definition of constrained absolute minimisers on subdomains is quite problematic in many respects, even for specific choices of constraints in the scalar case $N = 1$, which has a well developed L^∞ theory. In any case, the "non-intrinsic" method of L^p approximations, employed herein, is widely believed to select always the "best" L^∞ minimiser, which in the scalar unconstrained case can be shown to be indeed an absolute minimiser. In the vectorial case of $N \geq 2$, the situation is trickier and the "correct" localised minimality notion is still under discussion, even without any constraints being involved (see e.g. [12, 13] for work in this direction).

We now state our main results. Our notation is generally standard as e.g. in [73], or otherwise self-explanatory. As we will only be working with finite measures, we will be using the normalised L^p norms in which we replace the integral over the domain with the respective average. For our density functions f and g appearing in (3.1) we will assume the following:

$$\begin{cases} \text{Both } f \text{ and } g \text{ are Carathéodory functions, namely } f(\cdot,\eta,P),\ g(\cdot,\eta,P) \text{ are} \\ \text{measurable for all } (\eta,P), \text{ and } f(x,\cdot,\cdot),\ g(x,\cdot,\cdot) \text{ are continuous for a.e. } x. \end{cases} \tag{3.5}$$

$$\begin{cases} \text{Exists a continuous } \mathrm{C} : \overline{\Omega} \times \mathbb{R}^N \longrightarrow [0,\infty) \text{ and an } \alpha > 2\text{:} \\ \qquad\qquad \begin{cases} 0 \leq f(x,\eta,P) \leq \mathrm{C}(x,\eta)\Big(|P|^\alpha + 1\Big), \\ 0 \leq g(x,\eta,P) \leq \mathrm{C}(x,\eta)\Big(|P|^\alpha + 1\Big), \end{cases} \\ \text{for a.e. } x \text{ and all } (\eta, P). \end{cases} \tag{3.6}$$

$$\text{For a.e. } x \text{ and all } \eta,\ f(x,\eta,\cdot) \text{ and } g(x,\eta,\cdot) \text{ are quasiconvex on } \mathbb{R}^{N\times n}. \tag{3.7}$$

$$\begin{cases} \texttt{Either } f \texttt{ or } g \text{ is coercive, namely exist } c, C > 0 \text{ such that either} \\ \qquad\qquad f(x,\eta,P) \ \geq\ c|P|^\alpha - C, \\ \text{or} \\ \qquad\qquad g(x,\eta,P) \ \geq\ c|P|^\alpha - C, \\ \text{for a.e. } x \text{ and all } (\eta, P). \end{cases} \tag{3.8}$$

These assumptions are relatively standard in the Calculus of Variations. We note only that (3.7) is meant in the sense of Morrey quasiconvexity for integral functionals (as e.g. in [36]), not in the sense of level-convexity or of the L^∞ notion of "BJW-quasiconvexity" of Barron-Jensen-Wang in [17], nor in the sense of "$\mathcal{A}$-Young quasiconvexity" of Ansini-Prinari in [10]. This stronger notion of quasiconvexity is not necessary if one is interested in merely solving (3.4) by applying the direct method in $W^{1,\infty}$ *without deriving any PDEs*, and can indeed be weakened substantially, but it is needed for Theorems 3.1.2 and 3.1.3, so we simplify the exposition by assuming it at the outset. For our operator Q in (3.2) we will assume the following:

$$\mathrm{Q}^{-1}\Big(\{0\}\Big) \text{ is weakly closed in } W^{1,\overline{p}}_0(\Omega;\mathbb{R}^N). \tag{3.9}$$

This is a very feeble requirement for Q and there exist numerous explicit examples of interest that satisfy (3.9), see Section 3.4. For $1 \leq p < \infty$, we define the approximating L^p functionals $\mathrm{F}_p, \mathrm{G}_p : W^{1,\alpha p}_0(\Omega;\mathbb{R}^N) \longrightarrow \mathbb{R}$ by setting

$$\begin{cases} \mathrm{F}_p(u) := \left(\displaystyle -\!\!\!\!\!\!\int_\Omega f(\cdot, u, \mathrm{D}u)^p \, \mathrm{d}\mathcal{L}^n \right)^{1/p}, \\ \mathrm{G}_p(u) := \left(\displaystyle -\!\!\!\!\!\!\int_\Omega g(\cdot, u, \mathrm{D}u)^p \, \mathrm{d}\mathcal{L}^n \right)^{1/p}. \end{cases} \tag{3.10}$$

For $p = \infty$, F_∞ and G_∞ are given by (3.3). For each $p \in [1,\infty]$ and $G \geq 0$ fixed, we define the admissible minimisation class $\mathfrak{X}^p(\Omega)$ on which F_p is to be minimised, by setting

$$\mathfrak{X}^p(\Omega) \ :=\ \Big\{ v \in W^{1,\alpha p}_0(\Omega;\mathbb{R}^N) \ :\ \mathrm{G}_p(v) \leq G \text{ and } \mathrm{Q}(v) = 0 \Big\}. \tag{3.11}$$

Our first main result concerns the existence of F_p-minimisers in $\mathfrak{X}^p(\Omega)$ and the existence of F_∞-minimisers in $\mathfrak{X}^\infty(\Omega)$, obtained as subsequential limits as $p \to \infty$.

Theorem 3.1.1 (F_∞-minimisers, F_p-minimisers & convergence as $p \to \infty$)**.** *Suppose that the mappings f, g and Q satisfy the assumptions (3.5) through (3.9). If the next compatibility condition is satisfied*

$$\inf \Big\{ \mathrm{G}_\infty : \mathrm{Q}^{-1}\Big(\{0\}\Big) \cap W^{1,\infty}_0(\Omega;\mathbb{R}^N) \Big\} \,<\, G, \tag{3.12}$$

then, for any $p \in [\bar{p}, \infty]$, the functional F_p has a constrained minimiser u_p in the admissible class $\mathfrak{X}^p(\Omega)$, namely

$$\mathrm{F}_p\Big(u_p\Big) \,=\, \inf\Big\{ \mathrm{F}_p\Big(v\Big) \,:\, v \in \mathfrak{X}^p(\Omega) \Big\}. \tag{3.13}$$

Additionally, there exists a subsequence of indices $(p_j)_1^\infty$ such that, the sequence of respective F_{p_j}-minimisers satisfies $u_p \longrightarrow u_\infty$ uniformly on $\overline{\Omega}$, and $u_p \rightharpoonup u_\infty$ weakly in $W^{1,q}_0(\Omega;\mathbb{R}^N)$, for all $q \in (1,\infty)$ fixed, as $p_j \to \infty$. Finally, we have the convergence of minimum values $\mathrm{F}_p(u_p) \longrightarrow \mathrm{F}_\infty(u_\infty)$ as $p_j \to \infty$.

Note that, in view of (3.9), the compatibility condition (3.12) guarantees that $\mathfrak{X}^p(\Omega) \neq \emptyset$. Given the existence of constrained minimisers provided from Theorem 3.1.1 above, the next natural question concerns the deduction of necessary conditions satisfied by constrained minimisers in the form of PDEs. Firstly, we examine the case of $p < \infty$. The nonlinear constraint expressed by the zero level-set of the mapping Q in (3.11) gives rise to a Lagrange multiplier in the Euler-Lagrange equations. This can be inferred by employing well-known results on Lagrange multipliers, see for instance [94]. For this to be possible, though, one needs improved regularity of the mappings f, g and Q involved. We will suppose additionally that

$$\left\{ \begin{array}{l} \text{The partial derivatives } f_{\eta\eta}, f_{\eta P}, f_{P\eta}, f_{PP} \text{ of } f \text{ are continuous on} \\ \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n}, \text{ and for } \mathrm{C}, \alpha \text{ as in (3.6), we have the bounds} \\ \qquad \left\{ \begin{array}{l} |f_{\eta\eta}(x,\eta,P)| \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-2}+1\Big), \\ |f_{\eta P}(x,\eta,P)| \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-2}+1\Big), \\ |f_{PP}(x,\eta,P)| \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-2}+1\Big), \\ |f_{P\eta}(x,\eta,P)| \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-2}+1\Big), \end{array}\right. \\ \text{for all } (x,\eta,P). \end{array}\right. \tag{3.14}$$

$$
\left\{
\begin{array}{l}
\text{The partial derivatives } g_{\eta\eta}, g_{\eta P}, g_{P\eta}, g_{PP} \text{ of } g \text{ are continuous on} \\
\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n}, \text{ and for C}, \alpha \text{ as in (3.6), we have the bounds} \\
\qquad \left\{
\begin{array}{l}
|g_{\eta\eta}(x,\eta,P)| \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-2}+1\Big), \\
|g_{\eta P}(x,\eta,P)| \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-2}+1\Big), \\
|g_{PP}(x,\eta,P)| \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-2}+1\Big), \\
|g_{P\eta}(x,\eta,P)| \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-2}+1\Big),
\end{array}
\right. \\
\text{for all } (x,\eta,P).
\end{array}
\right. \tag{3.15}
$$

It follows that

$$
\left\{
\begin{array}{l}
\text{The partial derivatives } f_{\eta}, f_P, g_{\eta}, g_P \text{ of } f \text{ and } g \text{ are continuous on} \\
\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n}, \text{ and for C}, \alpha \text{ as in (3.6), we have the bounds} \\
\qquad \left\{
\begin{array}{l}
|f_{\eta}|(x,\eta,P) + |f_P|(x,\eta,P) \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-1}+1\Big), \\
|g_{\eta}|(x,\eta,P) + |g_P|(x,\eta,P) \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-1}+1\Big),
\end{array}
\right. \\
\text{for all } (x,\eta,P).
\end{array}
\right. \tag{3.16}
$$

Further, we will assume that:

$$
\left\{
\begin{array}{l}
\mathrm{Q} \text{ is continuously differentiable, and its Fréchet derivative} \\
\qquad\qquad (\mathrm{dQ})_{\overline{u}} \ : \ \ W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N) \longrightarrow \mathbf{E} \\
\text{has closed range in } \mathbf{E}, \text{ for any } \overline{u} \in \mathrm{Q}^{-1}(\{0\}) \subseteq W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N).
\end{array}
\right. \tag{3.17}
$$

Recall that no regularity was assumed for Q to obtain the existence of minimisers in Theorem 3.1.1. Finally, for the sake of brevity, for any $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, we will employ the following notation

$$
f[u] \ \equiv \ f(\cdot, u, \mathrm{D}u), \quad g[u] \ \equiv \ g(\cdot, u, \mathrm{D}u),
$$

and similar notation will be used for the compositions of f_η, f_P, g_η, g_P of f and g respectively, namely $f_\eta[u], g_\eta[u]$, etc. Further, "$\cdot$" and "$:$" will denote the standard inner products on $\mathbb{R}^N$ and $\mathbb{R}^{N\times n}$ respectively.

Theorem 3.1.2 (The equations in L^p)**.** *Suppose we are in the setting of Theorem 3.1.1 and assumptions (3.5) through (3.9) are satisfied. Suppose additionally that (3.14) through (3.17) are satisfied. Then, for any $p \in (\bar{p}, \infty)$, there exists Lagrange multipliers*

$$
\lambda_p \geq 0, \quad \mu_p \geq 0, \quad \psi_p \ \in \ \mathbf{E}^*, \tag{3.18}
$$

where $(\mathbf{E}^*, \|\cdot\|_*)$ *is the dual space of* $\mathbf{E}$*, such that not all vanish simultaneously:*

$$|\lambda_p| + |\mu_p| + \|\psi_p\|_* \neq 0. \tag{3.19}$$

Then, the minimiser $u_p \in \mathfrak{X}^p(\Omega)$ *satisfies the equation*

$$\begin{cases} \quad \lambda_p \displaystyle\fint_\Omega f[u_p]^{p-1}\Big(f_\eta[u_p]\cdot\phi \,+\, f_P[u_p] : \mathrm{D}\phi\Big)\,\mathrm{d}\mathcal{L}^n \\ + \,\mu_p \displaystyle\fint_\Omega g[u_p]^{p-1}\Big(g_\eta[u_p]\cdot\phi \,+\, g_P[u_p] : \mathrm{D}\phi\Big)\,\mathrm{d}\mathcal{L}^n \,=\, \Big\langle \psi_p, (\mathrm{dQ})_{u_p}(\phi)\Big\rangle, \end{cases} \tag{3.20}$$

for all test mappings $\phi \in W^{1,\alpha p}_0\big(\Omega;\mathbb{R}^N\big)$*, coupled by the additional condition*

$$\mu_p\Big(\mathrm{G}_p(u_p) - G\Big) = 0. \tag{3.21}$$

Note that condition (3.21) implies that if $\mathrm{G}_p(u_p) < G$, namely if the L^∞-energy constraint is not realised (i.e. the minimiser u_p lies in the interior of the sublevel set $\{\mathrm{G}_p \leq G\}$), then $\mu_p = 0$ and hence the associated multiplier vanishes.

Now we consider the case of $p = \infty$. In this case, there is not a simple analogue of the divergence structure Euler-Lagrange equations. The equations are derived by an appropriate passage to limits as $p \to \infty$ in Theorem 3.1.2. A standard approach in Calculus of Variations in L^∞ has been to derive Aronsson-type PDE systems, which are non-divergence counterparts to the Euler-Lagrange equations, as e.g. done in [11] for the case of $n = 1$ and $\mathrm{Q} \equiv 0$. However, Aronsson-type systems are always non-divergence and far less tractable than their divergence counterparts. In fact, in the vectorial case they have discontinuous coefficients and are fully nonlinear in the higher order case (see for instance [35] and [60, 61, 72] for this evolving line of development regarding the direct study of generalised solutions to Aronsson systems).

Nevertheless, there exists an alternative approach which allows to derive *divergence structure* PDE systems as necessary conditions. The starting point of this idea is based on the use of a different scaling in the Euler-Lagrange equations in L^p and has already born substantial fruit in [30, 63, 64, 65, 66, 68, 69]. There is however a toll to be paid for this “forcing” of divergence structure: certain non-uniquely determined measures arise as auxiliary parameters in the coefficients of the PDE system, which depend nonlinearly on the minimisers. For more details on the historical origins of this alternative approach to deriving L^∞ equations for variational problems we refer to [65].

Our final main result therefore concerns the satisfaction of necessary conditions for the constrained minimiser in L^∞ constructed in Theorem 3.1.1. For this result we will need to impose some natural additional hypotheses. These hypotheses, although they restrict considerably the classes of f, g, Q that were utilised in order to prove existence of minimisers, they do nonetheless include the interesting case of F_∞ being the L^∞ norm of the gradient. Firstly, let us introduce some convenient notation and rewrite (3.20) in a way which will be more appropriate for the statement and the subsequent proof. By introducing for each $p \in (\bar{p}, \infty)$ the non-negative Radon measures $\sigma_p, \tau_p \in \mathcal{M}(\overline{\Omega})$ given by

$$
\begin{aligned}
\sigma_p \;&:=\; \left\{ \begin{array}{ll} \dfrac{1}{\mathcal{L}^n(\Omega)}\left(\dfrac{f[u_p]}{\mathrm{F}_p(u_p)}\right)^{p-1} \mathcal{L}^n \llcorner_\Omega, & \text{if } \mathrm{F}_p(u_p) > 0, \\ 0, & \text{if } \mathrm{F}_p(u_p) = 0, \end{array} \right. \\
\tau_p \;&:=\; \left\{ \begin{array}{ll} \dfrac{1}{\mathcal{L}^n(\Omega)}\left(\dfrac{g[u_p]}{\mathrm{G}_p(u_p)}\right)^{p-1} \mathcal{L}^n \llcorner_\Omega, & \text{if } \mathrm{G}_p(u_p) > 0, \\ 0, & \text{if } \mathrm{G}_p(u_p) = 0, \end{array} \right.
\end{aligned}
\tag{3.22}
$$

and the scaled multipliers

$$
\begin{aligned}
\hat{\lambda}_p \;&:=\; \left\{ \begin{array}{ll} \lambda_p \mathrm{F}_p(u_p)^{p-1}, & \text{if } \mathrm{F}_p(u_p) > 0, \\ \lambda_p, & \text{if } \mathrm{F}_p(u_p) = 0, \end{array} \right. \\
\hat{\mu}_p \;&:=\; \left\{ \begin{array}{ll} \mu_p \mathrm{G}_p(u_p)^{p-1}, & \text{if } \mathrm{G}_p(u_p) > 0, \\ \mu_p, & \text{if } \mathrm{G}_p(u_p) = 0, \end{array} \right.
\end{aligned}
\tag{3.23}
$$

we can rewrite (3.20) as

$$
\left\{
\begin{array}{l}
\quad \hat{\lambda}_p \displaystyle\int_\Omega \Big(f_\eta[u_p] \cdot \phi \,+\, f_P[u_p] : \mathrm{D}\phi \Big)\, \mathrm{d}\sigma_p \\
+\, \hat{\mu}_p \displaystyle\int_\Omega \Big(g_\eta[u_p] \cdot \phi \,+\, g_P[u_p] : \mathrm{D}\phi \Big)\, \mathrm{d}\tau_p \;=\; \Big\langle \psi_p, (\mathrm{dQ})_{u_p}(\phi) \Big\rangle.
\end{array}
\right.
\tag{3.24}
$$

Further, let us set

$$
R_p \;:=\; \hat{\lambda}_p + \hat{\mu}_p + \|\psi_p\|_* \tag{3.25}
$$

and note that, by virtue of Theorem 3.1.2 and Definition (3.23), we have that $R_p > 0$. We may then define the new rescaled multipliers

$$
\Lambda_p \;:=\; \frac{\hat{\lambda}_p}{R_p} \in [0,1], \quad \mathrm{M}_p \;:=\; \frac{\hat{\mu}_p}{R_p} \in [0,1], \quad \Psi_p \;:=\; \frac{\psi_p}{R_p} \in \bar{\mathbb{B}}_1^{\mathbf{E}^*}(0), \tag{3.26}
$$

where $\bar{\mathbb{B}}_1^{\mathbf{E}^*}(0)$ is the closed unit ball in $\mathbf{E}^*$. Let us finally set

$$
C^1_0\big(\overline{\Omega}; \mathbb{R}^N\big) \;:=\; W^{1,\infty}_0(\Omega; \mathbb{R}^N) \bigcap C^1\big(\mathbb{R}^n; \mathbb{R}^N\big).
$$

Now we state the additional assumptions which we need to impose:

$$\mathbf{E} \text{ is a separable Banach space.} \tag{3.27}$$

$$\left\{\begin{array}{l}
\text{The restriction of the differential } (u,v) \mapsto (\mathrm{dQ})_u(v), \text{ considered as}\\
\qquad\qquad \mathrm{dQ} \ : \ \ \mathrm{Q}^{-1}(\{0\}) \times W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N) \longrightarrow \mathbf{E},\\
\text{satisfies the following conditions:}\\
\text{If } u_m \rightharpoonup u \text{ in } \mathrm{Q}^{-1}(\{0\}) \text{ as } m\to\infty, \text{ and } \phi \in W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N), \text{ then}\\
\qquad\qquad \left\{\begin{array}{l} (\mathrm{dQ})_{u_m}(u_m) \longrightarrow (\mathrm{dQ})_u(u),\\ (\mathrm{dQ})_{u_m}(\phi) \longrightarrow (\mathrm{dQ})_u(\phi), \end{array}\right.\\
\ \text{as } m\to\infty.
\end{array}\right. \tag{3.28}$$

The above assumption requires that dQ be weakly-strongly continuous on the diagonal of $\mathrm{Q}^{-1}(\{0\}) \times \mathrm{Q}^{-1}(\{0\})$ and on subsets of the form $\mathrm{Q}^{-1}(\{0\}) \times \{\phi\}$, when $W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N) \times W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$ is endowed with its weak topology and $\mathbf{E}$ with its norm topology. We assume further that:

$$\left\{\begin{array}{l}
\text{(i) } g \text{ does not depend on } P, \text{ namely } g(x,\eta,P) = g(x,\eta),\\
\text{(ii) } f \text{ is quadratic in } P \text{ and independent of } \eta, \text{ namely}\\
\qquad\qquad f(x,\eta,P) = \mathbf{A}(x) : P\otimes P,\\
\text{for some continuous positive symmetric fourth order tensor}\\
\mathbf{A} : \overline{\Omega} \longrightarrow \mathbb{R}^{N\times n} \otimes \mathbb{R}^{N\times n}, \text{ which satisfies}\\
\qquad \mathbf{A}(x) : P\otimes P > 0 \ , \quad \mathbf{A}(x) : P\otimes Q = \mathbf{A}(x) : Q\otimes P,\\
\text{for all } x \in \overline{\Omega} \text{ and all } P,Q \in \mathbb{R}^{N\times n} \setminus \{0\}.
\end{array}\right. \tag{3.29}$$

The above requirements are compatible with the previous assumptions on f. In fact, by [65, Lemma 4, p. 8] and our earlier assumptions, the positivity and symmetry requirements for $\mathbf{A}$ are superfluous and can be deduced by merely assuming that f is quadratic in P (up to a replacement of $\mathbf{A}$ by its symmetrisation), but we have added them to (3.29) for simplicity. We may finally state our last principal result.

Theorem 3.1.3 (The equations in L^∞)**.** *Suppose we are in the setting of Theorem 3.1.2 and that the same assumptions are satisfied. Additionally we assume that* (3.27) *through* (3.29) *hold true. Then, there exist*

$$\Lambda_\infty \in [0,1], \qquad \mathrm{M}_\infty \in [0,1], \quad \Psi_\infty \in \bar{\mathbb{B}}^{\mathbf{E}^*}_1(0), \tag{3.30}$$

which are Lagrange multipliers associated with the constrained minimisation problem (3.4). *There also exist Radon measures*

$$\sigma_\infty \in \mathcal{M}(\overline{\Omega}), \quad \tau_\infty \in \mathcal{M}(\overline{\Omega}), \tag{3.31}$$

and a Borel measurable mapping $\mathrm{D}u^\star_\infty : \overline{\Omega} \longrightarrow \mathbb{R}^{N\times n}$ *which is a version of* $\mathrm{D}u_\infty \in L^\infty(\Omega;\mathbb{R}^{N\times n})$, *such that the minimiser* $u_\infty \in \mathfrak{X}^\infty(\Omega)$ *satisfies the equation*

$$\Lambda_\infty \int_{\overline{\Omega}} f_P(\cdot, \mathrm{D}u^\star_\infty) : \mathrm{D}\phi \,\mathrm{d}\sigma_\infty \,+\, \mathrm{M}_\infty \int_{\overline{\Omega}} g_\eta(\cdot, u_\infty)\cdot\phi\,\mathrm{d}\tau_\infty \,=\, \Big\langle \Psi_\infty, (\mathrm{dQ})_{u_\infty}(\phi)\Big\rangle, \tag{3.32}$$

for all test maps $\phi \in C^1_0\Big(\overline{\Omega};\mathbb{R}^N\Big)$, *coupled by the condition*

$$\mathrm{M}_\infty\Big(\mathrm{G}_\infty(u_\infty) - G\Big) = 0. \tag{3.33}$$

Additionally, the map $\mathrm{D}u^\star_\infty$ *can be represented (modulo Lebesgue null sets) as follows:*

$$\left\{\begin{array}{l} \textit{For any sequence } (v_j)_1^\infty \subseteq C^1_0\Big(\overline{\Omega};\mathbb{R}^N\Big) \textit{ satisfying} \\[1ex] \qquad\qquad \left\{\begin{array}{l} \displaystyle\lim_{j\to\infty} \|v_j - u_\infty\|_{(W^{1,1}_0\cap L^\infty)(\Omega)} \,=\, 0, \\ \displaystyle\quad\limsup_{j\to\infty} \mathrm{F}_\infty(v_j) \,\leq\, \mathrm{F}_\infty(u_\infty), \end{array}\right. \\[1ex] \textit{exist a subsequence } (j_k)_1^\infty \textit{ such that} \\[1ex] \mathrm{D}u^\star_\infty(x) \,=\left\{\begin{array}{ll} \displaystyle\lim_{k\to\infty} \mathrm{D}v_{j_k}(x), & \textit{if the limit exists,} \\ 0, & \textit{otherwise.} \end{array}\right. \end{array}\right. \tag{3.34}$$

(Such an explicit sequence $(v_j)_1^\infty$ *is constructed in the proof.) Finally, the Lagrange multipliers* $\Lambda_\infty, \mathrm{M}_\infty, \Psi_\infty$ *and the measures* $\sigma_\infty, \tau_\infty$ *can be approximated as follows:*

$$\left\{\begin{array}{lll} \Psi_p \overset{*}{\rightharpoonup} \Psi_\infty, & \textit{in } \bar{\mathbb{B}}^{\mathbf{E}^*}_1(0), \\ \Lambda_p \longrightarrow \Lambda_\infty, & \textit{in } [0,1], \\ \mathrm{M}_p \longrightarrow \mathrm{M}_\infty, & \textit{in } [0,1], \end{array}\right. \tag{3.35}$$

and

$$\left\{\begin{array}{ll} \sigma_p \overset{*}{\rightharpoonup} \sigma_\infty, & \textit{in } \mathcal{M}(\overline{\Omega}), \\ \tau_p \overset{*}{\rightharpoonup} \tau_\infty, & \textit{in } \mathcal{M}(\overline{\Omega}), \end{array}\right. \tag{3.36}$$

along a subsequence $p_j \to \infty$.

The weak interpretation of (3.32) is

$$-\Lambda_\infty \mathrm{div}\big(f_P(\cdot, \mathrm{D}u_\infty^\star)\sigma_\infty\big) \,+\, \mathrm{M}_\infty\, g_\eta(\cdot, u_\infty)\tau_\infty \;=\; \Big\langle \Psi_\infty, (\mathrm{dQ})_{u_\infty}\Big\rangle,$$

in $\Big(C_0^1\big(\overline{\Omega};\mathbb{R}^N\big)\Big)^*$, up to the identifications

$$\Big\langle \Psi_\infty, (\mathrm{dQ})_{u_\infty}\Big\rangle \equiv \Big\langle \Psi_\infty, (\mathrm{dQ})_{u_\infty}(\cdot)\Big\rangle, \quad g_\eta \equiv g_\eta \cdot (\cdot), \quad f_P \equiv (\cdot)\cdot f_P.$$

Note that in Theorem 3.1.3, the equations obtained depend on certain measures not a priori known explicitly. Therefore, their significance is understood to be largely theoretical, rather than computational. For the proof of this result, we will utilise some machinery developed in the recent paper [65] for some related work on generalised ∞-eigenvalue problems. The main points of this approach are recalled in the course of the proof, for the convenience of the reader.

We conclude this lengthy introduction with some comments concerning the composition of this paper. In Sections 3.2 and 3.3 we establish our main results, Theorems 3.1.1, 3.1.2 and 3.1.3. In Section 3.4 we provide a rather detailed list of explicit large classes of nonlinear operators Q to which our results apply.

3.2 Minimisers of L^p problems and convergence as $p \to \infty$

In this section we demonstrate Theorem 3.1.1. The proof is a consequence of the next two propositions, utilising the direct method of the Calculus of Variations.

Proposition 3.2.1. *In the setting of Theorem 3.1.1 and under the same assumptions, for any $p \in [\bar{p}, \infty)$, the functional F_p has a constrained minimiser $u_p \in \mathfrak{X}^p(\Omega)$, as claimed in (3.13).*

Proof of Proposition 3.2.1. Fix $p \geq \bar{p} > n$. We begin by illustrating that $\mathfrak{X}^p(\Omega) \neq \emptyset$. First note that by the compatibility condition (3.12), the finiteness of the infimum implies that $\mathrm{Q}^{-1}\big(\{0\}\big) \cap W_0^{1,\infty}(\Omega;\mathbb{R}^N) \neq \emptyset$. Further, there exists $u_0 \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ with $\mathrm{Q}(u_0) = 0$ such that $\mathrm{G}_\infty(u_0) < G$. Hence, by Hölder inequality we have

$$\mathrm{G}_p(u_0) \;=\; \bigg(-\!\!\!\!\!\!\int_\Omega g(\cdot, u_0, \mathrm{D}u_0)^p \,\mathrm{d}\mathcal{L}^n\bigg)^{1/p} \leq\; \Big\| g(\cdot, u_0, \mathrm{D}u_0)\Big\|_{L^\infty(\Omega)} \;=\; \mathrm{G}_\infty(u_0) \;<\; G.$$

Consequently, in view of (3.11), both constraints are satisfied by u_0, hence $u_0 \in \mathfrak{X}^p(\Omega) \neq \emptyset$. Next, note that f^p is a (Morrey) quasiconvex function. To see this, let $h : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ be an arbitrary quasiconvex function, in our case we will take $h(P) = f(x, \eta, P)$ for fixed (x, η). Then, by assumption (3.7), for any $\phi \in W_0^{1,\infty}(U; \mathbb{R}^N)$ with $U \Subset \mathbb{R}^n$ open and $P \in \mathbb{R}^{N\times n}$ fixed,

$$h(P) \leq -\!\!\!\!\!\!\int_U h(P + \mathrm{D}\phi)\, \mathrm{d}\mathcal{L}^n.$$

Hence, by Jensen's inequality and the convexity of $t \mapsto t^p$, we conclude that

$$h(P)^p \leq \left(-\!\!\!\!\!\!\int_U h(P + \mathrm{D}\phi)\, \mathrm{d}\mathcal{L}^n \right)^p \leq -\!\!\!\!\!\!\int_U h(P + \mathrm{D}\phi)^p\, \mathrm{d}\mathcal{L}^n.$$

We now proceed to bound f^p. By (3.6), we estimate

$$0 \leq f(x, \eta, P)^p \leq \mathrm{C}(x, \eta)^p \Big(1 + |P|^\alpha\Big)^p \leq 2^{p-1} \mathrm{C}(x, \eta)^p \Big(1 + |P|^{\alpha p}\Big).$$

By standard results (see [36]), F_p is weakly lower semicontinuous on $W_0^{1,\alpha p}(\Omega; \mathbb{R}^N)$. Let $(u^{(i)})_1^\infty \subseteq \mathfrak{X}^p(\Omega)$ denote a minimising sequence. By virtue of (3.6) we have $f \geq 0$, hence clearly $\inf_{i\in\mathbb{N}} \mathrm{F}_p(u^{(i)}) \geq 0$. We now show that the infimum is finite. To this aim, by (3.6) we estimate

$$\begin{aligned}
\inf_{i\in\mathbb{N}} \mathrm{F}_p(u^{(i)}) &\leq \mathrm{F}_p(u_0) \\
&= \left(-\!\!\!\!\!\!\int_\Omega f(\cdot, u_0, \mathrm{D}u_0)^p\, \mathrm{d}\mathcal{L}^n \right)^{1/p} \\
&\leq \left(-\!\!\!\!\!\!\int_\Omega \Big(\mathrm{C}(\cdot, u_0)(1 + |\mathrm{D}u_0|^\alpha) \Big)^p \mathrm{d}\mathcal{L}^n \right)^{1/p},
\end{aligned}$$

which yields

$$\begin{aligned}
\inf_{i\in\mathbb{N}} \mathrm{F}_p(u^{(i)}) &\leq \Big\| \mathrm{C}(\cdot, u_0)\Big(1 + |\mathrm{D}u_0|^\alpha\Big) \Big\|_{L^\infty(\Omega)} \\
&\leq \|\mathrm{C}(\cdot, u_0)\|_{L^\infty(\Omega)} \|(1 + |\mathrm{D}u_0|^\alpha)\|_{L^\infty(\Omega)} \\
&\leq \|\mathrm{C}(\cdot, u_0)\|_{L^\infty(\Omega)} \Big(1 + \|\mathrm{D}u_0\|^\alpha_{L^\infty(\Omega)}\Big) \\
&< \infty.
\end{aligned}$$

Hence, the infimum is indeed finite. Now we show that under assumption (3.8), the minimising sequence $(u^{(i)})_1^\infty$ is bounded in $W^{1,\alpha p}(\Omega; \mathbb{R}^N)$. Let h symbolise either f or g, whichever is coercive. Hence, since h is coercive, we have

$$-\!\!\!\!\!\!\int_\Omega \Big| h(\cdot, u^{(i)}, \mathrm{D}u^{(i)}) + C \Big|^p \mathrm{d}\mathcal{L}^n \geq c -\!\!\!\!\!\!\int_\Omega |\mathrm{D}u^{(i)}|^{\alpha p}\, \mathrm{d}\mathcal{L}^n.$$

By using the Poincaré and Hölder inequalities, we infer that

$$C + \left(⨍_{\Omega} |h(\cdot, u^{(i)}, \mathrm{D}u^{(i)})|^p \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}} \geq c' \, \|u^{(i)}\|^{\alpha}_{W^{1,\alpha p}(\Omega)}$$

for some new constant $c' > 0$ which is independent of $i \in \mathbb{N}$. If $h = f$, then by the previously derived estimates we have the uniform bound

$$\|u^{(i)}\|^{\alpha}_{W^{1,\alpha p}(\Omega)} \leq \frac{1}{c'}\Big(C + \mathrm{F}_p(u^{(i)}) \Big) \leq \frac{C}{c'} + \frac{1}{c'} \|\mathrm{C}(\cdot, u_0)\|_{L^\infty(\Omega)} \Big(1 + \|\mathrm{D}u_0\|^{\alpha}_{L^\infty(\Omega)} \Big),$$

and if $h = g$, then by the isosupremic constraint we have the uniform bound

$$\|u^{(i)}\|^{\alpha}_{W^{1,\alpha p}(\Omega)} \leq \frac{1}{c'}\Big(C + \mathrm{G}_p(u^{(i)}) \Big) \leq \frac{C + G}{c'}.$$

In either case, we have that $(u^{(i)})_1^\infty$ is weakly precompact in $W^{1,\alpha p}_0(\Omega;\mathbb{R}^N)$. By passing to a subsequence if necessary, standard strong and weak compactness arguments imply that there exists a map $u_p \in W^{1,\alpha p}_0(\Omega;\mathbb{R}^N)$ and a subsequence denoted again by $(u^{(i)})_1^\infty$ such that

$$\begin{cases} u^{(i)} \longrightarrow u_p, & \text{in } L^{\alpha p}(\Omega;\mathbb{R}^N), \\ \mathrm{D}u^{(i)} \rightharpoonup \mathrm{D}u_p, & \text{in } L^{\alpha p}(\Omega;\mathbb{R}^{N\times n}), \end{cases}$$

as $i \to \infty$. Further, since $p > n$, by the Morrey estimate we have that $(u^{(i)})_1^\infty$ is also bounded in $C^{0,\gamma}(\overline{\Omega}, \mathbb{R}^N)$ for $\gamma < 1 - n/(\alpha p)$. By the compact embedding of Hölder spaces, we conclude that

$$u^{(i)} \longrightarrow u_p \quad \text{in } C(\overline{\Omega};\mathbb{R}^N),$$

as $i \to \infty$. It remains to show that $\mathfrak{X}^p(\Omega)$ is weakly closed in $W^{1,\alpha p}_0(\Omega;\mathbb{R}^N)$. To this end, we need to show that the constraints determine a weakly closed subset of $W^{1,\alpha p}_0(\Omega;\mathbb{R}^N)$. Firstly note that by assumptions (3.5)-(3.7), G_p is a weakly lower semi-continuous functional on $W^{1,\alpha p}_0(\Omega;\mathbb{R}^N)$. This can be seen by an analogous argument to that used to show that F_p is weakly lower-continuous. Since $(u^{(i)})_1^\infty \subseteq \mathfrak{X}^p(\Omega)$, we have $\mathrm{G}_p(u^{(i)}) \leq G$ for all $i \in \mathbb{N}$. Therefore, since $u^{(i)} \rightharpoonup u_p$ in $W^{1,\alpha p}_0(\Omega;\mathbb{R}^N)$ as $i \to \infty$ and G_p is weakly lower-continuous, we infer that

$$\mathrm{G}_p(u_p) \leq \liminf_{i\to\infty} \mathrm{G}_p(u^{(i)}) \leq G.$$

Hence, $\mathrm{G}_p(u_p) \leq G$. Recall now that, in view of assumption (3.9), $\mathrm{Q}^{-1}(\{0\})$ is a weakly closed subset of $W^{1,\bar{p}}(\Omega;\mathbb{R}^N)$, where $\bar{p} > n$. We now show that $\mathrm{Q}^{-1}(\{0\})$ is also a weakly closed subset of $W^{1,\alpha p}_0(\Omega;\mathbb{R}^N)$, which will complete the claim that $\mathfrak{X}^p(\Omega)$ is weakly closed in $W^{1,\alpha p}_0(\Omega;\mathbb{R}^N)$. Note first that since $(u^{(i)})_1^\infty \subseteq \mathfrak{X}^p(\Omega)$, we have that $\mathrm{Q}(u^{(i)}) = 0$ for all $i \in \mathbb{N}$. Recalling that $u^{(i)} \longrightarrow u_p$ in $W^{1,\alpha p}_0(\Omega;\mathbb{R}^N)$ as $i \to \infty$ and that $\alpha p > \bar{p}$ because by assumption $\alpha > 1$, we deduce that $u^{(i)} \longrightarrow u_p$ in $W^{1,\bar{p}}(\Omega;\mathbb{R}^N)$ as well, as $i \to \infty$. Since $\mathrm{Q}^{-1}(\{0\})$ is weakly closed in $W^{1,\bar{p}}(\Omega;\mathbb{R}^N)$, we infer that $\mathrm{Q}(u_p) = 0$, as desired. Thus, for each $p \in [\bar{p}, \infty)$, $u_p \in \mathfrak{X}^p(\Omega)$, and

$$\mathrm{F}_p(u_p) \,\leq\, \liminf_{i\to\infty} \mathrm{F}_p(u^{(i)}) \,=\, \inf\Big\{\mathrm{F}_p \,:\, \mathfrak{X}^p(\Omega)\Big\}.$$

The proposition ensues. □

Our next result below establishes the existence of constrained minimisers for F_∞ and the approximation by minimisers of the F_p functionals as $p \to \infty$, therefore completing the proof of Theorem 3.1.1.

Proposition 3.2.2. *In the setting of Theorem 3.1.1 and under the same assumptions, the functional F_∞ has a constrained minimiser $u_\infty \in \mathfrak{X}^\infty(\Omega)$, as claimed in (3.13). Additionally, the claimed modes of convergence hold true for a subsequence of minimisers $(u_{p_j})_{j=1}^\infty$ as $j \to \infty$.*

Proof of Proposition 3.2.2. We continue from the proof of Proposition 3.2.1. The element $u_0 \in \mathfrak{X}^p(\Omega)$ provides an energy bound uniform in p, and also $u_0 \in \mathfrak{X}^\infty(\Omega)$. Fix $p, q > 1$ with $p \geq q \geq \bar{p}$. By the Hölder inequality, minimality and the definition of the constrained class, we have the estimates

$$\left\{\begin{array}{l} \mathrm{F}_q(u_p) \,\leq\, \mathrm{F}_p(u_p) \,\leq\, \mathrm{F}_p(u_0) \,\leq\, \|\mathrm{C}(\cdot, u_0)\|_{L^\infty(\Omega)}\Big(1 + \|\mathrm{D}u_0\|^\alpha_{L^\infty(\Omega)}\Big), \\ \mathrm{G}_q(u_p) \,\leq\, \mathrm{G}_p(u_p) \leq G, \end{array}\right.$$

with right hand side bounds which are uniform in p, q. We now argue in a similar fashion to that used in the proof of Proposition 3.2.1. We first show that under assumption (3.8), the family of minimisers $(u_p)_{p\geq\bar{p}}$ is bounded in $W^{1,q}(\Omega;\mathbb{R}^N)$, for any $q \in (1,\infty)$ fixed. Let h symbolise either f or g, whichever is coercive. We then have that

$$-\!\!\!\!\!\!\int_\Omega \Big|h(\cdot, u_p, \mathrm{D}u_p) + C\Big|^q \,\mathrm{d}\mathcal{L}^n \,\geq\, c -\!\!\!\!\!\!\int_\Omega |\mathrm{D}u_p|^{\alpha q} \,\mathrm{d}\mathcal{L}^n.$$

Since $\alpha > 1$, by the Hölder inequality we infer

$$C \,+\, \left(-\!\!\!\!\!\!\int_\Omega |h(\cdot, u_p, \mathrm{D}u_p)|^q \,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{q}} \,\geq\, c^{\frac{1}{q}} \,\|\mathrm{D}u_p\|^\alpha_{L^q(\Omega)},$$

for the constants $c, C > 0$ of (3.8) (which are independent of p and q). If $h = f$, by applying our earlier estimates we deduce the uniform bound

$$\|\mathrm{D}u_p\|^{\alpha}_{L^q(\Omega)} \,\leq\, \frac{1}{c}\Big(C\,+\,\mathrm{F}_q(u_p)\Big) \,\leq\, \frac{C}{c}\,+\,\frac{1}{c}\|\mathrm{C}(\cdot,u_0)\|_{L^\infty(\Omega)}\Big(1+\|\mathrm{D}u_0\|^{\alpha}_{L^\infty(\Omega)}\Big).$$

If $h = g$, then again as in our earlier estimates we have the uniform bound

$$\|\mathrm{D}u_p\|^{\alpha}_{L^q(\Omega)} \,\leq\, \frac{1}{c}\Big(C\,+\,\mathrm{G}_q(u_p)\Big) \,\leq\, \frac{C\,+\,G}{c}.$$

In either case, we see that under (3.8), our estimates above imply that

$$\|\mathrm{D}u_p\|_{L^q(\Omega)} \,\leq\, K,$$

for some constant $K > 0$ independent of p, q. Further, by the Poincaré inequality, we deduce that

$$\|u_p\|_{W^{1,q}(\Omega)} \,\leq\, K\Big(1+C(q)\Big),$$

where $C(q)$ is the constant of the Poincaré inequality in $W^{1,q}_0(\Omega;\mathbb{R}^N)$. Hence, the sequence of minimisers $(u_p)_{p\geq\bar{p}}$ is bounded in $W^{1,q}_0(\Omega;\mathbb{R}^N)$ for any fixed $q\in(1,\infty)$, and therefore it is weakly precompact in this collection of spaces. By a standard diagonal argument, there exists a sequence $(p_j)_1^\infty$ and a mapping

$$u_\infty \,\in \bigcap_{\bar{p}<q<\infty} W^{1,q}_0(\Omega;\mathbb{R}^n),$$

such that $u_p \longrightarrow u_\infty$ in $W^{1,q}_0(\Omega;\mathbb{R}^n)$ as $p_j\to\infty$, for any fixed $q\in(\bar{p},\infty)$. By standard compactness arguments in Sobolev and Hölder spaces, we infer that

$$\left\{\begin{array}{ll} u_p \longrightarrow u_\infty, & \text{in } C\Big(\overline{\Omega};\mathbb{R}^N\Big),\\ \mathrm{D}u_p \longrightarrow \mathrm{D}u_\infty, & \text{in } L^q(\Omega;\mathbb{R}^{N\times n}), \end{array}\right.$$

as $p_j\to\infty$, for any $q\in(\bar{p},\infty)$. We will now show that $u_\infty\in\mathfrak{X}^\infty(\Omega)$. In view of (3.11), we need to show that $u_\infty\in W^{1,\infty}_0(\Omega;\mathbb{R}^N)$ and that $\mathrm{G}_\infty(u_\infty)\leq G$ and also $\mathrm{Q}(u_\infty)=0$. By the weak lower semi-continuity of the L^q norm for $q\geq\bar{p}$ fixed, we have

$$\|\mathrm{D}u_\infty\|_{L^q(\Omega)} \,\leq\, \liminf_{p_j\to\infty}\|\mathrm{D}u_p\|_{L^q(\Omega)} \,\leq\, K.$$

By letting $q\to\infty$, this yields that $\mathrm{D}u_\infty\in L^\infty(\Omega;\mathbb{R}^N)$. By the Poincaré inequality in $W^{1,\infty}_0(\Omega;\mathbb{R}^N)$, we infer that $u_\infty\in W^{1,\infty}_0(\Omega;\mathbb{R}^N)$. Next, since $\mathrm{G}_p(u_p)\leq G$ for all $p\in(\bar{p},\infty)$, via the Hölder inequality and weak lower semi-continuity, we have

$$\mathrm{G}_\infty(u_\infty)=\lim_{q\to\infty}\mathrm{G}_q(u_\infty)\leq\liminf_{q\to\infty}\left(\liminf_{p_j\to\infty}\mathrm{G}_q(u_p)\right)\,\leq\,\liminf_{p_j\to\infty}\mathrm{G}_p(u_p)\,\leq\,G,$$

yielding that indeed $\mathrm{G}_\infty(u_\infty) \leq G$. We now show that $\mathrm{Q}(u_\infty) = 0$. We have already shown in Proposition 3.2.1 that $\mathrm{Q}^{-1}(\{0\})$ is a weakly closed subset of $W_0^{1,q}(\Omega;\mathbb{R}^N)$ for any $q \in (\bar{p},\infty)$. Since $\mathrm{Q}(u_p) = 0$ for all $p \geq \bar{p}$ and $u_p \rightharpoonup u_\infty$ in $W_0^{1,q}(\Omega;\mathbb{R}^N)$ as $p_j \to \infty$, we deduce that $\mathrm{Q}(u_\infty) = 0$, as desired.

It remains to show that u_∞ is indeed a minimiser of F_∞ in $\mathfrak{X}^\infty(\Omega)$, and additionally that the energies converge. Fix an arbitrary $u \in \mathfrak{X}^\infty(\Omega)$. By minimality and by noting that $\mathfrak{X}^\infty(\Omega) \subseteq \mathfrak{X}^p(\Omega)$ for any $p \in [\bar{p},\infty]$, we have the estimate

$$
\begin{aligned}
\mathrm{F}_\infty(u_\infty) &= \lim_{q\to\infty} \mathrm{F}_q(u_\infty) \\
&\leq \liminf_{q\to\infty} \left(\liminf_{p_j\to\infty} \mathrm{F}_q(u_p) \right) \\
&\leq \liminf_{p_j\to\infty} \mathrm{F}_p(u_p) \\
&\leq \limsup_{p_j\to\infty} \mathrm{F}_p(u_p) \\
&\leq \limsup_{p_j\to\infty} \mathrm{F}_p(u) \\
&= \mathrm{F}_\infty(u),
\end{aligned}
$$

for any $u \in \mathfrak{X}^\infty(\Omega)$. Hence u_∞ is a minimiser of F_∞ over $\mathfrak{X}^\infty(\Omega)$ and (3.4) follows. The choice $u = u_\infty$ in the above implies $\mathrm{F}_p(u_p) \longrightarrow \mathrm{F}_\infty(u_\infty)$, as $p_j \to \infty$. This completes the proof of the proposition and therefore of Theorem 3.1.1. □

3.3 The equations for constrained minimisers in L^p and in L^∞

In this section we establish the proofs of Theorem 3.1.2 and Theorem 3.1.3. The proof of the former is a relatively simple consequence of deep results in the generalised Kuhn-Tucker theory from [94], whilst the proof of the latter is based on applying an appropriate limiting process to the former result through compactness estimates.

Proof of Theorem 3.1.2. Fix $p \in [\bar{p},\infty)$. We begin with the simplifying observation that, the minimisation problem (3.13) can be rewritten as

$$
\frac{1}{p}\mathrm{F}_p(u_p)^p = \left\{ \frac{1}{p}\mathrm{F}_p(u)^p \;:\; u \in W_0^{1,\alpha p}(\Omega;\mathbb{R}^N),\ \frac{1}{p}\mathrm{G}_p(u)^p - \frac{G^p}{p} \leq 0 \ \&\ \mathrm{Q}(u) = 0 \right\}.
$$

This reformulation is a labour-saving device, drastically shortening the proof of this result. In view of assumption (3.16), first we will show that the following functionals are Fréchet differentiable

$$
\begin{aligned}
\frac{1}{p}(\mathrm{F}_p)^p &\;:\; W_0^{1,\alpha p}\big(\Omega;\mathbb{R}^N\big) \longrightarrow \mathbb{R},\\
\frac{1}{p}(\mathrm{G}_p)^p - \frac{G^p}{p} &\;:\; W_0^{1,\alpha p}\big(\Omega;\mathbb{R}^N\big) \longrightarrow \mathbb{R}.
\end{aligned}
$$

A direct computation gives the next formal expressions for their Gateaux derivatives

$$
\begin{aligned}
\left(\mathrm{d}\Big[\frac{1}{p}(\mathrm{F}_p)^p\Big]\right)_u(v) &= ⨍_\Omega f[u]^{p-1}\Big(f_\eta[u]\cdot v + f_P[u] : \mathrm{D}v\Big)\,\mathrm{d}\mathcal{L}^n,\\
\left(\mathrm{d}\Big[\frac{1}{p}(\mathrm{G}_p)^p - \frac{G^p}{p}\Big]\right)_u(v) &= ⨍_\Omega g[u]^{p-1}\Big(g_\eta[u]\cdot v + g_P[u] : \mathrm{D}v\Big)\,\mathrm{d}\mathcal{L}^n,
\end{aligned}
$$

for all $u, v \in W_0^{1,\alpha p}(\Omega;\mathbb{R}^N)$. We will now show the above formal expressions indeed define Fréchet derivatives, by employing relatively standard estimates through the Hölder inequality. We argue only for $\frac{1}{p}(\mathrm{F}_p)^p$, as the estimates for $\frac{1}{p}\Big((\mathrm{G}_p)^p - G^p\Big)$ are identical. Since $\alpha > 1$ and $p \geq \bar{p} > n$, by Morrey's estimate we have

$$
\begin{aligned}
&\left| ⨍_\Omega f[u]^{p-1}\Big(f_\eta[u]\cdot v + f_P[u] : \mathrm{D}v\Big)\,\mathrm{d}\mathcal{L}^n \right|\\
&\quad\leq ⨍_\Omega |f[u]|^{p-1}\Big(|f_\eta[u]||v| + |f_P[u]||\mathrm{D}v|\Big)\,\mathrm{d}\mathcal{L}^n\\
&\quad\leq ⨍_\Omega C(\cdot,u)^p\Big(1+|\mathrm{D}u|^\alpha\Big)^{p-1}\Big(1+|\mathrm{D}u|^{\alpha-1}\Big)(|v|+|\mathrm{D}v|)\,\mathrm{d}\mathcal{L}^n\\
&\quad\leq 2^p\Big\||v|C(\cdot,u)^p\Big\|_{L^\infty(\Omega)} ⨍_\Omega \Big(1+|\mathrm{D}u|^{\alpha-1}+|\mathrm{D}u|^{\alpha p-\alpha}+|\mathrm{D}u|^{\alpha p-1}\Big)\,\mathrm{d}\mathcal{L}^n\\
&\qquad + 2^p\Big\|C(\cdot,u)^p\Big\|_{L^\infty(\Omega)} ⨍_\Omega \Big(1+|\mathrm{D}u|^{\alpha-1}+|\mathrm{D}u|^{\alpha p-\alpha}+|\mathrm{D}u|^{\alpha p-1}\Big)|\mathrm{D}v|\,\mathrm{d}\mathcal{L}^n\\
&\quad\leq 2^p\Big\||v|C(\cdot,u)^p\Big\|_{L^\infty(\Omega)}\Big(1+\|\mathrm{D}u\|^{\alpha-1}_{L^{\alpha p}(\Omega)}+\|\mathrm{D}u\|^{\alpha p-\alpha}_{L^{\alpha p}(\Omega)}+\|\mathrm{D}u\|^{\alpha p-1}_{L^{\alpha p}(\Omega)}\Big)\\
&\qquad + 2^p\Big\|C(\cdot,u)^p\Big\|_{L^\infty(\Omega)}\Bigg(\|\mathrm{D}v\|_{L^{\alpha p}(\Omega)}\\
&\qquad + \sum_{t\in\{\alpha-1,\alpha p-\alpha,\alpha p-1\}} \|\mathrm{D}u\|^t_{L^{\frac{t\alpha p}{\alpha p-1}}(\Omega)}\|\mathrm{D}v\|_{L^{\alpha p}(\Omega)}\Bigg).
\end{aligned}
$$

Therefore the expression for the Gateaux derivative exists in L^1, for test functions in $W_0^{1,\alpha p}\big(\Omega;\mathbb{R}^N\big)$. We now demonstrate these functionals are Fréchet differentiable. Suppose

$\mathrm{F} \in C^2(\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n})$, with $\mathrm{F}[u] \equiv \mathrm{F}(\cdot, u, \mathrm{D}u)$, where

$$\mathrm{E}(u) = \fint_\Omega \mathrm{F}[u] \,\mathrm{d}\mathcal{L}^n,$$

and

$$(\mathrm{dE})_u(v) = \fint_\Omega \mathrm{F}_\eta[u] \cdot v + \mathrm{F}_P[u] : \mathrm{D}v \,\mathrm{d}\mathcal{L}^n,$$

for all $u, v \in W^{1,\alpha p}_0(\Omega; \mathbb{R}^N)$. As F is arbitrary we can choose $\mathrm{F} = f^p$ (to investigate our functionals of interest), such that f satisfies (3.14). We have

$$\begin{aligned}
\mathrm{F}_P &= p f^{p-1} f_P, \\
\mathrm{F}_{PP} &= p f^{p-2}\Big(f f_{PP} + (p-1) f_P \otimes f_P\Big).
\end{aligned}$$

By (3.14), it follows that

$$\begin{aligned}
|f_P(x,\eta,P)| &\leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha-1} + 1\Big), \\
|f(x,\eta,P)| &\leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha} + 1\Big),
\end{aligned}$$

for some new continuous functions at each step. Hence,

$$|\mathrm{F}(x,\eta,P)| \leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha p} + 1\Big).$$

Additionally,

$$\begin{aligned}
|\mathrm{F}_P(x,\eta,P)| &\leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha(p-1)} + 1\Big)\Big(|P|^{\alpha-1} + 1\Big) \\
&\leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha p-1} + 1\Big).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|\mathrm{F}_{PP}(x,\eta,P)| &\leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha(p-2)} + 1\Big)\Big[\Big(|P|^{\alpha} + 1\Big)\Big(|P|^{\alpha-2} + 1\Big) + \Big(|P|^{2(\alpha-1)} + 1\Big)\Big] \\
&\leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha p-2} + 1\Big).
\end{aligned}$$

The derivatives with respect to η are symmetric, thus

$$\begin{aligned}
|\mathrm{F}_\eta(x,\eta,P)| &\leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha p-1} + 1\Big), \\
|\mathrm{F}_{\eta\eta}(x,\eta,P)| &\leq \mathrm{C}(x,\eta)\Big(|P|^{\alpha p-2} + 1\Big).
\end{aligned}$$

We can produce comparable bounds for the mixed derivatives,

$$\mathrm{F}_{\eta P} = pf^{p-2}\Big((p-1)f_P f_\eta + f f_{\eta P}\Big)$$

and

$$\begin{aligned}|\mathrm{F}_{\eta P}(x,\eta,P)| &\le \mathrm{C}(x,\eta)\Big(|P|^{\alpha p}+1\Big)^{p-2}\Big[\Big(|P|^{\alpha-1}+1\Big)^2 + \Big(|P|^{\alpha}+1\Big)\Big(|P|^{\alpha-2}+1\Big)\Big]\\ &\le \mathrm{C}(x,\eta)\Big(|P|^{\alpha p-2}+1\Big).\end{aligned}$$

Analogously, we have

$$|\mathrm{F}_{P\eta}(x,\eta,P)| \le \mathrm{C}(x,\eta)\Big(|P|^{\alpha p-2}+1\Big).$$

Utilising these bounds, we intend to show that E is Fréchet differentiable, specifically

$$|\mathrm{E}(u+v) - \mathrm{E}(u) - (\mathrm{dE})_u(v)| \le o\left(\|v\|_{W^{1,\alpha p}(\Omega)}\right).$$

By the mean value theorem,

$$\begin{aligned}&|\mathrm{E}(u+v) - \mathrm{E}(u) - (\mathrm{dE})_u(v)| = \left| \fint_\Omega \mathrm{F}[u+v] - \mathrm{F}[u] - \Big(\mathrm{F}_\eta[u]\cdot v + \mathrm{F}_P[u] : \mathrm{D}v\Big)\right| \mathrm{d}\mathcal{L}^n\\ &= \left| \fint_\Omega \int_0^1 \mathrm{F}_\eta[u+\lambda v] - \mathrm{F}_\eta[u]\,\mathrm{d}\lambda\cdot v + \int_0^1 \mathrm{F}_P[u+\lambda v] - \mathrm{F}_P[u]\,\mathrm{d}\lambda : \mathrm{D}v\,\mathrm{d}\mathcal{L}^n\right|\\ &\le \fint_\Omega \left|\int_0^1 \mathrm{F}_\eta[u+\lambda v] - \mathrm{F}_\eta[u]\ \mathrm{d}\lambda\cdot v\right|\mathrm{d}\mathcal{L}^n + \fint_\Omega\left|\int_0^1 \mathrm{F}_P[u+\lambda v] - \mathrm{F}_P[u]\,\mathrm{d}\lambda : \mathrm{D}v\right|\mathrm{d}\mathcal{L}^n.\end{aligned}$$

Let us restrict our attention to the first integral,

$$\begin{aligned}&\fint_\Omega \left|\int_0^1 \mathrm{F}_\eta[u+\lambda v] - \mathrm{F}_\eta[u]\,\mathrm{d}\lambda\cdot v\right|\mathrm{d}\mathcal{L}^n\\ &\le \fint_\Omega\left|\int_0^1\int_0^1 \mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v]\cdot\lambda v + \mathrm{F}_{\eta P}[u+(\lambda+\mu)v] : \lambda\mathrm{D}v\ \mathrm{d}\mu\ \mathrm{d}\lambda\right| |v|\,\mathrm{d}\mathcal{L}^n\\ &\le \left(\fint_\Omega\left|\int_0^1\int_0^1 \mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v]\cdot\lambda v + \mathrm{F}_{\eta P}[u+(\lambda+\mu)v] : \lambda\mathrm{D}v\ \mathrm{d}\mu\ \mathrm{d}\lambda\right|^{\frac{\alpha p}{\alpha p-1}}\mathrm{d}\mathcal{L}^n\right)^{\frac{\alpha p-1}{\alpha p}}\\ &\quad\cdot \|v\|_{L^{\alpha p}(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\fint_{\Omega} \int_0^1 \int_0^1 |\mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v] \cdot \lambda v + \mathrm{F}_{\eta P}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v|^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \\
&\quad \cdot \|v\|_{L^{\alpha p}(\Omega)} \\
&\leq \left(\fint_{\Omega} \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v] \cdot \lambda v| + |\mathrm{F}_{\eta P}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \\
&\quad \cdot \|v\|_{L^{\alpha p}(\Omega)} \\
&\leq C_0 \Bigg(\fint_{\Omega} \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v] \cdot \lambda v| \Big)^{\frac{\alpha p}{\alpha p-1}} \\
&\quad + \Big(|\mathrm{F}_{\eta P}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \Bigg)^{\frac{\alpha p-1}{\alpha p}} \|v\|_{L^{\alpha p}(\Omega)} \\
&= C_0 \Bigg(\fint_{\Omega} \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v] \cdot \lambda v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \\
&\quad + \fint_{\Omega} \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta P}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \Bigg)^{\frac{\alpha p-1}{\alpha p}} \|v\|_{L^{\alpha p}(\Omega)} \\
&\leq C_1 \Bigg[\left(\fint_{\Omega} \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v] \cdot \lambda v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \\
&\quad + \left(\fint_{\Omega} \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta P}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \Bigg] \|v\|_{L^{\alpha p}(\Omega)}.
\end{aligned}
$$

We can ascertain an analogous bound for the second integral,

$$
\begin{aligned}
&\fint_{\Omega} \left| \int_0^1 \mathrm{F}_P[u+\lambda v] - \mathrm{F}_P[u] \,\mathrm{d}\lambda : \mathrm{D}v \right| \mathrm{d}\mathcal{L}^n \\
&\leq C_1 \Bigg[\left(\fint_{\Omega} \int_0^1 \int_0^1 \Big(|\mathrm{F}_{P\eta}[u+(\lambda+\mu)v] \cdot \lambda v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \\
&\quad + \left(\fint_{\Omega} \int_0^1 \int_0^1 \Big(|\mathrm{F}_{PP}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \Bigg] \\
&\quad \cdot \|\mathrm{D}v\|_{L^{\alpha p}(\Omega)}.
\end{aligned}
$$

Combining both of these we bounds, we obtain,

$$
\begin{aligned}
&\fint_\Omega \left| \int_0^1 \mathrm{F}_\eta[u+\lambda v] - \mathrm{F}_\eta[u]\,\mathrm{d}\lambda \cdot v \right| \mathrm{d}\mathcal{L}^n + \fint_\Omega \left| \int_0^1 \mathrm{F}_P[u+\lambda v] - \mathrm{F}_P[u]\,\mathrm{d}\lambda : \mathrm{D}v \right| \mathrm{d}\mathcal{L}^n \\
&\leq C_1 \Bigg[\left(\fint_\Omega \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v]\cdot\lambda v| \Big)^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \\
&\quad + \left(\fint_\Omega \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta P}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v| \Big)^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \Bigg] \|v\|_{L^{\alpha p}(\Omega)} \\
&\quad + C_1 \Bigg[\left(\fint_\Omega \int_0^1 \int_0^1 \Big(|\mathrm{F}_{P\eta}[u+(\lambda+\mu)v]\cdot\lambda v| \Big)^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \\
&\quad + \left(\fint_\Omega \int_0^1 \int_0^1 \Big(|\mathrm{F}_{PP}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v| \Big)^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \Bigg] \|\mathrm{D}v\|_{L^{\alpha p}(\Omega)}.
\end{aligned}
$$

We proceed to bound the first term,

$$
\begin{aligned}
&C_1 \left(\fint_\Omega \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v]\cdot\lambda v| \Big)^{\frac{\alpha p}{\alpha p-1}} \mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \|v\|_{L^{\alpha p}(\Omega)} \\
&\leq C_1 \left(\fint_\Omega \int_0^1 \int_0^1 \left(C(x,\eta) \left(\Big(|\mathrm{D}u| + |(\lambda+\mu)\mathrm{D}v| \Big)^{\alpha p-2} + 1 \right) \right)^{\frac{\alpha p}{\alpha p-1}} |v|^{\frac{\alpha p}{\alpha p-1}}\,\mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \\
&\quad \cdot \|v\|_{L^{\alpha p}(\Omega)} \\
&\leq \|c_1(\cdot,\eta)\|_{L^\infty(\Omega)} \left(\fint_\Omega \Big(|\mathrm{D}u|^{\alpha p-2} + |\mathrm{D}v|^{\alpha p-2} + 1 \Big)^{\frac{\alpha p}{\alpha p-1}} |v|^{\frac{\alpha p}{\alpha p-1}}\,\mathrm{d}\mu\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \|v\|_{L^{\alpha p}(\Omega)} \\
&\leq \|c_1(\cdot,\eta)\|_{L^\infty(\Omega)} \left(\fint_\Omega \Big(|\mathrm{D}u|^{\alpha p-2} + |\mathrm{D}v|^{\alpha p-2} + 1 \Big)^{\frac{\alpha pt}{\alpha p-1}}\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha pt}} \left(\fint_\Omega |v|^{\frac{t'\alpha p}{\alpha p-1}}\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha pt'}} \\
&\quad \cdot \|v\|_{L^{\alpha p}(\Omega)} \\
&\leq \|c_2(\cdot,\eta)\|_{L^\infty(\Omega)} \left(\fint_\Omega |\mathrm{D}u|^{\frac{(\alpha p-2)\alpha pt}{\alpha p-1}} + |\mathrm{D}v|^{\frac{(\alpha p-2)\alpha pt}{\alpha p-1}} + 1\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha pt}} \left(\fint_\Omega |v|^{\frac{t'\alpha p}{\alpha p-1}}\,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha pt'}} \\
&\quad \cdot \|v\|_{L^{\alpha p}(\Omega)} \\
&= \|c_2(\cdot,\eta)\|_{L^\infty(\Omega)} \left(\fint_\Omega |\mathrm{D}u|^{\frac{(\alpha p-2)\alpha pt}{\alpha p-1}}\,\mathrm{d}\mathcal{L}^n + \fint_\Omega |\mathrm{D}v|^{\frac{(\alpha p-2)\alpha pt}{\alpha p-1}}\,\mathrm{d}\mathcal{L}^n + \fint_\Omega \mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha pt}}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \left(\fint_\Omega |v|^{\frac{t'\alpha p}{\alpha p-1}} \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p t'}} \|v\|_{L^{\alpha p}(\Omega)} \\
&\le \|c_3(\cdot,\eta)\|_{L^\infty(\Omega)} \left[\left(\fint_\Omega |\mathrm{D}u|^{\frac{(\alpha p-2)\alpha p t}{\alpha p-1}} \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p t}} + \left(\fint_\Omega |\mathrm{D}v|^{\frac{(\alpha p-2)\alpha p t}{\alpha p-1}} \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p t}} + 1 \right] \\
&\cdot \left(\fint_\Omega |v|^{\frac{t'\alpha p}{\alpha p-1}} \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p t'}} \|v\|_{L^{\alpha p}(\Omega)}.
\end{aligned}
$$

Now select t, such that both the following inequalities hold

$$
\begin{cases}
\dfrac{\alpha p t(\alpha p-2)}{(\alpha p-1)} \le \alpha p, \\
\dfrac{\alpha p t}{(\alpha p-1)(t-1)} \le \alpha p.
\end{cases}
$$

Correspondingly, choosing $t = \frac{\alpha p-1}{\alpha p-2}$. Consequently,

$$
\begin{aligned}
&C_1 \left(\fint_\Omega \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v] \cdot \lambda v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \|v\|_{L^{\alpha p}(\Omega)} \\
&\le \|c_3(\cdot,\eta)\|_{L^\infty(\Omega)} \Big(\|\mathrm{D}u\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + \|\mathrm{D}v\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + 1 \Big) \|v\|^2_{L^{\alpha p}(\Omega)}.
\end{aligned}
$$

We can replicate this approach, for each of the four second derivatives, furthermore,

$$
\begin{aligned}
&|\mathrm{E}(u+v) - \mathrm{E}(u) - (\mathrm{dE})_u(v)| \\
&\le C_1 \left[\left(\fint_\Omega \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta\eta}[u+(\lambda+\mu)v] \cdot \lambda v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \right. \\
&\quad \left. + \left(\fint_\Omega \int_0^1 \int_0^1 \Big(|\mathrm{F}_{\eta P}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \right] \|v\|_{L^{\alpha p}(\Omega)} \\
&\quad + C_1 \left[\left(\fint_\Omega \int_0^1 \int_0^1 \Big(|\mathrm{F}_{P\eta}[u+(\lambda+\mu)v] \cdot \lambda v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \right. \\
&\quad \left. + \left(\fint_\Omega \int_0^1 \int_0^1 \Big(|\mathrm{F}_{PP}[u+(\lambda+\mu)v] : \lambda \mathrm{D}v| \Big)^{\frac{\alpha p}{\alpha p-1}} \,\mathrm{d}\mu \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{L}^n \right)^{\frac{\alpha p-1}{\alpha p}} \right] \|\mathrm{D}v\|_{L^{\alpha p}(\Omega)} \\
&\le \|c_3(\cdot,\eta)\|_{L^\infty(\Omega)} \Big(\|\mathrm{D}u\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + \|\mathrm{D}v\|^{\alpha p-2}_{L^{\alpha p}(\Omega)} + 1 \Big) \|v\|^2_{L^{\alpha p}(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&+\|c_3(\cdot,\eta)\|_{L^\infty(\Omega)}\Big(\|\mathrm{D}u\|^{\alpha p-2}_{L^{\alpha p}(\Omega)}+\|\mathrm{D}v\|^{\alpha p-2}_{L^{\alpha p}(\Omega)}+1\Big)\|\mathrm{D}v\|_{L^{\alpha p}(\Omega)}\|v\|_{L^{\alpha p}(\Omega)}\\
&+\|c_3(\cdot,\eta)\|_{L^\infty(\Omega)}\Big(\|\mathrm{D}u\|^{\alpha p-2}_{L^{\alpha p}(\Omega)}+\|\mathrm{D}v\|^{\alpha p-2}_{L^{\alpha p}(\Omega)}+1\Big)\|v\|_{L^{\alpha p}(\Omega)}\|\mathrm{D}v\|_{L^{\alpha p}(\Omega)}\\
&+\|c_3(\cdot,\eta)\|_{L^\infty(\Omega)}\Big(\|\mathrm{D}u\|^{\alpha p-2}_{L^{\alpha p}(\Omega)}+\|\mathrm{D}v\|^{\alpha p-2}_{L^{\alpha p}(\Omega)}+1\Big)\|\mathrm{D}v\|^2_{L^{\alpha p}(\Omega)}\\
=\;&\|c_3(\cdot,\eta)\|_{L^\infty(\Omega)}\Big(\|\mathrm{D}u\|^{\alpha p-2}_{L^{\alpha p}(\Omega)}+\|\mathrm{D}v\|^{\alpha p-2}_{L^{\alpha p}(\Omega)}+1\Big)\\
&\cdot\Big(\|v\|^2_{L^{\alpha p}(\Omega)}+2\|\mathrm{D}v\|_{L^{\alpha p}(\Omega)}\|v\|_{L^{\alpha p}(\Omega)}+\|\mathrm{D}v\|^2_{L^{\alpha p}(\Omega)}\Big)\\
=\;&\|c_3(\cdot,\eta)\|_{L^\infty(\Omega)}\Big(\|\mathrm{D}u\|^{\alpha p-2}_{L^{\alpha p}(\Omega)}+\|\mathrm{D}v\|^{\alpha p-2}_{L^{\alpha p}(\Omega)}+1\Big)\|v\|^2_{W^{1,\alpha p}(\Omega)}\\
\le\;& o\Big(\|v\|_{W^{1,\alpha p}(\Omega)}\Big).
\end{aligned}
$$

This estimate establishes that the functional $\frac{1}{p}(\mathrm{F}_p)^p$ (and therefore $\frac{1}{p}\Big((\mathrm{G}_p)^p-G^p\Big)$) is indeed Fréchet differentiable.

We now show that the equations that the constrained minimiser satisfies take the form as given in (3.20) and (3.21). Given the Fréchet derivatives and our assumption (3.17) on the range of dQ, we can invoke the generalised Kuhn-Tucker theory. By applying [94, Theorem 48.B, p.p. 416-417] with (in the book's notation)

$$
\begin{aligned}
&F_0:=\frac{1}{p}(\mathrm{F}_p)^p,\quad F_1:=\frac{1}{p}(\mathrm{G}_p)^p-\frac{G^p}{p},\quad F_3:=\mathrm{Q},\\
&X=U=N_2:=W^{1,\alpha p}_0\Big(\Omega;\mathbb{R}^N\Big),\quad Y:=\mathbf{E},\quad n=2,
\end{aligned}
$$

and by noting that N_2 herein is the entire (vector) space $W^{1,\alpha p}_0\Big(\Omega;\mathbb{R}^N\Big)$, we readily infer the claims made in (3.18)-(3.21). The result ensues. $\square$

Now we establish our last main result.

Proof of Theorem 3.1.3. The proof is divided into several steps.

Step 1. We first confirm that the measures defined by (3.22) are indeed finite, and show that their total variations are bounded uniformly in $p\in(\bar{p},\infty)$. This will imply the convergence modes of (3.36) for some limiting $\mu_\infty,\nu_\infty\in\mathcal{M}(\overline{\Omega})$ along a subsequence $(p_j)_1^\infty$ as $j\to\infty$, as a consequence of the sequential weak* precompactness of bounded sets in the space of Radon measures. Indeed, if $\mathrm{F}_p(u_p) > 0$, then since $f \geq 0$ we have

$$
\begin{aligned}
\|\sigma_p\|(\overline{\Omega}) \ &= \ \sigma_p(\overline{\Omega}) \\
&= \ \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega \frac{f(\cdot, \mathrm{D}u_p)^{p-1}}{\mathrm{F}_p(u_p)^{p-1}} \, \mathrm{d}\mathcal{L}^n \\
&= \ \frac{1}{\mathrm{F}_p(u_p)^{p-1}} \ -\!\!\!\!\!\!\int_\Omega f(\cdot, \mathrm{D}u_p)^{p-1} \, \mathrm{d}\mathcal{L}^n \\
&\leq \ \frac{1}{\mathrm{F}_p(u_p)^{p-1}} \bigg(-\!\!\!\!\!\!\int_\Omega f(\cdot, \mathrm{D}u_p)^p \, \mathrm{d}\mathcal{L}^n \bigg)^{\frac{p-1}{p}} \\
&= \ 1,
\end{aligned}
$$

whilst if $\mathrm{F}_p(u_p) = 0$, then trivially $\|\sigma_p\|(\overline{\Omega}) = 0$. In both cases, $\|\sigma_p\|(\overline{\Omega}) \leq 1$ for all $p \in (\bar{p}, \infty)$. The estimate for $\|\tau_p\|(\overline{\Omega})$ is completely analogous, yielding $\|\tau_p\|(\overline{\Omega}) \leq 1$ for all $p \in (\bar{p}, \infty)$.

Step 2. By using assumption (3.29) and definition (3.22), we have the following differential identity: for any fixed $v \in C^1_0\big(\overline{\Omega}; \mathbb{R}^N\big)$ and any $p \in (\bar{p}, \infty)$ we have

$$
\begin{aligned}
\int_\Omega f(\cdot, \mathrm{D}v - \mathrm{D}u_p) \, \mathrm{d}\sigma_p \ &= \int_\Omega f(\cdot, \mathrm{D}v) \, \mathrm{d}\sigma_p - \int_\Omega f(\cdot, \mathrm{D}u_p) \, \mathrm{d}\sigma_p \\
&\quad + \int_\Omega f_P(\cdot, \mathrm{D}u_p) : (\mathrm{D}u_p - \mathrm{D}v) \, \mathrm{d}\sigma_p.
\end{aligned}
$$

Indeed, by using that $f_P(x, P) \ = \ \mathbf{A}(x) : \big((\cdot) \otimes P + P \otimes (\cdot)\big)$, we may compute

$$
\begin{aligned}
\int_\Omega f(\cdot, \mathrm{D}v - \mathrm{D}u_p) \, \mathrm{d}\sigma_p \ &= \ \int_\Omega \mathbf{A} : (\mathrm{D}v - \mathrm{D}u_p) \otimes (\mathrm{D}v - \mathrm{D}u_p) \, \mathrm{d}\sigma_p \\
&= \int_\Omega \mathbf{A} : \mathrm{D}v \otimes \mathrm{D}v \, \mathrm{d}\sigma_p \ - \int_\Omega \mathbf{A} : \mathrm{D}u_p \otimes \mathrm{D}u_p \, \mathrm{d}\sigma_p \\
&\quad + \int_\Omega \mathbf{A} : \Big((\mathrm{D}u_p - \mathrm{D}v) \otimes \mathrm{D}u_p + \mathrm{D}u_p \otimes (\mathrm{D}u_p - \mathrm{D}v) \Big) \, \mathrm{d}\sigma_p \\
&= \int_\Omega f(\cdot, \mathrm{D}v) \, \mathrm{d}\sigma_p - \int_\Omega f(\cdot, \mathrm{D}u_p) \, \mathrm{d}\sigma_p \ + \int_\Omega f_P(\cdot, \mathrm{D}u_p) : (\mathrm{D}u_p - \mathrm{D}v) \, \mathrm{d}\sigma_p.
\end{aligned}
$$

We also note that the above established identity holds true over $\overline{\Omega}$ as well, because $\sigma_p(\partial\Omega) = \tau_p(\partial\Omega) = 0$.

Step 3. For any fixed $p \in (\bar{p}, \infty)$, by using (3.22)-(3.26) and (3.29), we may rewrite (3.20) (obtained in Theorem 3.1.2) as

$$
\Lambda_p \int_{\overline{\Omega}} f_P(\cdot, \mathrm{D}u_p) : \mathrm{D}\phi \, \mathrm{d}\sigma_p \ + \ \mathrm{M}_p \int_{\overline{\Omega}} g_\eta(\cdot, u_p) \cdot \phi \, \mathrm{d}\tau_p \ = \ \Big\langle \Psi_p, (\mathrm{dQ})_{u_p}(\phi) \Big\rangle,
$$

for all test maps $\phi \in W_0^{1,\alpha p}(\Omega;\mathbb{R}^N)$, whilst we also have that

$$\Lambda_p \in [0,1], \ \ \mathrm{M}_p \in [0,1] \ \text{ and } \ \Psi_p \in \bar{\mathbb{B}}_1^{\mathbf{E}^*}(0).$$

Further, by assumption (3.27), the weak* topology of the dual space $\mathbf{E}^*$ is sequentially (pre-) compact on bounded sets. Thus, the previous memberships imply that, upon passing to a further subsequence as $j \to \infty$, symbolised again by $(p_j)_1^\infty$, there exist

$$\Lambda_\infty \in [0,1], \ \ \mathrm{M}_\infty \in [0,1] \ \text{ and } \ \Psi_\infty \in \bar{\mathbb{B}}_1^{\mathbf{E}^*}(0),$$

such that the modes of convergence (3.35) hold true as $p_j \to \infty$.

Step 4. By Steps 2 and 3, for $\phi := u_p - v$, where $v \in C_0^1\big(\overline{\Omega};\mathbb{R}^N\big)$ is an arbitrary fixed map, for any fixed $p \in (\bar{p},\infty)$ we have the identity

$$\begin{aligned}\Lambda_p \int_{\overline{\Omega}} f(\cdot,\mathrm{D}u_p - \mathrm{D}v)\,\mathrm{d}\sigma_p = &\ \Big\langle \Psi_p, (\mathrm{dQ})_{u_p}(u_p - v)\Big\rangle \\ &- \mathrm{M}_p \int_{\Omega} g_\eta(\cdot,u_p)\cdot(u_p - v)\,\mathrm{d}\tau_p \,+\, \Lambda_p\bigg(\int_{\Omega} f(\cdot,\mathrm{D}v)\,\mathrm{d}\sigma_p - \int_{\Omega} f(\cdot,\mathrm{D}u_p)\,\mathrm{d}\sigma_p\bigg).\end{aligned}$$

Step 5. For any fixed $p \in (\bar{p},\infty)$ and $v \in C_0^1\big(\overline{\Omega};\mathbb{R}^N\big)$, we have the relations

$$\int_{\overline{\Omega}} f(\cdot,\mathrm{D}u_p - \mathrm{D}v)\,\mathrm{d}\sigma_p \,\geq\, \alpha_0 \int_{\overline{\Omega}} \Big|\mathrm{D}u_p - \mathrm{D}v\Big|^2\,\mathrm{d}\sigma_p \tag{3.37}$$

and

$$\int_{\overline{\Omega}} f(\cdot,\mathrm{D}u_p)\,\mathrm{d}\sigma_p \,=\, \mathrm{F}_p(u_p), \tag{3.38}$$

where we have symbolised

$$\alpha_0 \,:=\, \min_{x\in\overline{\Omega}}\bigg\{\min_{|Q|=1}\mathbf{A}(x) : Q\otimes Q\bigg\} \,>\, 0.$$

Let us first establish (3.38), beginning with the case that $\mathrm{F}_p(u_p) > 0$. By definition (3.22) and assumption (3.29), we may compute

$$\begin{aligned}\int_{\Omega} f(\cdot,\mathrm{D}u_p)\,\mathrm{d}\sigma_p \,&=\, \frac{1}{\mathcal{L}^n(\Omega)}\int_{\Omega} f(\cdot,\mathrm{D}u_p)\frac{f(\cdot,\mathrm{D}u_p)^{p-1}}{\mathrm{F}_p(u_p)^{p-1}}\,\mathrm{d}\mathcal{L}^n \\ &=\, \frac{1}{\mathrm{F}_p(u_p)^{p-1}} ⨍_{\Omega} f(\cdot,\mathrm{D}u_p)^p\,\mathrm{d}\mathcal{L}^n \\ &=\, \mathrm{F}_p(u_p),\end{aligned}$$

whilst for $\mathrm{F}_p(u_p) = 0$ the equality follows trivially. To establish (3.37), it suffices to note that by assumption (3.29) and by the variational representation of the minimum eigenvalue of the symmetric linear operator $\mathbf{A}(x) : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}^{N\times n}$, we have that $\alpha_0 > 0$ and the inequality

$$\alpha_0 |Q|^2 \leq \mathbf{A}(x) : Q \otimes Q$$

for all $x \in \overline{\Omega}$ and all $Q \in \mathbb{R}^{N\times n}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^{N\times n}$.

Step 6. By Steps 1, 3 and 5, and by using that $\mathrm{F}_p(u_p) \longrightarrow \mathrm{F}_\infty(u_\infty)$ as $p_j \to \infty$ (as shown in Theorem 3.1.1), we may invoke Hutchinson's theory of measure-function pairs, in particular [57, Sec. 4, Def. 4.1.1, 4.1.2, 4.2.1 and Th. 4.4.2], to infer that there exists a map

$$V_\infty \in L^2\Big(\overline{\Omega}, \sigma_\infty; \mathbb{R}^{N\times n}\Big)$$

such that, along perhaps a further subsequence $(p_j)_1^\infty$ we have

$$\mathrm{D}u_p \sigma_p \overset{*}{\rightharpoonup} V_\infty \sigma_\infty \ \text{ in } \mathcal{M}\Big(\overline{\Omega}; \mathbb{R}^{N\times n}\Big),$$

as $p_j \to \infty$, with the property that

$$\int_{\overline{\Omega}} \Phi(\cdot, V_\infty)\, \mathrm{d}\sigma_\infty \leq \liminf_{p_j\to\infty} \int_{\overline{\Omega}} \Phi(\cdot, \mathrm{D}u_p)\, \mathrm{d}\sigma_p$$

for any $\Phi \in C\Big(\overline{\Omega} \times \mathbb{R}^{N\times n}\Big)$ such that $\Phi(x,\cdot)$ is convex and of quadratic growth, for all $x \in \overline{\Omega}$. Further, in view of assumptions (3.16), (3.17), (3.28), (3.29), and the modes of convergence established in Theorem 3.1.1 together with the convergence $\Psi_p \overset{*}{\rightharpoonup} \Psi_\infty$ in $\mathbf{E}^*$ as $p_j \to \infty$, we have that

$$\begin{aligned} \Big\langle \Psi_p, (\mathrm{dQ})_{u_p}(\phi)\Big)\Big\rangle &\longrightarrow \Big\langle \Psi_\infty, (\mathrm{dQ})_{u_\infty}(\phi)\Big\rangle \quad \text{ in } \mathbb{R}, \\ \Big(f_P(\cdot, \mathrm{D}u_p) : \mathrm{D}\phi\Big)\sigma_p &\overset{*}{\rightharpoonup} \Big(f_P(\cdot, V_\infty) : \mathrm{D}\phi\Big)\sigma_\infty \ \text{ in } \mathcal{M}\Big(\overline{\Omega}\Big), \end{aligned}$$

for any fixed $\phi \in C^1_0\Big(\overline{\Omega}; \mathbb{R}^N\Big) \subseteq W^{1,\bar{p}}_0(\Omega; \mathbb{R}^N)$. Hence, we may let $p_j \to \infty$ in Step 3, to deduce the equation

$$\Lambda_\infty \int_{\overline{\Omega}} f_P(\cdot, V_\infty) : \mathrm{D}\phi\, \mathrm{d}\sigma_\infty + \mathrm{M}_\infty \int_{\overline{\Omega}} g_\eta(\cdot, u_\infty)\cdot \phi\, \mathrm{d}\tau_\infty = \Big\langle \Psi_\infty, (\mathrm{dQ})_{u_\infty}(\phi)\Big\rangle,$$

for any fixed $\phi \in C^1_0\Big(\overline{\Omega}; \mathbb{R}^N\Big)$. Further, by letting $p_j \to \infty$ in (3.21) we deduce that

$$\mathrm{M}_\infty\Big(\mathrm{G}_\infty(u_\infty) - G\Big) = 0.$$

Step 7. The equations established in Step 6 will complete the proof of the theorem, upon establishing that

$$V_\infty \,=\, \mathrm{D}u^\star_\infty \quad \sigma_\infty\text{-a.e. on } \overline{\Omega},$$

where $\mathrm{D}u^\star_\infty : \overline{\Omega} \longrightarrow \mathbb{R}^{N\times n}$ is some Borel measurable mapping which is a version of $\mathrm{D}u_\infty \in L^\infty(\Omega;\mathbb{R}^{N\times n})$, namely such that

$$\mathrm{D}u_\infty \,=\, \mathrm{D}u^\star_\infty \quad \mathcal{L}^n\text{-a.e. on } \overline{\Omega},$$

(recall that $\partial\Omega$ is a nullset for the Lebesgue measure $\mathcal{L}^n$). The remaining steps are devoted to establishing this claim, together with the approximability properties claimed in (3.34) for some sequence of mappings $(v_j)_1^\infty \subseteq C^1_0\big(\overline{\Omega};\mathbb{R}^N\big)$, which will be constructed explicitly.

Step 8. If $\Lambda_\infty = 0$, then Step 6 completes the proof of Theorem 3.1.3 as the first term involving V_∞ vanishes. Hence, we may henceforth assume that $\Lambda_\infty > 0$. Therefore, by passing perhaps to a further subsequence if necessary, we may assume that

$$\Lambda_{p_j} \,\geq\, \frac{\Lambda_\infty}{2} \,>\, 0, \text{ for all } j \in \mathbb{N}.$$

Step 9. By Steps 3, 4, 5 and 8, the absolute continuity $\tau_p << \mathcal{L}^n \llcorner_\Omega$ and the bounds $0 \leq \mathrm{M}_p \leq 1$ and $0 \leq \tau_p\big(\overline{\Omega}\big) \leq 1$, we have the estimate

$$\begin{aligned}
\frac{\alpha_0\Lambda_\infty}{2}\int_{\overline{\Omega}}\big|\mathrm{D}u_p - \mathrm{D}v\big|^2\,\mathrm{d}\sigma_p \;\leq\; & \Big\langle \Psi_p,\, (\mathrm{dQ})_{u_p}(u_p) - (\mathrm{dQ})_{u_p}(v)\Big\rangle \\
& +\, \|u_p - v\|_{L^\infty(\Omega)}\Big\{\sup_{j\in\mathbb{N}}\big\|g_\eta(\cdot,u_{p_j})\big\|_{L^\infty(\Omega)}\Big\} \\
& +\, \Lambda_p\bigg(\int_{\overline{\Omega}} f(\cdot,\mathrm{D}v)\,\mathrm{d}\sigma_p \,-\, \mathrm{F}_p(u_p)\bigg),
\end{aligned}$$

along the sequence $(p_j)_1^\infty$. By letting $j \to \infty$ in the above estimate, in view of Steps 1, 3 and 6 (for the choice $\Phi(x,Q) := |\mathrm{D}v(x) - Q|^2$) and assumption (3.28), we infer that

$$\begin{aligned}
\frac{\alpha_0\Lambda_\infty}{2}\int_{\overline{\Omega}}\big|V_\infty - \mathrm{D}v\big|^2\,\mathrm{d}\sigma_\infty \;\leq\; & \Big\langle \Psi_\infty,\, (\mathrm{dQ})_{u_\infty}(u_\infty) - (\mathrm{dQ})_{u_\infty}(v)\Big\rangle \\
& +\, \|u_\infty - v\|_{L^\infty(\Omega)}\Big\{\sup_{j\in\mathbb{N}}\big\|g_\eta(\cdot,u_{p_j})\big\|_{L^\infty(\Omega)}\Big\} \\
& +\, \Lambda_\infty\bigg(\int_{\overline{\Omega}} f(\cdot,\mathrm{D}v)\,\mathrm{d}\sigma_\infty \,-\, \mathrm{F}_\infty(u_\infty)\bigg),
\end{aligned}$$

for any fixed mapping $v \in C^1_0\big(\overline{\Omega};\mathbb{R}^N\big)$.

Step 10. Let $(v_j)_1^\infty \subseteq C^1_0\big(\overline{\Omega};\mathbb{R}^N\big)$ be any sequence of mappings satisfying the assumptions in (3.34). We claim that there exists a subsequence of indices $(j_k)_1^\infty$ such that

$$\left\{\begin{array}{lll} \mathrm{D}v_j \longrightarrow V_\infty, & \text{in } L^2\big(\overline{\Omega},\sigma_\infty;\mathbb{R}^{N\times n}\big), & \text{and } \sigma_\infty\text{-a.e. on } \overline{\Omega}, \\ \mathrm{D}v_j \longrightarrow \mathrm{D}u_\infty, & \text{in } L^q\big(\Omega;\mathbb{R}^{N\times n}\big),\ q\in[1,\infty), & \text{and } \mathcal{L}^n\text{-a.e. on } \Omega, \end{array}\right.$$

as $j_k \to \infty$. Therefore, if $\mathrm{D}u^\star_\infty$ is defined as in (3.34), then from the above we infer

$$\begin{cases} \mathrm{D}u^\star_\infty \,=\, V_\infty, & \sigma_\infty\text{-a.e. on } \overline{\Omega}, \\ \mathrm{D}u^\star_\infty \,=\, \mathrm{D}u_\infty, & \mathcal{L}^n\text{-a.e. on } \Omega, \end{cases}$$

which completes the proof (subject to showing that at least one sequence of mapping $(v_j)_1^\infty$ with the desired properties exists). Let us now establish the above claims. If $(v_j)_1^\infty$ satisfies (3.34), then by Step 5 and the resulting bound

$$\alpha_0\|\mathrm{D}v_j\|^2_{L^\infty(\Omega)} \,\leq\, \mathrm{F}_\infty(v_j) \,\leq\, \mathrm{F}_\infty(u_\infty) \,+\, o(1)_{j\to\infty}$$

in conjunction with the Vitali convergence theorem, it follows that

$$\left\{\begin{array}{ll} v_j \longrightarrow u_\infty, & \text{in } L^\infty(\Omega;\mathbb{R}^N), \\ \mathrm{D}v_j \longrightarrow \mathrm{D}u_\infty, & \text{in } L^q(\Omega;\mathbb{R}^{N\times n}),\ q\in[1,\infty), \end{array}\right.$$

along a subsequence of indices $(j_k)_1^\infty$ as $k\to\infty$. Consequently, by the estimate of Step 9, we have

$$\begin{split} \frac{\alpha_0\Lambda_\infty}{2}\int_{\overline{\Omega}}\Big|V_\infty - \mathrm{D}v_j\Big|^2\mathrm{d}\sigma_\infty \,\leq\, & \Big\langle \Psi_\infty,\,(\mathrm{dQ})_{u_\infty}(u_\infty) - (\mathrm{dQ})_{u_\infty}(v_j)\Big\rangle \\ & +\,\|u_\infty - v_j\|_{L^\infty(\Omega)}\bigg\{\sup_{j\in\mathbb{N}}\Big\|g_\eta(\cdot,u_{p_j})\Big\|_{L^\infty(\Omega)}\bigg\} \\ & +\,\Lambda_\infty\bigg(\Big\{\sup_{\overline{\Omega}} f(\cdot,\mathrm{D}v_j)\Big\}\,\sigma_\infty\big(\overline{\Omega}\big) \,-\, \mathrm{F}_\infty(u_\infty)\bigg). \end{split}$$

Since by Step 1 we have $0\leq\sigma_\infty\big(\overline{\Omega}\big)\leq 1$, by noting that $\sup_{\overline{\Omega}} f(\cdot,\mathrm{D}v_j) = \mathrm{F}_\infty(v_j)$ (due to the continuity of $\mathrm{D}v_j$ on $\overline{\Omega}$) and that $v_j \longrightarrow u_\infty$ strongly in $W^{1,\bar p}_0(\Omega;\mathbb{R}^N)$, from the last estimate and assumption (3.34) we deduce that

$$\limsup_{j_k\to\infty}\int_{\overline{\Omega}}\Big|V_\infty - \mathrm{D}v_j\Big|^2\mathrm{d}\sigma_\infty \,\leq\, 0.$$

Step 11. To complete the proof of Theorem 3.1.3, it remains to show that at least one sequence of mapping $(v_j)_1^\infty \subseteq C_0^1\big(\overline{\Omega};\mathbb{R}^N\big)$ exists, which satisfies the modes of convergence required by (3.34). To this end we utilise (for the first time) the assumption that the bounded domain Ω has Lipschitz boundary $\partial\Omega$, and we invoke the regularisation scheme introduced and utilised in the recent paper [65]. This method is based on results on the geometry of Lipschitz domains proved in Hofmann-Mitrea-Taylor [56], and is inspired by the regularisation schemes employed in Ern-Guermond [40]. If $\mathrm{n} \in L^\infty(\partial\Omega,\mathcal{H}^{n-1};\mathbb{R}^n)$ be the outer unit normal vector field on $\partial\Omega$, by [56, Sec. 2, 4], there exists a smooth vector field $\xi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ which is globally transversal to n on $\partial\Omega$ with respect to the surface measure, namely exists $c>0$ such that

$$\xi\cdot \mathrm{n} \geq c, \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega.$$

Additionally, ξ can be chosen to satisfy $|\xi|\equiv 1$ in an open tubular neighbourhood $\{\mathrm{dist}(\cdot,\partial\Omega) < r\}$ around $\partial\Omega$ for some $r>0$, whilst vanishing on $\{\mathrm{dist}(\cdot,\partial\Omega) > 2r\}$. (In the special case that $\partial\Omega$ happens to be a compact C^∞ manifold, then we can simply choose ξ to be a smooth extension of n, and the transversality condition is satisfied with $c=1$.) Further, it can be shown that there exists $\varepsilon_0,\ell>0$ such that, for all $\varepsilon\in(0,\varepsilon_0)$ we have

$$\inf_{x\in\partial\Omega} \mathrm{dist}\Big(x+\varepsilon\ell\xi(x),\, \partial\Omega\Big) \geq\, 2\varepsilon.$$

We now define our adapted global mollifiers, taken from [65]. Let us select any function $\varrho \in C_c^\infty(\mathbb{B}_1(0))$ which satisfies $\varrho\geq 0$ and $\|\varrho\|_{L^1(\mathbb{R}^n)} = 1$ (for instance the "standard" mollifying kernel as in [73]). For any $v\in L^\infty(\Omega;\mathbb{R}^N)$, extended to $\mathbb{R}^n\setminus\Omega$ by zero, we define for any $\varepsilon\in(0,\varepsilon_0)$ the map $\mathrm{K}^\varepsilon v:\mathbb{R}^n \longrightarrow \mathbb{R}^N$, by setting

$$(\mathrm{K}^\varepsilon v)(x) \,:=\, \int_{\mathbb{R}^n} v\Big(x+\varepsilon\ell\xi(x)-\varepsilon y\Big)\,\varrho(y)\,\mathrm{d}y.$$

Then, by [65, Prop. 12, p. 18], for any $u\in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ and $\varepsilon\in(0,\varepsilon_0)$:

- We have that $\mathrm{K}^\varepsilon u \in C_0^\infty\Big(\Omega;\mathbb{R}^N\Big)$, and the identity

$$\mathrm{D}(\mathrm{K}^\varepsilon u) \,=\, \mathrm{K}^\varepsilon(\mathrm{D}u) \,+\, \varepsilon\ell(\mathrm{K}^\varepsilon(\mathrm{D}u))(\mathrm{D}\xi)^\top,$$

everywhere on $\overline{\Omega}$.

- We have $\mathrm{K}^\varepsilon u \longrightarrow u$ in $W_0^{1,q}(\Omega;\mathbb{R}^N)$ for all $q\in[1,\infty)$, and in $C^\gamma(\overline{\Omega};\mathbb{R}^N)$ for all $\gamma\in(0,1)$, as $\varepsilon\to 0$. Further, $\mathrm{K}^\varepsilon u \overset{*}{\rightharpoonup} u$ in $W_0^{1,\infty}(\Omega;\mathbb{R}^N)$, as $\varepsilon\to 0$.

• For any $\Theta \in C\Big(\overline{\Omega} \times \mathbb{R}^{N\times n}\Big)$, satisfying for any $x \in \overline{\Omega}$ that $\Theta(x,\cdot)$ is convex on $\mathbb{R}^{N\times n}$ with $\Theta(x,\cdot) \geq \Theta(x,0) = 0$, and also that the partial derivative Θ_P exists and is continuous on $\overline{\Omega} \times \mathbb{R}^{N\times n}$, we will show that there exists a modulus of continuity $\omega \in C\Big([0,\infty);[0,\infty)\Big)$ with $\omega(0) = 0$ which is independent of ε and x, such that

$$\Theta\Big(x, \mathrm{D}(\mathrm{K}^\varepsilon u)(x)\Big) \;\leq\; \underset{\Omega\cap\mathbb{B}_\varepsilon(x+\varepsilon\ell\xi(x))}{\operatorname{ess\,sup}} \Theta(\cdot, \mathrm{D}u) \;+\; \omega(\varepsilon),$$

for any $x \in \Omega$. In particular, the above estimate implies that

$$\Big\|\Theta\Big(\cdot, \mathrm{D}(\mathrm{K}^\varepsilon u)\Big)\Big\|_{L^\infty(\Omega)} \;\leq\; \|\Theta(\cdot, \mathrm{D}u)\|_{L^\infty(\Omega)} \;+\; \omega(\varepsilon).$$

(This was already established in [65], but without x-dependence for Θ.) As a result,

$$\limsup_{\varepsilon\to 0} \Big\|\Theta\Big(\cdot, \mathrm{D}(\mathrm{K}^\varepsilon u)\Big)\Big\|_{L^\infty(\Omega)} \;\leq\; \|\Theta(\cdot, \mathrm{D}u)\|_{L^\infty(\Omega)}.$$

Let us now establish the claimed estimate. For any fixed $R > 0$ such that

$$R \;>\; \Big(\|\mathrm{D}\xi\|_{L^\infty(\mathbb{R}^n)} + 1\Big)\Big(\|\mathrm{D}u\|_{L^\infty(\Omega)} + 1\Big),$$

we set

$$\omega(t) \;:=\; t\ell R\|\Theta_P\|_{C(\overline{\Omega}\times\bar{\mathbb{B}}_R(0))} \;+\; \sup_{\substack{|Q|\leq R \\ |z-x|\leq t\ell R}} \Big|\Theta\Big(x, Q\Big) - \Theta\Big(z, Q\Big)\Big|, \quad t \geq 0.$$

Then, we have

$$\begin{aligned}
\Theta\Big(x, \mathrm{D}(\mathrm{K}^\varepsilon u)(x)\Big) &= \Theta\Big(x, \mathrm{K}^\varepsilon(\mathrm{D}u)(x) \;+\; \varepsilon\ell\mathrm{K}^\varepsilon(\mathrm{D}u)(x)(\mathrm{D}\xi(x))^\top\Big) \\
&\leq \Theta\Big(x, \mathrm{K}^\varepsilon(\mathrm{D}u)(x)\Big) + \|\Theta_P(x,\cdot)\|_{C(\bar{\mathbb{B}}_R(0))}\Big\|\varepsilon\ell\mathrm{K}^\varepsilon(\mathrm{D}u)(\mathrm{D}\xi)^\top\Big\|_{L^\infty(\Omega)} \\
&\leq \Theta\Big(x, \mathrm{K}^\varepsilon(\mathrm{D}u)(x)\Big) + \varepsilon\ell\|\Theta_P\|_{C(\overline{\Omega}\times\bar{\mathbb{B}}_R(0))}\|\mathrm{D}u\|_{L^\infty(\Omega)}\|\mathrm{D}\xi\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \Theta\Big(x, \mathrm{K}^\varepsilon(\mathrm{D}u)(x)\Big) \;+\; \varepsilon\ell R\|\Theta_P\|_{C(\overline{\Omega}\times\bar{\mathbb{B}}_R(0))},
\end{aligned}$$

for any $x \in \Omega$. Further, since $\varrho\mathcal{L}^n$ is a probability measure on $\mathbb{R}^n$, by Jensen's inequality, we have

$$
\begin{aligned}
\Theta\Big(x, \mathrm{K}^{\varepsilon}(\mathrm{D}u)(x)\Big) \;&=\; \Theta\left(x, \int_{\mathbb{R}^n} \mathrm{D}u\Big(x+\varepsilon\ell\xi(x)-\varepsilon y\Big)\,\varrho(y)\,\mathrm{d}y\right)\\
&\leq\; \int_{\mathbb{R}^n} \Theta\Big(x, \mathrm{D}u\Big(x+\varepsilon\ell\xi(x)-\varepsilon y\Big)\Big)\varrho(y)\,\mathrm{d}y\\
&\leq\; \operatorname*{ess\,sup}_{y\in\mathbb{B}_1(0)} \Theta\Big(x, \mathrm{D}u\Big(x+\varepsilon\ell\xi(x)-\varepsilon y\Big)\Big)\\
&=\; \operatorname*{ess\,sup}_{z\in\mathbb{B}_\varepsilon(x+\varepsilon\ell\xi(x))} \Theta\Big(x, \mathrm{D}u(z)\Big)\\
&\leq\; \operatorname*{ess\,sup}_{z\in\mathbb{B}_\varepsilon(x+\varepsilon\ell\xi(x))} \Theta\Big(z, \mathrm{D}u(z)\Big)\\
&\qquad + \sup_{\substack{|Q|\leq\|\mathrm{D}u\|_{L^\infty(\Omega)}\\ |z-x|\leq\varepsilon\ell\|\mathrm{D}\xi\|_{L^\infty(\mathbb{R}^n)}}} \Big|\Theta\Big(x,Q\Big)-\Theta\Big(z,Q\Big)\Big|,
\end{aligned}
$$

for any $x\in\Omega$. By using that $\mathrm{D}u\equiv 0$ on $\mathbb{R}^n\setminus\Omega$ and our assumptions on Θ, the previous two estimates yield that

$$
\begin{aligned}
\Theta\Big(x, \mathrm{D}(\mathrm{K}^{\varepsilon}u)(x)\Big) \;&\leq\; \operatorname*{ess\,sup}_{\mathbb{B}_\varepsilon(x+\varepsilon\ell\xi(x))} \Theta(\cdot,\mathrm{D}u) \,+\, \omega(\varepsilon)\\
&=\; \max\left\{ \operatorname*{ess\,sup}_{\Omega\cap\mathbb{B}_\varepsilon(x+\varepsilon\ell\xi(x))} \Theta(\cdot,\mathrm{D}u),\; \operatorname*{ess\,sup}_{\mathbb{B}_\varepsilon(x+\varepsilon\ell\xi(x))\setminus\Omega} \Theta(\cdot,\mathrm{D}u)\right\} \,+\, \omega(\varepsilon)\\
&=\; \max\left\{ \operatorname*{ess\,sup}_{\Omega\cap\mathbb{B}_\varepsilon(x+\varepsilon\ell\xi(x))} \Theta(\cdot,\mathrm{D}u),\; \operatorname*{ess\,sup}_{\mathbb{B}_\varepsilon(x+\varepsilon\ell\xi(x))\setminus\Omega} \Theta(\cdot,0)\right\} \,+\, \omega(\varepsilon)\\
&=\; \operatorname*{ess\,sup}_{\Omega\cap\mathbb{B}_\varepsilon(x+\varepsilon\ell\xi(x))} \Theta(\cdot,\mathrm{D}u) \,+\, \omega(\varepsilon),
\end{aligned}
$$

for any $x\in\Omega$.

In conclusion, the above establish the existence of an approximating sequence as in (3.34), by taking

$$
v_j \;:=\; \mathrm{K}^{\varepsilon_j}u_\infty \in\, C^1_0\Big(\overline{\Omega};\mathbb{R}^N\Big), \quad j\in\mathbb{N},
$$

along any infinitesimal sequence $(\varepsilon_j)_1^\infty$ satisfying $\varepsilon_j\to 0$, for the choice $\Theta := f$, which is admissible because of our hypotheses (3.5), (3.6) and (3.29).

Step 12. By putting together all the previous, the proof of Theorem 3.1.3 has now been completed. □

3.4 Explicit classes of nonlinear operators

In this section we provide various examples of nonlinear operators Q as in (3.2), satisfying our assumptions. Note that each our main results, Theorems 3.1.1, 3.1.2 and 3.1.3, have been obtained with progressively stronger assumption on the operator Q which expresses one of constraints in the admissible class. For the sake of clarity, in the next table we list in a concise way which assumptions are required to be satisfied by Q, in order to obtain the corresponding result (assuming that f, g satisfy separately their respective required assumptions, the table concerns Q solely).

(3.9)	$\Rrightarrow$ Theorem 3.1.1
(3.9) & (3.17)	$\Rrightarrow$ Theorem 3.1.2
(3.9) & (3.17) & (3.27) & (3.28)	$\Rrightarrow$ Theorem 3.1.3

3.4.1 Pointwise constraints, unilateral constraints and inclusions

The nonlinear operator of (3.2) we are using in the admissible class of (3.4), can include the following model cases:

Case 1. $\Pi(x, u(x)) = 0$ for a.e. $x \in \Omega$, where $\Pi : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is given.

Case 2. $\Pi(x, u(x)) \leq 0$ for a.e. $x \in \Omega$, where $\Pi : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is given.

Case 3. $u(x) \in \mathcal{K}$ for a.e. $x \in \Omega$, where $\mathcal{K} \subseteq \mathbb{R}^N$ is a given closed set.

Constraints as in Case 1 are sometimes called holonomic (see for instance [11]). We now elaborate on the assumptions required to be fulfilled in each of these cases.

Proposition 3.4.1 (Case 1)**.** *Let* $\Pi \in C^1(\overline{\Omega} \times \mathbb{R}^N; \mathbb{R}^M)$*,* $M \in \mathbb{N}$*. By defining*

$$\mathrm{Q} \ : \ W_0^{1,\bar{p}}(\Omega; \mathbb{R}^N) \longrightarrow L^1(\Omega; \mathbb{R}^M), \qquad \mathrm{Q}(u) \ := \ \Pi(\cdot, u), \tag{3.39}$$

and setting $\mathbf{E} := L^1(\Omega; \mathbb{R}^M)$*, we have the following:*

(i) The zero set of Q *equals*

$$\mathrm{Q}^{-1}(\{0\}) \ = \ \Big\{ v \in W_0^{1,\bar{p}}(\Omega; \mathbb{R}^N) \ : \ \Pi(x, v(x)) = 0, \ a.e. \ x \in \Omega \Big\},$$

and assumption (3.9) *is always satisfied.*

(ii) If for any $x \in \Omega$ *we have*

$$\Big\{\Pi(x,\cdot)=0\Big\} \subseteq \Big\{\Pi_\eta(x,\cdot)=0\Big\},$$

namely when all points in the zero set are critical points, then Q *satisfies* (3.17)*.*

(iii) Assumptions (3.27) *and* (3.28) *are always satisfied.*

The choice of $\mathbf{E}$ is deliberately made "as large as possible", as then the Lagrange multipliers of Theorems 3.1.2 and 3.1.3 are valued in the smaller space $\mathbf{E}^* = L^\infty(\Omega;\mathbb{R}^M)$.

Proof of Proposition 3.4.1. (i) Follows directly from the definitions, by the continuity of Π and by Morrey's estimate, because $\bar{p} > n$.

(ii) Indeed, since

$$(\mathrm{dQ})_u(\phi) \,=\, \Pi_\eta(\cdot,u)\cdot\phi,$$

if $u \in \mathrm{Q}^{-1}(\{0\})$, then $\Pi(\cdot,u)=0$ a.e. on Ω and therefore $\Pi_\eta(\cdot,u)=0$ a.e. on Ω, which implies that $(\mathrm{dQ})_u = 0$, hence its image is the closed trivial subspace $\{0\} \subseteq L^1(\Omega;\mathbb{R}^M)$.

(iii) Note first that $L^1(\Omega;\mathbb{R}^M)$ is separable. Also, if we have $u_m \rightharpoonup u$ and $\phi_m \rightharpoonup \phi$ in $W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$ as $m\to\infty$, then by Morrey's theorem and the compactness of the imbedding of Hölder spaces we have $u_m \longrightarrow u$ and also $\phi_m \longrightarrow \phi$ in $C\big(\overline{\Omega};\mathbb{R}^N\big)$ as $m\to\infty$. Hence, we have as $m\to\infty$ that

$$(\mathrm{dQ})_{u_m}(\phi_m) = \Pi_\eta(\cdot,u_m)\cdot\phi_m \,\longrightarrow\, \Pi_\eta(\cdot,u)\cdot\phi = (\mathrm{dQ})_u(\phi),$$

in $C\big(\overline{\Omega};\mathbb{R}^M\big)$, which a fortiori implies strong convergence in $L^1(\Omega;\mathbb{R}^M)$. $\square$

We note that the proof of (iii) above is immediate if one assumes the additional hypothesis of (ii), since then $(\mathrm{dQ})_{u_m} = 0$ for any sequence $(u_m)_1^\infty \subseteq \mathrm{Q}^{-1}(\{0\})$.

Proposition 3.4.2 (Case 2)**.** *Let* $\Pi \in C^1(\overline{\Omega}\times\mathbb{R}^N)$ *and let us define* $\pi:\mathbb{R}\longrightarrow\mathbb{R}$ *by*

$$\pi(t) \,:=\, \left\{\begin{array}{ll} 0, & t\leq 0,\\ t^2, & t>0. \end{array}\right. \tag{3.40}$$

By defining the operator

$$\mathrm{Q} \;:\; W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N) \longrightarrow L^1(\Omega), \qquad \mathrm{Q}(u) \,:=\, \pi(\Pi(\cdot,u)),$$

for $\mathbf{E} := L^1(\Omega)$*, we have the following:*

(i) The zero set of Q *equals*

$$\mathrm{Q}^{-1}(\{0\}) \;=\; \Big\{ v \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \;:\; \Pi(x, v(x)) \leq 0, \;\, a.e. \;\, x \in \Omega \Big\},$$

and assumption (3.9) *is always satisfied.*

(ii) If for any $x \in \Omega$ *it holds that*

$$\Big\{\Pi(x,\cdot) = 0\Big\} \;\subseteq\; \Big\{\Pi_\eta(x,\cdot) = 0\Big\},$$

then Q *satisfies assumption* (3.17)*.*

(iii) Assumptions (3.27) *and* (3.28) *are always satisfied.*

Proof of Proposition 3.4.2. (i) Follows as in the proof of Proposition 3.4.1(i), upon noting that $\{\pi \leq 0\} = (-\infty, 0]$.

(ii) Since

$$(\mathrm{dQ})_u(\phi) \;=\; \pi'\Big(\Pi(\cdot,u)\Big)\Pi_\eta(\cdot,u)\cdot\phi,$$

if $u \in \mathrm{Q}^{-1}(\{0\})$, then $\Pi(\cdot,u) \leq 0$ a.e. on Ω and therefore $\pi'(\Pi(\cdot,u)) = 0$ a.e. on Ω because $\{\pi' = 0\} = (-\infty, 0]$, which implies that $(\mathrm{dQ})_u = 0$, hence its image is the closed trivial subspace $\{0\} \subseteq L^1(\Omega;\mathbb{R}^M)$ and (3.17) is satisfied.

(iii) Similar to the proof of Proposition 3.4.1(iii), using the different expression for the differential dQ as above. $\square$

Proposition 3.4.3 (Case 3)**.** *Let* $\mathcal{K} \subseteq \mathbb{R}^N$ *be a closed set with* $\mathcal{K} \neq \emptyset$*. Then, there exists* $\Pi \in C^\infty(\mathbb{R}^N)$ *satisfying* $\mathcal{K} = \{\Pi = 0\} \subseteq \{\Pi_\eta = 0\}$*. Further, if one defines*

$$\mathrm{Q} \;:\; W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \longrightarrow L^1(\Omega), \qquad \mathrm{Q}(u) \;:=\; \Pi(u),$$

and $\mathbf{E} := L^1(\Omega)$*, then we have*

$$\mathrm{Q}^{-1}(\{0\}) \;=\; \Big\{ v \in W_0^{1,\bar{p}}(\Omega;\mathbb{R}^N) \;:\; v(x) \in \mathcal{K}, \;\, a.e. \;\, x \in \Omega \Big\},$$

and Q *satisfies* (3.9)*,* (3.17)*,* (3.27) *and* (3.28)*.*

Proof of Proposition 3.4.3. It is well-known that for every such set $\mathcal{K}$, there exists a function $\Pi \in C^\infty(\mathbb{R}^N)$ with the claimed properties. A proof of this fact can be found e.g. in [82, Sec. 1.1.13, p. 25] (the claimed inclusion is not explicitly stated, but follows from the method of proof by the smooth Urysohn lemma). The rest follows from Proposition 3.4.1. $\square$

3.4.2 Integral and isoperimetric constraints

The nonlinear operator of (3.2) can also cover the following important case of constraint:

$$\int_\Omega h(\cdot, u, \mathrm{D}u)\, \mathrm{d}\mathcal{L}^n \leq H,$$

when $h : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ and $H \in \mathbb{R}$ are given.

Proposition 3.4.4. *Let $h : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ satisfy the assumptions (3.5)-(3.7) and (3.16) that f, g are assumed to satisfy, with $\alpha \leq \bar{p}$. Let also $H \in \mathbb{R}$ be given and let $\pi : \mathbb{R} \longrightarrow \mathbb{R}$ be as in (4.2). Then, by defining the operator*

$$\mathrm{Q} \ : \ W^{1,\bar{p}}_0(\Omega; \mathbb{R}^N) \longrightarrow \mathbb{R}, \qquad \mathrm{Q}(u) \, := \, \pi\left(\int_\Omega h(\cdot, u, \mathrm{D}u)\, \mathrm{d}\mathcal{L}^n - H\right),$$

and setting $\mathbf{E} := \mathbb{R}$, we have the following:

(i) The zero set of Q equals

$$\mathrm{Q}^{-1}(\{0\}) \, = \, \left\{ v \in W^{1,\bar{p}}_0(\Omega; \mathbb{R}^N) \ : \ \int_\Omega h(\cdot, v, \mathrm{D}v)\, \mathrm{d}\mathcal{L}^n \leq H \right\}$$

and assumption (3.9) is satisfied.

(ii) Q *satisfies (3.17), (3.27) and (3.28).*

Proof of Proposition 3.4.4. (i) If $\mathrm{Q}(u_m) = 0$ and $u_m \rightharpoonup u$ in $W^{1,\bar{p}}_0(\Omega; \mathbb{R}^N)$ as $m \to \infty$, then since $\{\pi \leq 0\} = (-\infty, 0]$, we have

$$\int_\Omega h(\cdot, u_m, \mathrm{D}u_m)\, \mathrm{d}\mathcal{L}^n - H \, \leq \, 0.$$

Since h satisfies (3.5)-(3.7) for $\alpha \leq \bar{p}$, by standard results (see e.g. [36]), the functional $u \mapsto \|h(\cdot, u, \mathrm{D}u)\|_{L^1(\Omega)}$ is weakly lower-semicontinuous in $W^{1,\bar{p}}_0(\Omega; \mathbb{R}^N)$. Hence

$$\int_\Omega h(\cdot, u, \mathrm{D}u)\, \mathrm{d}\mathcal{L}^n - H \, \leq \, \liminf_{m\to\infty} \int_\Omega h(\cdot, u_m, \mathrm{D}u_m)\, \mathrm{d}\mathcal{L}^n - H \, \leq \, 0.$$

Therefore, $\mathrm{Q}(u) = 0$, yielding that $\mathrm{Q}^{-1}(\{0\})$ is weakly closed and hence (3.9) is satisfied.

(ii) By a computation, the Gateaux derivative of Q is given by

$$(\mathrm{dQ})_u(\phi) = \pi'\Big(\int_\Omega h(\cdot, u, \mathrm{D}u)\, \mathrm{d}\mathcal{L}^n - H\Big) \int_\Omega \Big[h_\eta(\cdot, u, \mathrm{D}u) \cdot \phi \, + \, h_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \Big]\, \mathrm{d}\mathcal{L}^n,$$

and assumption (3.16) for h implies that dQ is (jointly) continuous on $W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N) \times W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$. Further, if $u \in \mathrm{Q}^{-1}(\{0\})$, then by part (i) we have

$$\int_\Omega h(\cdot,u,\mathrm{D}u)\,\mathrm{d}\mathcal{L}^n - H \,\leq\, 0,$$

and therefore the first factor of $(\mathrm{dQ})_u(\phi)$ vanishes because $\{\pi' = 0\} = (-\infty, 0]$. Thus, $(\mathrm{dQ})_u = 0$ when $u \in \mathrm{Q}^{-1}(\{0\})$, and hence its image is the closed trivial subspace $\{0\} \subseteq \mathbb{R}$, yielding that (3.17) is satisfied.

(iii) For any sequences $u_m \rightharpoonup u$ in $\mathrm{Q}^{-1}(\{0\}) \subseteq W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$ and $\phi_m \rightharpoonup \phi$ in $W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$ as $m \to \infty$, by part (ii) we have

$$(\mathrm{dQ})_{u_m}(\phi_m) \,=\, 0 \longrightarrow 0 \,=\, (\mathrm{dQ})_u(\phi)$$

as $m \to \infty$, hence (3.27) and (3.28) are satisfied. $\square$

3.4.3 Quasilinear second order differential constraints

The operator Q of (3.2) can also cover the case of various types of nontrivial PDE constraints. As an example, we discuss the case of quasilinear divergence second order systems of PDE of the form

$$\mathrm{div}\Big(A(\cdot,u,\mathrm{D}u)\Big) \,=\, B(\cdot,u,\mathrm{D}u) \quad \text{in } \Omega, \tag{3.41}$$

where the coefficients maps $A : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}^{N\times n}$ and $B : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}^N$ are given. Given the plethora of possibilities on the assumptions for such systems, the discussion in this subsection is less formal and is only aimed as a general indication of the admissible choices for Q.

Suppose that A, B are C^1 and satisfy appropriate growth bounds, and also that $P \mapsto A(\cdot,\cdot,P)$ a monotone map, and that the set of weak solutions to the system (3.41) is strongly precompact in $W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$. A sufficient conditions for strong precompactness in $W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$ for the set of weak solutions is for example a global $C^{1,\gamma}$ or a $W^{2,1+\gamma}$ a priori uniform bound on the set of solutions, for some $\gamma \in (0,1)$. Appropriate assumptions on the coefficients A, B that allow the derivation of such a priori bounds can be found e.g. in [55] for $N = 1$ and in [52] for $N \geq 2$. Then, by defining the operator

$$\mathrm{Q} \;:\; W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N) \longrightarrow W^{-1,\bar{p}'}(\Omega;\mathbb{R}^N)$$

as

$$\langle \mathrm{Q}(u), \psi \rangle \,:=\, \int_\Omega \Big[A(\cdot,u,\mathrm{D}u) : \mathrm{D}\psi \,+\, B(\cdot,u,\mathrm{D}u)\cdot\psi\Big]\,\mathrm{d}\mathcal{L}^n,$$

and setting also $\mathbf{E} := W^{-1,\bar{p}'}(\Omega;\mathbb{R}^N)$, assumptions (3.9), (3.17), (3.27) and (3.28) are satisfied, with

$$\mathrm{Q}^{-1}(\{0\}) \,=\, \Big\{ u \in W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N) \;:\; \mathrm{div}\Big(A(\cdot,u,\mathrm{D}u)\Big) = B(\cdot,u,\mathrm{D}u) \text{ weakly in } \Omega \Big\}.$$

Note first that the expression of $\mathrm{Q}^{-1}(\{0\})$ is immediate by the definition of the differential operator Q. Next, note that by assumption, for any sequence of weak solutions $(u_m)_1^\infty \subseteq W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$ to (3.41), there exists $u \in W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$ such that $u_m \longrightarrow u$ strongly along a subsequence $m_j \to \infty$. By applying this to any sequence $(u_m)_1^\infty \subseteq \mathrm{Q}^{-1}(\{0\})$ (namely sequence of solutions) for which $u_m \rightharpoonup u$ as $m \to \infty$, by passing to the limit in the weak formulation for fixed $\psi \in W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$, which reads

$$\int_\Omega \Big[A(\cdot,u_m,\mathrm{D}u_m) : \mathrm{D}\psi \,+\, B(\cdot,u_m,\mathrm{D}u_m)\cdot\psi\Big]\,\mathrm{d}\mathcal{L}^n \,=\, 0,$$

we get that $u \in \mathrm{Q}^{-1}(\{0\})$, as the convergence is in fact strong. Hence, (3.9) is satisfied. Further, under appropriate bounds, the operator Q is Fréchet differentiable and

$$\begin{aligned}\Big\langle (\mathrm{dQ})_u(\phi),\psi\Big\rangle \,=\, &\int_\Omega \Big[A_\eta(\cdot,u,\mathrm{D}u)\cdot\phi \,+\, A_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi\Big] : \mathrm{D}\psi\,\mathrm{d}\mathcal{L}^n\\ &+\int_\Omega \Big[B_\eta(\cdot,u,\mathrm{D}u)\cdot\phi \,+\, B_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi\Big]\cdot\psi\,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

To see that the image of $(\mathrm{dQ})_u : W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N) \longrightarrow \mathrm{E}$ is closed for any fixed $u \in \mathrm{Q}^{-1}(\{0\})$, let $(T_m)_1^\infty \subseteq \mathrm{Rg}\big((\mathrm{dQ})_u\big) \subseteq \mathrm{E}$ be a sequence in the range with $T_m \longrightarrow T$ strongly in $\mathbf{E}$ as $m \to \infty$. Since $T_m \in \mathrm{Rg}\big((\mathrm{dQ})_u\big)$, exists $\phi_m \in W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$ solving the following linear second order system

$$\begin{aligned}-\mathrm{div}\Big(A_\eta(\cdot,u,\mathrm{D}u)\cdot\phi_m \,+\, A_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi_m\Big)&\\ +\, B_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi_m \,+\, B_\eta(\cdot,u,\mathrm{D}u)\cdot\phi_m \,&=\, T_m.\end{aligned}$$

By the monotonicity of the above system (due to our earlier assumption), under appropriate conditions one has a uniform bound in $W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$, yielding the weak compactness of the sequence of solutions $(\phi_m)_1^\infty$, which establishes the closedness of $\mathrm{Rg}\big((\mathrm{dQ})_u\big) \subseteq \mathbf{E}$ and (3.17) ensues.

Finally, for any sequence $(u_m)_1^\infty \subseteq \mathrm{Q}^{-1}(\{0\})$ satisfying $u_m \rightharpoonup u$ as $m \to \infty$ and any $\phi \in W^{1,\bar{p}}_0(\Omega;\mathbb{R}^N)$, there exists $m_j \to \infty$ such that $u_m \longrightarrow u$ as $m_j \to \infty$. These facts imply that $(\mathrm{dQ})_{u_m}(u_m) \longrightarrow (\mathrm{dQ})_u(u)$ and also $(\mathrm{dQ})_{u_m}(\phi) \longrightarrow (\mathrm{dQ})_u(\phi)$, both strongly in $\mathbf{E}$ as $m \to \infty$. Hence, (3.27) and (3.28) are satisfied.

3.4.4 Null Lagrangians and determinant constraints

We close this paper with the observation that Theorem 3.1.1 holds true even when Q expresses a fully nonlinear pointwise Jacobian determinant constraint, or even a more general pointwise PDE constraint driven by a null Lagrangian. As an explicit example, let $n = N$ and consider the differential operator

$$\mathrm{Q} \ : \ W^{1,\bar{p}}_0(\Omega;\mathbb{R}^n) \longrightarrow W^{-1,(\bar{p}/n)'}(\Omega),$$

by setting

$$\mathrm{Q}(u) \ := \ \det(\mathrm{D}u) - h,$$

for a fixed $h \in L^{\bar{p}/n}(\Omega)$, satisfying the necessary compatibility condition

$$\int_\Omega h \,\mathrm{d}\mathcal{L}^n \ = \ 0.$$

We also take

$$\mathbf{E} \ := \ W^{-1,(\bar{p}/n)'}(\Omega) \ = \ \Big(W^{1,\bar{p}/n}_0(\Omega)\Big)^*.$$

Then, we have

$$\mathrm{Q}^{-1}(\{0\}) \ = \ \Big\{ u \in W^{1,\bar{p}}_0(\Omega;\mathbb{R}^n) \ : \ \det(\mathrm{D}u) = h \text{ a.e. in } \Omega \Big\}.$$

It follows that (3.9) is satisfied by the well-known property of weak continuity for Jacobian determinants (see e.g. [36, Th. 8.20, p. 395]). However, the situation is more complicated regarding the satisfaction of the remaining assumptions. If additionally $n = 2$, then (3.27) and (3.28) are also satisfied. Indeed, since

$$(\mathrm{dQ})_u(\phi) \ = \ \mathrm{cof}(\mathrm{D}u) : \mathrm{D}\phi,$$

and since for $u = \phi$ we have the identity

$$(\mathrm{dQ})_u(u) \ = \ \mathrm{cof}(\mathrm{D}u) : \mathrm{D}u \ = \ n\det(\mathrm{D}u),$$

for any $(u_m)_1^\infty \subseteq \mathrm{Q}^{-1}(\{0\})$ with $u_m \rightharpoonup u$ as $m \to \infty$, we have

$$(\mathrm{dQ})_{u_m}(u_m) \ = \ n\det(\mathrm{D}u_m) \ \rightharpoonup \ n\det(\mathrm{D}u) \ = \ (\mathrm{dQ})_u(u)$$

in $L^{\bar{p}/2}(\Omega)$ as $m \to \infty$, whilst for any $\phi \in W^{1,\bar{p}}_0(\Omega;\mathbb{R}^2)$ we have

$$(\mathrm{dQ})_{u_m}(\phi) \ \rightharpoonup \ (\mathrm{dQ})_u(\phi)$$

in $L^{\bar{p}/2}(\Omega)$ as $m \to \infty$, by the linearity of the cofactor operator when $n = 2$. Then, the compactness of the imbedding

$$L^{\bar{p}/2}(\Omega) \Subset W^{-1,(\bar{p}/2)'}(\Omega)$$

implies that the above modes of convergence are in fact strong in $\mathbf{E} = W^{-1,(\bar{p}/2)'}(\Omega)$. However, it is not clear when assumption (3.17) is satisfied, or when (3.28) is satisfied in the case that $n \geq 3$. This means Theorems 3.1.2 and 3.1.3 as they stand do not apply to the case of Jacobian constraints. This does *not* mean that it is impossible to derive the associated equations, it merely means that in this case of such a highly nonlinear constraint a different specialised method of proof is required.

Chapter 4

Generalised Second Order Vectorial ∞-Eigenvalue Problem

4.1 Introduction and main results

Let $n, N \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \Subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary $\partial\Omega$. In this paper we are interested in studying nonlinear second order L^∞ eigenvalue problems. Specifically, we investigate the problem of finding a minimising map $u_\infty : \overline{\Omega} \longrightarrow \mathbb{R}^N$, that solves

$$\begin{aligned} \|f(\mathrm{D}^2 u_\infty)\|_{L^\infty(\Omega)} \,=\, \inf\Big\{ &\|f(\mathrm{D}^2 v)\|_{L^\infty(\Omega)} \;: \\ &v \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N),\ \|g(v,\mathrm{D}v)\|_{L^\infty(\Omega)} = 1\Big\}. \end{aligned} \tag{4.1}$$

Additionally, we pursue the necessary conditions that these constrained minimisers must satisfy, in the form of PDEs. In the above, $f : \mathbb{R}^{N\times n^2}_s \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ are given functions that will be required to satisfy some natural assumptions, to be discussed later in this section. We merely note now that $\mathbb{R}^{N\times n^2}_s$ symbolises the symmetric tensor space $\mathbb{R}^N \otimes (\mathbb{R}^n \vee \mathbb{R}^n)$ wherein the hessians of twice differentiable maps $u : \Omega \longrightarrow \mathbb{R}^N$ are valued. The functional Sobolev space $W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$ appearing above will taken to be either of:

$$\left\{ \begin{array}{l} W^{2,\infty}_{\mathrm{C}}(\Omega;\mathbb{R}^N) := W^{2,\infty}_0(\Omega;\mathbb{R}^N), \\ W^{2,\infty}_{\mathrm{H}}(\Omega;\mathbb{R}^N) := W^{2,\infty} \cap W^{1,\infty}_0(\Omega;\mathbb{R}^N). \end{array} \right. \tag{4.2}$$

The space $W^{2,\infty}_{\mathrm{C}}(\Omega;\mathbb{R}^N)$ encompasses the case of so-called clamped boundary conditions, which can be seen as first order Dirichlet or as coupled Dirichlet-Neumann conditions, requiring $|u| = |\mathrm{D}u| = 0$ on $\partial\Omega$. On the other hand, $W^{2,\infty}_{\mathrm{H}}(\Omega;\mathbb{R}^N)$ encompasses the so-called hinged boundary conditions, which are zeroth order Dirichlet conditions, requiring $|u| = 0$ on $\partial\Omega$. This is standard terminology for such problems, see e.g. [69].

Problem (4.1) lies within the Calculus of Variations in L^∞, a modern area, initiated by Gunnar Aronsson in the 1960s. Since then this field has undergone a substantial transformation. There are some general complications one must be wary of when tackling L^∞ variational problems. For example, the L^∞ norm is generally not Gateaux differentiable, therefore the analogue of the Euler-Lagrange equations cannot be derived directly by considering variations. Any supremal functional also has issues with locality in terms of minimisation on subdomains. Further, the space itself lacks some fundamental functional analytic properties, such as reflexivity and separability. Higher order problems and problems involving constraints present additional difficulties and have been studied even more sparsely, see e.g. [11, 15, 30, 31, 63, 64, 66, 65, 68, 72]. In fact, this paper is an extension of [65] to the second order case, and generalises part of the results corresponding to the existence of minimisers and the satisfaction of PDEs from [69]. In turn, the paper [65] generalised results on the scalar case of eigenvalue problems for the ∞-Laplacian ([58, 59]). For various interesting results, see for instance [10, 11, 26, 27, 76, 80, 81, 84].

The vectorial and higher order nature of the problem we are considering herein precludes the use of standard methods, such as viscosity solutions (see e.g. [61] for a pedagogical introduction). However, we overcome these difficulties by approximating by corresponding L^p problems for finite p case and let $p \to \infty$. The intuition for using this technique is based on the rudimentary idea that, for a fixed L^∞ function on a set of finite measure, its L^p norm tends to its L^∞ norm as $p \to \infty$. This technique is rather standard for L^∞ problems, and in the vectorial higher order case we consider herein is essentially the only method known. Even the very recent intrinsic duality method of [26] is limited to scalar-valued first order problems.

To state our main result, we now introduce the required hypotheses for the functions f and g:

$$
\left\{
\begin{array}{l}
(a)\ f \in C^1(\mathbb{R}_s^{N\times n^2}).\\
(b)\ f \text{ is (Morrey) quasiconvex.}\\
(c)\ \text{There exist } 0 < C_1 \leq C_2 \text{ such that, for all } X \in \mathbb{R}_s^{N\times n^2} \setminus \{0\},\\
\qquad\qquad 0 < C_1 f(X) \ \leq\ \partial f(X) : X \ \leq\ C_2 f(X).\\
(d)\ \text{There exist } C_3, ..., C_6 > 0, \alpha > 1 \text{ and } \beta \leq 1 \text{ such that, for all } X \in \mathbb{R}_s^{N\times n^2},\\
\qquad\qquad -C_3 + C_4|X|^\alpha \ \leq\ f(X) \leq C_5|X|^\alpha + C_6,\\
\qquad\qquad |\partial f(X)| \ \leq\ C_5 f(X)^\beta + C_6.
\end{array}
\right.
\tag{4.3}
$$

$$
\left\{
\begin{array}{l}
(a)\ g \in C^1(\mathbb{R}^N \times \mathbb{R}^{N\times n}).\\
(b)\ g \text{ is coercive, i.e for any } (\eta, P) \in \Big(\mathbb{R}^N \times \mathbb{R}^{N\times n}\Big) \setminus \{(0,0)\} \text{ we have}\\
\qquad\qquad \lim\limits_{t\to\infty} g(t\eta, tP) = \infty.\\
(c)\ \text{There exist } 0 < C_7 \leq C_8 \text{ such that, for all } (\eta, P) \in \Big(\mathbb{R}^N \times \mathbb{R}^{N\times n}\Big)\setminus\{(0,0)\},\\
\qquad\qquad 0 < C_7\, g(\eta, P) \ \leq\ \partial_\eta g(\eta, P)\cdot \eta + \partial_P g(\eta, P) : P \ \leq\ C_8\, g(\eta, P).
\end{array}
\right.
\tag{4.4}
$$

In the above, $\partial f(X)$ denotes the the derivative of f whilst $\partial_\eta g$ and $\partial_P g$ signifies the respective partial derivatives. Additionally ":" and " $\cdot$ " represent the Euclidean inner products. The terminology of (Morrey) quasiconvex refers to the standard notion for integral functionals (see e.g. [36, 95]), namely

$$
F(X) \ \leq\ \fint_\Omega F(X + \mathrm{D}^2\phi)\, \mathrm{d}\mathcal{L}^n, \qquad \forall\ \phi \in W_0^{2,\infty}(\Omega; \mathbb{R}^N),\ \forall\ X \in \mathbb{R}_s^{N\times n^2}.
$$

We note that herein we will be using the following function space symbolisations:

$$
\begin{aligned}
C^2_{\mathrm{B}}(\overline{\Omega}; \mathbb{R}^N) \ &:=\ C^2(\overline{\Omega}; \mathbb{R}^N) \cap W^{2,\infty}_{\mathrm{B}}(\Omega; \mathbb{R}^N),\\
W^{2,p}_{\mathrm{C}}(\Omega; \mathbb{R}^N) \ &:=\ W^{2,p}_0(\Omega; \mathbb{R}^N), \quad p \in [1, \infty),\\
W^{2,p}_{\mathrm{H}}(\Omega; \mathbb{R}^N) \ &:=\ W^{2,p} \cap W^{1,p}_0(\Omega; \mathbb{R}^N), \quad p \in [1, \infty),
\end{aligned}
$$

Further, we will be using the rescaled L^p norms for $p \in [1, \infty)$, given by

$$
\|h\|_{L^p(\Omega)} \ :=\ \left(\frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega |h|^p\, \mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}} \ =\ \left(\fint_\Omega |h|^p\, \mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}}.
$$

Finally, we observe that (4.3)(c), implies that $f > 0$ on $\mathbb{R}^{N\times n^2}_s \setminus \{0\}$, $f(0) = 0$ and f is radially increasing, meaning that $t \mapsto f(tX)$ is increasing on $(0,\infty)$ for any fixed $X \in \mathbb{R}^{N\times n^2}_s \setminus \{0\}$. Similarly, (4.4)(c) implies that $g > 0$ on $(\mathbb{R}^N \times \mathbb{R}^{N\times n}) \setminus \{(0,0)\}$, $g(0,0) = 0$ and g is radially increasing on $\mathbb{R}^N \times \mathbb{R}^{N\times n}$, namely $t \mapsto g(t\eta, tP)$ is increasing on $(0,\infty)$ for any fixed $(\eta, P) \in (\mathbb{R}^N \times \mathbb{R}^{N\times n}) \setminus \{(0,0)\}$.

Below is our main result, in which we consider both cases of boundary conditions simultaneously.

Theorem 4.1.1. *Suppose that the assumptions (4.3) and (4.4) hold true. Then:*
(A) The problem (4.1) has a solution $u_\infty \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$.
(B) There exist Radon measures

$$\mathrm{M}_\infty \in \mathcal{M}\Big(\overline{\Omega};\mathbb{R}^{N\times n^2}_s\Big), \quad \nu_\infty \in \mathcal{M}(\overline{\Omega}),$$

such that

$$\int_{\overline{\Omega}} \mathrm{D}^2\phi : \mathrm{dM}_\infty = \Lambda_\infty \int_{\overline{\Omega}} \Big(\partial_\eta g(u_\infty, \mathrm{D}u_\infty)\cdot\phi \,+\, \partial_P g(u_\infty,\mathrm{D}u_\infty) : \mathrm{D}\phi\Big)\,\mathrm{d}\nu_\infty \tag{4.5}$$

for all test maps $\phi \in C^2_{\mathrm{B}}(\overline{\Omega};\mathbb{R}^N)$, *where*

$$\Lambda_\infty \,=\, \Big\| f(\mathrm{D}^2(u_\infty)\Big\|_{L^\infty(\Omega)} > 0. \tag{4.6}$$

(C) The quadruple $(u_\infty, \Lambda_\infty, \mathrm{M}_\infty, \nu_\infty)$ *satisfies the following approximation properties: there exists a sequence of exponents* $(p_j)_1^\infty \subseteq (n/\alpha)$ *where* $p_j \to \infty$ *as* $j \to \infty$, *and for any* p, *a quadruple*

$$(u_p, \Lambda_p, \mathrm{M}_p, \nu_p) \;\in\; W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N) \times [0,\infty) \times \mathcal{M}\Big(\overline{\Omega};\mathbb{R}^{N\times n^2}_s\Big) \times \mathcal{M}(\overline{\Omega}),$$

such that

$$\left\{\begin{array}{lll} u_p \longrightarrow u_\infty, & \text{in } C^1\Big(\overline{\Omega};\mathbb{R}^N\Big), & \\ \mathrm{D}^2 u_p \rightharpoonup \mathrm{D}^2 u_\infty, & \text{in } L^q\Big(\Omega;\mathbb{R}^{N\times n^2}_s\Big), \text{ for all } q \in (1,\infty), & \\ \Lambda_p \longrightarrow \Lambda_\infty, & \text{in } [0,\infty), & \\ \mathrm{M}_p \overset{*}{\rightharpoonup} \mathrm{M}_\infty, & \text{in } \mathcal{M}\Big(\overline{\Omega};\mathbb{R}^{N\times n^2}_s\Big), & \\ \nu_p \overset{*}{\rightharpoonup} \nu_\infty, & \text{in } \mathcal{M}(\overline{\Omega}), & \end{array}\right. \tag{4.7}$$

as $p \to \infty$ *along* $(p_j)_1^\infty$. *Further,* u_p *solves the constrained minimisation problem*

$$\|f(\mathrm{D}^2 u_p)\|_{L^p(\Omega)} \,=\, \inf\Big\{\|f(\mathrm{D}^2 v)\|_{L^p(\Omega)} \;:\; v \in W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N),\ \|g(v,\mathrm{D}v)\|_{L^p(\Omega)} = 1\Big\}, \tag{4.8}$$

and (u_p, Λ_p) *satisfies*

$$
\left\{
\begin{aligned}
& \fint_\Omega f(\mathrm{D}^2 u_p)^{p-1} \partial f(\mathrm{D}^2 u_p) : \mathrm{D}^2 \phi \, \mathrm{d}\mathcal{L}^n \\
&= (\Lambda_p)^p \fint_\Omega g(u_p, \mathrm{D}u_p)^{p-1} \Big(\partial_\eta g(u_p, \mathrm{D}u_p) \cdot \phi + \partial_P g(u_p, \mathrm{D}u_p) : \mathrm{D}\phi \Big) \, \mathrm{d}\mathcal{L}^n
\end{aligned}
\right.
\tag{4.9}
$$

for all test maps $\phi \in W^{2,\alpha p}_{\mathrm{B}}(\Omega; \mathbb{R}^N)$. *Finally, the measures* M_p, ν_p *are given by*

$$
\left\{
\begin{aligned}
\mathrm{M}_p &= \frac{1}{\mathcal{L}^n(\Omega)} \left(\frac{f(\mathrm{D}^2 u_p)}{\Lambda_p} \right)^{p-1} \partial f(\mathrm{D}^2 u_p) \, \mathcal{L}^n \llcorner_\Omega, \\
\nu_p &= \frac{1}{\mathcal{L}^n(\Omega)} \, g(u_p, \mathrm{D}u_p)^{p-1} \, \mathcal{L}^n \llcorner_\Omega.
\end{aligned}
\right.
\tag{4.10}
$$

We note that one could pursue optimality in Theorem 4.1.1 (A) by using L^∞ versions of quasiconvexity, as developed by Barron-Jensen-Wang [17] but adapted to this higher order case, in regards to the existence of L^∞ minimisers. However, for parts (B) and (C) of Theorem 4.1.1 regarding the necessary PDE conditions, we do need Morrey quasiconvexity, as we rely essentially on the existence of solutions to the corresponding Euler-Lagrange equations and the theory of Lagrange multipliers in the finite p case. Further, the measures $\mathrm{M}_\infty, \nu_\infty$ *depend* on the minimiser u_∞ in a non-linear fashion, hence one more could perhaps symbolise them more concisely as $\mathrm{M}_\infty(u_\infty), \nu_\infty(u_\infty)$. Consequently, the significance of these equations is currently understood to be mostly of conceptual value, rather than of computational nature. However, it is possible to obtain further information about the underlying structure of these parametric measure coefficients. This requires techniques such as measure function pairs and mollifications up to the boundary as in [31, 57, 65], but to keep the presentation as simple as possible, we refrain from pursuing this considerably more technical endeavour, which also requires stronger assumptions.

4.2 Proofs

In this section we establish Theorem 4.1.1. Its proof is not labeled explicitly, but will be completed by proving a combination of smaller subsidiary results, including a selection of lemmas and propositions.

Before introducing the approximating problem (the L^p case for finite p), we need to establish a convergence result, which shows that the admissible classes of the p-problems are non-empty. It is required because the function g appearing in the constraint is not assumed to be homogeneous, therefore a standard scaling argument does not suffice.

Lemma 4.2.1. *For any $v \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N) \setminus \{0\}$, there exists $(t_p)_{p\in(n/\alpha,\infty]}$ with $t_p \to t_\infty$ as $p \to \infty$, such that*

$$\left\| g\Big(t_p v, t_p \mathrm{D}v\Big) \right\|_{L^p(\Omega)} = 1,$$

for all $p \in (n/\alpha, \infty]$. Further, if $\|g(v, \mathrm{D}v)\|_{L^\infty(\Omega)} = 1$, then $t_\infty = 1$.

Proof of Lemma 4.2.1. Fix $v \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N) \setminus \{0\}$ and define

$$\rho_\infty(t) := \max_{x\in\overline{\Omega}} g\Big(tv(x), t\mathrm{D}v(x)\Big), \qquad t \geq 0.$$

It follows that $\rho_\infty(0) = 0$ and ρ_∞ is continuous on $[0,\infty)$. We will now show that ρ_∞ is strictly increasing. We first show it is non-decreasing. For any $s > 0$ and $(\eta, P) \in \mathbb{R}^N \times \mathbb{R}^{N\times n} \setminus \{(0,0)\}$, our assumption (4.4)(c) implies

$$\begin{aligned} 0 \;&<\; \frac{C_7 g(s\eta, sP)}{s} \\ &\leq\; \partial_\eta g(s\eta, sP)\cdot\eta \,+\, \partial_P g(s\eta, sP) : P \\ &=\; \partial_{(\eta,P)} g(s\eta, sP) : (\eta, P) \\ &=\; \frac{\mathrm{d}}{\mathrm{d}s}\Big(g(s\eta, sP)\Big), \end{aligned}$$

thus $s \mapsto g(s\eta, sP)$ is increasing on $(0,\infty)$. Hence, for any $x \in \overline{\Omega}$ and $t > s \geq 0$ we have $g(sv(x), s\mathrm{D}v(x)) \geq g(tv(x), t\mathrm{D}v(x))$, which yields,

$$\rho_\infty(s) = \max_{x\in\overline{\Omega}} g\Big(sv(x), s\mathrm{D}v(x)\Big) \;\leq\; \max_{x\in\overline{\Omega}} g\Big(tv(x), t\mathrm{D}v(x)\Big) = \rho_\infty(t).$$

We proceed to demonstrate that $t \mapsto \rho_\infty(t)$ is actually strictly monotonic over $(0,\infty)$. Fix $t_0 > 0$. By Danskin's theorem [37], the derivative from the right $\rho'(t_0^+)$ exists, and is given by the formula

$$\rho'_\infty(t_0^+) \;=\; \max_{x\in\Omega_{t_0}} \Big\{ \partial_{(\eta,P)} g(t_0 v(x), t_0 \mathrm{D}v(x)) : \Big(v(x), \mathrm{D}v(x)\Big) \Big\},$$

where

$$\Omega_{t_0} \;:=\; \Big\{ \overline{x} \in \overline{\Omega} \;:\; \rho_\infty(t_0) = g\Big(t_0 v(\overline{x}), t_0 \mathrm{D}v(\overline{x})\Big) \Big\}.$$

Hence, by (4.4)(c) we estimate

$$
\begin{aligned}
\rho_\infty'(t_0^+) &= \frac{1}{t_0}\max_{x\in\Omega_{t_0}}\Big\{\partial_{(\eta,P)}g(t_0v(x),t_0\mathrm{D}v(x)) : \Big(t_0v(x),t_0\mathrm{D}v(x)\Big)\Big\} \\
&\geq \frac{C_7}{t_0}\max_{x\in\Omega_{t_0}} g\Big(t_0v(x),t_0\mathrm{D}v(x)\Big) \\
&= \frac{C_7}{t_0}\rho_\infty(t_0) \\
&> 0.
\end{aligned}
$$

This implies that ρ_∞ is strictly increasing on $(0,\infty)$. Next, recall that g is coercive by assumption (4.4)(b), namely $g(s\eta,sP)\to\infty$ as $s\to\infty$, for fixed $(\eta,P)\neq(0,0)$. Thus, for any fixed point $\overline{x}\in\Omega$ with $(v(\overline{x}),\mathrm{D}v(\overline{x}))\neq(0,0)$, which exists because by assumption $v\not\equiv 0$, we have

$$
\lim_{t\to\infty}\rho_\infty(t)\geq\lim_{t\to\infty}g(tv(\overline{x}),t\mathrm{D}v(\overline{x}))=\infty.
$$

Since $\rho_\infty(0)=0$ and $\rho_\infty(t)\to\infty$ as $t\to\infty$, by continuity and the intermediate value theorem, there exists a number $t_\infty>0$ such that $\rho_\infty(t_\infty)=1$, that is

$$
\Big\|g\Big(t_\infty v,t_\infty\mathrm{D}v\Big)\Big\|_{L^\infty(\Omega)}=1.
$$

If $\|g(v,\mathrm{D}v)\|_{L^\infty(\Omega)}=1$, then $t_\infty=1$, as a result of the strict monotonicity of ρ_∞. Now let us fix $p\in(n/\alpha,\infty)$ and define the continuous function

$$
\rho_p(t) := \fint_\Omega g(tv,t\mathrm{D}v)^p\,\mathrm{d}\mathcal{L}^n,\quad t\geq 0.
$$

Since $g(0,0)=0$, it follows that $\rho_p(0)=0$ and that

$$
\rho_p(t) = \frac{1}{\mathcal{L}^n(\Omega)}\int_{\{(v,\mathrm{D}v)\neq(0,0)\}}g(tv,t\mathrm{D}v)^p\,\mathrm{d}\mathcal{L}^n.
$$

By Morrey's theorem and our assumptions, we have that $v\in C^1(\overline{\Omega};\mathbb{R}^N)\setminus\{0\}$, therefore $\mathcal{L}^n\Big(\{(v,\mathrm{D}v)\neq(0,0)\}\Big)>0$. Consider the family of functions $\{g(tv,t\mathrm{D}v)^p\}_{t>0}$, defined on $\{(v,\mathrm{D}v)\neq(0,0)\}\subseteq\Omega$. By the monotonicity of $s\mapsto g(s\eta,sP)$ on $(0,\infty)$ for $(\eta,P)\neq(0,0)$, for $s<t$ we have

$$
g(sv,s\mathrm{D}v)^p\leq g(tv,t\mathrm{D}v)^p,\ \text{ on }\{(v,\mathrm{D}v)\neq(0,0)\}.
$$

Since $g(tv, t\mathrm{D}v)^p \to \infty$ pointwise on $\{(v, \mathrm{D}v) \neq (0,0)\}$ as $t \to \infty$, by the monotone convergence theorem, we infer that

$$\int_{\{(v,\mathrm{D}v)\neq(0,0)\}} g(tv, t\mathrm{D}v)^p \,\mathrm{d}\mathcal{L}^n \longrightarrow \infty,$$

as $t \to \infty$. As a consequence, $\rho_p(t) \to \infty$ as $t \to \infty$. Since $\rho_p(0) = 0$, by the intermediate value theorem there exists $t_p > 0$ such that $\rho_p(t_p) = 1$, namely

$$\Big\| g(t_p v, t_p \mathrm{D}v) \Big\|_{L^p(\Omega)} = 1.$$

For the sake of contradiction, suppose that $t_p \not\to t_\infty$, as $p \to \infty$. In this case, there exists a subsequence $(t_{p_j})_1^\infty \subseteq (n/\alpha, \infty)$ and $t_0 \in [0, t_\infty) \cup (t_\infty, \infty]$ such that $t_{p_j} \to t_0$ as $j \to \infty$. Further, $(t_{p_j})_1^\infty$ can assumed to be either monotonically increasing or decreasing. We first prove that t_0 is finite. If $t_0 = \infty$, then the sequence $(t_{p_j})_1^\infty$ can be selected to be monotonically increasing. Therefore, by arguing as before, $g(t_{p_j} v, t_{p_j}\mathrm{D}v) \nearrow \infty$ as $j \to \infty$, pointwise on $\{(v, \mathrm{D}v) \neq (0,0)\}$, and the monotone convergence theorem provides the contradiction

$$1 \,=\, \lim_{j\to\infty} -\!\!\!\!\!\!\int_\Omega g(t_{p_j} v, t_{p_j}\mathrm{D}v)^{p_j} \,\mathrm{d}\mathcal{L}^n \,=\, -\!\!\!\!\!\!\int_\Omega \lim_{j\to\infty} g(t_{p_j} v, t_{p_j}\mathrm{D}v)^{p_j} \,\mathrm{d}\mathcal{L}^n \,=\, \infty.$$

Consequently, we have that $t_0 \in [0, t_\infty) \cup (t_\infty, \infty)$. Since $(t_{p_j} v, t_{p_j}\mathrm{D}v) \to (t_0 v, t_0 \mathrm{D}v)$ uniformly on $\overline{\Omega}$ as $j \to \infty$, we calculate

$$\begin{aligned} 1 \,&=\, \Big\| g(t_{p_j} v, t_{p_j}\mathrm{D}v) \Big\|_{L^{p_j}(\Omega)} \\ &=\, \Big\| g(t_0 v, t_0 \mathrm{D}v) \Big\|_{L^{p_j}(\Omega)} \,+\, \mathrm{o}(1)_{j\to\infty} \\ &=\, \Big\| g(t_0 v, t_0 \mathrm{D}v) \Big\|_{L^{\infty}(\Omega)} \,+\, \mathrm{o}(1)_{j\to\infty} \\ &=\, \rho_\infty(t_0) \,+\, \mathrm{o}(1)_{j\to\infty}. \end{aligned}$$

By passing to the limit as $j \to \infty$, we obtain a contradiction if $t_\infty \neq t_0$, because ρ_∞ is a strictly increasing function and $\rho_\infty(t_\infty) = 1$. In conclusion, $t_p \to t_\infty$ as $p \to \infty$. $\square$

Utilising the above result we can now show existence for the approximating minimisation problem for $p < \infty$.

Lemma 4.2.2. *For any $p > n/\alpha$, the minimisation problem (4.9) has a solution $u_p \in W^{2,\alpha p}_{\mathrm{B}}(\Omega; \mathbb{R}^N)$.*

Proof of Lemma 4.2.2. Let us fix $p \in (n/\alpha, \infty)$ and $v_0 \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$ where $v_0 \not\equiv 0$. By application of Lemma 4.2.1, there exists $t_p > 0$ such that $\|g(t_p v_0, t_p \mathrm{D} v_0)\|_{L^p(\Omega)} = 1$ implying that $t_p v_0$ is indeed an element of the admissible class of (4.9). Hence, we deduce that the admissible class is non empty. Further, by assumption (4.3)(b), f is (Morrey) quasiconvex. We now confirm that f^p is also (Morrey) quasiconvex function, as a consequence of Jensen's inequality: for any fixed $X \in \mathbb{R}^{N\times n^2}_s$ and any $\phi \in W^{2,\infty}_0(\Omega;\mathbb{R}^N)$, we have

$$f^p(X) \leq \left(\fint_\Omega f(X + \mathrm{D}^2\phi)\, \mathrm{d}\mathcal{L}^n \right)^p \leq \fint_\Omega f(X+\mathrm{D}^2\phi)^p\, \mathrm{d}\mathcal{L}^n.$$

By assumption by assumption (4.3)(d), we have for some new $C_5(p), C_6(p) > 0$ that

$$f(X)^p \, \leq \, C_5(p)|X|^{\alpha p} + C_6(p),$$

for any $X \in \mathbb{R}^{N\times n^2}_s$. Moreover, by [95, Theorem 3.6] we have that the functional $v \mapsto \|f(\mathrm{D}^2 v)\|_{L^p(\Omega)}$ is weakly lower semi-continuous on $W^{2,\alpha p}(\Omega;\mathbb{R}^N)$ and therefore the same is true over the closed subspace $W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$. Let $(u_i)_1^\infty$ be a minimising sequence for (4.9). As $f \geq 0$, it is clear that $\inf_{i\in\mathbb{N}} \|f(\mathrm{D}^2 u_i)\|_{L^p(\Omega)} \geq 0$. Since the admissible class is non-empty, the infimum is finite. Additionally, by (4.3)(d), we have the bound

$$\begin{aligned}
\inf_{i\in\mathbb{N}} \|f(\mathrm{D}^2 u_i)\|_{L^p(\Omega)} &\leq \left\| f\Big(\mathrm{D}^2(t_p v_0)\Big)\right\|_{L^p(\Omega)} \\
&\leq \left\| C_5 \Big| t_p \mathrm{D}^2 v_0\Big|^\alpha + C_6 \right\|_{L^\infty(\Omega)} \\
&\leq C_5 (t_p)^\alpha \|\mathrm{D}^2 v_0\|^\alpha_{L^\infty(\Omega)} + C_6 \\
&< \infty.
\end{aligned}$$

Now we show that the functional is coercive in $W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$, arguing separately for either case of boundary conditions. By assumption (4.3)(d) and the Poincaré inequality, for any $u \in W^{2,\alpha p}_{\mathrm{C}}(\Omega;\mathbb{R}^N)$ (satisfying $|u| = |\mathrm{D}u| = 0$ on $\partial\Omega$), we have

$$\left(\fint_\Omega \Big| f(\mathrm{D}^2 u) + C_3 \Big|^p \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}} \geq C_4 \left(\fint_\Omega |\mathrm{D}^2 u|^{\alpha p}\, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}} \, \geq C_4' \|u\|^\alpha_{W^{1,\alpha p}(\Omega)},$$

for a new constant $C_4' = C_4(p) > 0$. Hence, for any $u \in W^{2,\alpha p}_{\mathrm{C}}(\Omega;\mathbb{R}^N)$,

$$\|f(\mathrm{D}^2 u)\|_{L^p(\Omega)} \, \geq \, C_4' \Big(\|u\|_{W^{2,\alpha p}(\Omega)} \Big)^\alpha - C_3. \tag{4.11}$$

The above estimate is also true when $u \in W^{2,\alpha p}_{\mathrm{H}}(\Omega;\mathbb{R}^N)$, but since in this case we have only $|u| = 0$ on $\partial\Omega$, it requires an additional justification. By the Poincaré-Wirtinger inequality involving averages, for any $u \in W^{2,\alpha p}_{\mathrm{H}}(\Omega;\mathbb{R}^N)$ we have

$$\left\| \mathrm{D}u - \fint_\Omega \mathrm{D}u \, \mathrm{d}\mathcal{L}^n \right\|_{L^{\alpha p}(\Omega)} \leq C\|\mathrm{D}^2 u\|_{L^{\alpha p}(\Omega)},$$

where $C = C(\alpha, p, \Omega) > 0$ is a constant. Since $|u| = 0$ on $\partial\Omega$, by the Gauss-Green theorem we have

$$\int_\Omega \mathrm{D}u \, \mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} u \otimes \hat{n} \, \mathrm{d}\mathcal{H}^{n-1} = 0,$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. In conclusion,

$$\big\|\mathrm{D}u\|_{L^{\alpha p}(\Omega)} \leq C\|\mathrm{D}^2 u\|_{L^{\alpha p}(\Omega)},$$

for any $u \in W^{2,\alpha p}_{\mathrm{H}}(\Omega;\mathbb{R}^N)$. The above estimate together with the standard Poincaré inequality applied to u itself allow to infer that (4.11) holds for any $u \in W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$ in both cases of boundary conditions. Returning to our minimising sequence, by standard compactness results, exists $u_p \in W^{2,\alpha p}_{\mathrm{H}}(\Omega;\mathbb{R}^N)$ such that $u_i \longrightarrow u_p$ in $W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$, as $i \to \infty$ along a subsequence of indices. Additionally, by the Morrey estimate we have that $u_i \longrightarrow u_p$ in $C^1(\overline{\Omega};\mathbb{R}^N)$ as $i \to \infty$, along perhaps a further subsequence. Since $u \mapsto \|g(u, \mathrm{D}u)\|_{L^p(\Omega)}$ is weakly continuous on $W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$, the admissible class is weakly closed in $W^{2,\alpha p}(\Omega;\mathbb{R}^N)$ and hence we may pass to the limit in the constraint. By weak lower semicontinuity of the functional, it follows that a minimiser u_p which satisfies (4.9) does indeed exist. $\square$

Now we describe the necessary conditions (Euler-Lagrange equations) that approximating minimiser u_p must satisfy. These equations will involve a Lagrange multiplier, emerging from the constraint $\|g(\cdot, \mathrm{D}(\cdot))\|_{L^p(\Omega)} = 1$.

Lemma 4.2.3. *For any $p > n/\alpha$, let u_p be the minimiser of (4.9) procured by Lemma 4.2.2. Then, there exists $\lambda_p \in \mathbb{R}$ such that the pair (u_p, λ_p) satisfies the following PDE system*

$$\begin{aligned}
&\fint_\Omega f(\mathrm{D}^2 u_p)^{p-1} \partial f(\mathrm{D}^2 u_p) : \mathrm{D}^2 \phi \, \mathrm{d}\mathcal{L}^n \\
&= \lambda_p \fint_\Omega g(u_p, \mathrm{D}u_p)^{p-1} \Big(\partial_\eta g(u_p, \mathrm{D}u_p) \cdot \phi + \partial_P g(u_p, \mathrm{D}u_p) : \mathrm{D}\phi \Big) \, \mathrm{d}\mathcal{L}^n,
\end{aligned}$$

for all test maps $\phi \in W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$.

In particular, it follows that in both cases u_p is a weak solution in $W^{2,\alpha p}(\Omega;\mathbb{R}^N)$ to

$$\left\{\begin{array}{l} \mathrm{D}^2 : \Big(f(\mathrm{D}^2 u_p)^{p-1}\partial f(\mathrm{D}^2 u_p)\Big) \\ \;=\; \lambda_p \Big[g(u_p,\mathrm{D}u_p)^{p-1}\partial_\eta g(u_p,\mathrm{D}u_p) \;-\; \mathrm{div}\Big(g(u_p,\mathrm{D}u_p)^{p-1}\partial_P g(u_p,\mathrm{D}u_p)\Big)\Big], \end{array}\right. \tag{4.12}$$

where we have used the notation $\mathrm{D}^2 : F = \sum_{i,j=1}^n \mathrm{D}^2_{ij}F_{ij}$, when $F \in C^2(\Omega;\mathbb{R}^{n\times n})$, which is equivalent to the double divergence (applied once column-wise and once row-wise). Note that in the case of hinged boundary data, we have an additional natural boundary condition arising (since $\mathrm{D}u$ is free on $\partial\Omega$), we we will not make an particular use of this extra information in the sequel, therefore we refrain from discussing it explicitly.

Proof of Lemma 4.2.3. The result follows by standard results on Lagrange multipliers in Banach spaces (see e.g. [94, p. 278]), by utilising assumption (4.3)(d), which guarantees that the functional is Gateaux differentiable. ☐

Now we establish some further results regarding the family of eigenvalues.

Lemma 4.2.4. *Consider the family of pairs of eigenvectors-eigenvalues* $\{(u_p,\lambda_p)\}_{p>n/\alpha}$*, given by Lemma 4.2.3. Then, for any* $p > n/\alpha$*, there exists* $\Lambda_p > 0$ *such that*

$$\lambda_p = \big(\Lambda_p\big)^p > 0.$$

Further, by setting

$$L_p := \Big\| f(\mathrm{D}^2 u_p)\Big\|_{L^p(\Omega)},$$

we have the bounds

$$0 < \left(\frac{C_1}{C_8}\right)^{\frac{1}{p}} L_p \,\le\, \Lambda_p \,\le\, \left(\frac{C_2}{C_7}\right)^{\frac{1}{p}} L_p.$$

Proof of Lemma 4.2.4. We begin by showing that $L_p > 0$, namely the infimum over the admissible class of the p-approximating minimisation problem is strictly positive, owing to the constraint and our assumptions (4.3)-(4.4). Indeed, there is only one map $u \in W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$ for which $\|f(\mathrm{D}^2 u)\|_{L^p(\Omega)} = 0$, namely $u_0 \equiv 0$, but this is not an element of the admissible class since $\|g(u_0,\mathrm{D}u_0)\|_{L^p(\Omega)} = 0$. Now consider the Euler-Lagrange equations in Lemma 4.2.3 and select $\phi := u_p$, to obtain

$$\begin{aligned} &\fint_\Omega f(\mathrm{D}^2 u_p)^{p-1}\partial f(\mathrm{D}^2 u_p) : \mathrm{D}^2 u_p \,\mathrm{d}\mathcal{L}^n \\ &= \lambda_p \fint_\Omega g(u_p,\mathrm{D}u_p)^{p-1}\Big(\partial_\eta g(u_p,\mathrm{D}u_p)\cdot u_p + \partial_P g(u_p,\mathrm{D}u_p) : \mathrm{D}u_p\Big)\,\mathrm{d}\mathcal{L}^n. \end{aligned}$$

As $f, g \geq 0$ we can manipulate the respective assumptions (4.3)(c) and (4.4)(c) to produce the following bounds:

$$
\begin{aligned}
C_1 \fint_\Omega f(\mathrm{D}^2 u_p)^p \,\mathrm{d}\mathcal{L}^n \;&\leq\; \fint_\Omega f(\mathrm{D}^2 u_p)^{p-1} \partial f(\mathrm{D}^2 u_p) : \mathrm{D}^2 u_p \,\mathrm{d}\mathcal{L}^n \\
&\leq\; C_2 \fint_\Omega f(\mathrm{D}^2 u_p)^p \,\mathrm{d}\mathcal{L}^n,
\end{aligned}
$$

$$
\begin{aligned}
C_7 \fint_\Omega g(u_p, \mathrm{D}u_p)^p \,\mathrm{d}\mathcal{L}^n \leq\; &\fint_\Omega g(u_p, \mathrm{D}u_p)^{p-1} \Big(\partial_\eta g(u_p, \mathrm{D}u_p) \cdot u_p + \\
&+\; \partial_P g(u_p, \mathrm{D}u_p) : \mathrm{D}u_p \Big) \,\mathrm{d}\mathcal{L}^n \\
\leq\; &C_8 \fint_\Omega g(u_p, \mathrm{D}u_p)^p \,\mathrm{d}\mathcal{L}^n.
\end{aligned}
$$

The above two estimates, combined with the Euler-Lagrange equations, imply that $\lambda_p > 0$. Hence, we may therefore define $\Lambda_p := (\lambda_p)^{\frac{1}{p}} > 0$. We will now obtain the upper and lower bounds. We determine the lower bound as follows:

$$
\begin{aligned}
C_1 (L_p)^p \;&=\; C_1 \fint_\Omega f(\mathrm{D}^2 u_p)^p \,\mathrm{d}\mathcal{L}^n \\
&\leq\; \fint_\Omega f^{p-1}(\mathrm{D}^2 u_p) \partial f(\mathrm{D}^2 u_p) : \mathrm{D}^2 u_p \,\mathrm{d}\mathcal{L}^n \\
&=\; \lambda_p \fint_\Omega g(u_p, \mathrm{D}u_p)^{p-1} \Big(\partial_\eta g(u_p, \mathrm{D}u_p) \cdot \phi \;+\; \partial_P g(u_p, \mathrm{D}u_p) : \mathrm{D}u_p \Big) \,\mathrm{d}\mathcal{L}^n \\
&\leq\; \lambda_p C_8.
\end{aligned}
$$

Hence,

$$
\left(\frac{C_1}{C_8} \right)^{\frac{1}{p}} L_p \leq (\lambda_p)^{\frac{1}{p}} = \Lambda_p.
$$

The upper bound is determined analogously, by reversing the direction of the inequalities. Combining both bounds, we obtain the desired estimate. $\square$

Proposition 4.2.5. *There exists $(u_\infty, \Lambda_\infty) \in W^{2,\infty}_{\mathrm{B}}(\Omega; \mathbb{R}^N) \times (0, \infty)$ such that, along a sequence $(p_j)_1^\infty$ of exponents, we have*

$$
\left\{
\begin{array}{lll}
u_p \longrightarrow u_\infty, & \text{in } C^1\big(\overline{\Omega}; \mathbb{R}^N\big), & \\
\mathrm{D}^2 u_p \rightharpoonup \mathrm{D}^2 u_\infty, & \text{in } L^q(\Omega; \mathbb{R}^{N \times n^2}_s), & \text{for all } \; q \in (1, \infty), \\
\Lambda_p \longrightarrow \Lambda_\infty, & \text{in } [0, \infty), &
\end{array}
\right.
$$

as $p_j \to \infty$. Additionally, u_∞ solves the minimisation problem (4.1) and Λ_∞ is given by (4.6).

Proof of Proposition 4.2.5. Fix $p > n/\alpha, q \leq p$ and a map $v_0 \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N) \setminus \{0\}$. Then, by Lemma 4.2.1 there exists $(t_p)_{p\in(n/\alpha,\infty]} \subseteq (0,\infty)$ such that $t_p \to t_\infty$ as $p \to \infty$ and satisfying $\|g(t_p v_0, t_p \mathrm{D}v_0)\|_{L^p(\Omega)} = 1$ for all $p \in (n/\alpha,\infty]$. By Hölder's inequality and minimality, we have the following estimate

$$\begin{aligned}
\left\|f(\mathrm{D}^2 u_p)\right\|_{L^q(\Omega)} &\leq \left\|f(\mathrm{D}^2 u_p)\right\|_{L^p(\Omega)} \\
&\leq \left\|f(t_p \mathrm{D}^2 v_0)\right\|_{L^p(\Omega)} \\
&\leq \left\|f(t_p \mathrm{D}^2 v_0)\right\|_{L^\infty(\Omega)} \\
&\leq K + \left\|f(t_\infty \mathrm{D}^2 v_0)\right\|_{L^\infty(\Omega)} \\
&< \infty,
\end{aligned}$$

for some $K > 0$. By (4.3)(d), we have the bound $f^q(X) \geq C_4(q)|X|^{\alpha q} - C_3(q)$ for some constants $C_3(q), C_4(q) > 0$ and all $X \in \mathbb{R}^{N\times n^2}_s$. By the previous bound, we conclude that

$$\sup_{q\geq p} \|\mathrm{D}^2 u_p\|_{L^{\alpha q}(\Omega)} \leq C(q) < \infty,$$

for some q-dependent constant. By arguing as in the proof of Lemma 4.2.2 through the use of Poincaré inequalities, we can conclude in both cases of boundary conditions with the bound

$$\sup_{q\geq p} \|u_p\|_{W^{2,\alpha q}(\Omega)} \leq C(q) < \infty,$$

for a new q-dependent constant $C'(q) > 0$. Standard compactness in Sobolev spaces and a diagonal sequence argument imply the existence of a mapping

$$u_\infty \in \bigcap_{n/\alpha<p<\infty} W^{2,\alpha p}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$$

and a subsequence $(p_j)_1^\infty$ such that the desired modes of convergence hold true as $p_j \to \infty$ along this subsequence of indices. Fix a map $v \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$ satisfying the required constraint, namely $\|g(v,\mathrm{D}v)\|_{L^\infty(\Omega)} = 1$. In view of Lemma 4.2.1, there exists $(t_p)_{p\in(n/\alpha,\infty)} \subseteq (0,\infty)$ satisfying that $t_p \to 1$ as $p \to \infty$, and additionally $\|g(t_p v, t_p \mathrm{D}v)\|_{L^p(\Omega)} = 1$ for all $p > n/\alpha$. By Hölder's inequality, the definition of L_p and minimality, we have

$$\left\|f(\mathrm{D}^2 u_p)\right\|_{L^q(\Omega)} \leq \left\|f(\mathrm{D}^2 u_p)\right\|_{L^p(\Omega)} = L_p \leq \left\|f(t_p \mathrm{D}^2 v)\right\|_{L^p(\Omega)},$$

for any such v. By the weak lower semi-continuity of the functional on $W^{2,\alpha q}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$, we may let $p_j \to \infty$ to obtain

$$\begin{aligned}\Big\| f(\mathrm{D}^2 u_\infty)\Big\|_{L^q(\Omega)} \,&\leq\, \liminf_{p_j\to\infty} L_p\\ &\leq\, \limsup_{p_j\to\infty} L_p\\ &\leq\, \limsup_{p_j\to\infty} \|f(t_{p_j}\mathrm{D}^2 v)\|_{L^p(\Omega)}\\ &=\, \|f(\mathrm{D}^2 v)\|_{L^\infty(\Omega)}.\end{aligned}$$

Now we may let $q \to \infty$ in the above bound, hence producing

$$\Big\| f(\mathrm{D}^2 u_\infty)\Big\|_{L^\infty(\Omega)} \,\leq\, \liminf_{p_j\to\infty} L_p \,\leq\, \limsup_{p_j\to\infty} L_p \,\leq\, \|f(\mathrm{D}^2 v)\|_{L^\infty(\Omega)}.$$

for all mappings $v \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$ satisfying the constraint $\|g(v,\mathrm{D}v)\|_{L^\infty(\Omega)} = 1$. If we additionally show that in fact u_∞ satisfies the constraint in (4.1), then the above estimate shows both that it is the desired minimisers (by choosing $v := u_\infty$), and also that the sequence $(L_{p_j})_1^\infty$ converges to the infimum. Now we show that this is indeed the case. In view of assumption (4.3)(d), the previous estimate implies also that $\mathrm{D}^2 u_\infty \in L^\infty\Big(\Omega;\mathbb{R}^{N\times n^2}_s\Big)$, which together with Poincaré inequalities (as in the proof of Lemma 4.2.2) shows that in fact $u_\infty \in W^{2,\infty}_{\mathrm{B}}(\Omega;\mathbb{R}^N)$. By the continuity of the function g and the fact that $u_p \longrightarrow u_\infty$ in $C^1\Big(\overline{\Omega};\mathbb{R}^N\Big)$, we have

$$\begin{aligned}1 \,&=\, \|g(u_p,\mathrm{D}u_p)\|_{L^p(\Omega)}\\ &=\, \|g(u_\infty,\mathrm{D}u_\infty)\|_{L^p(\Omega)} \,+\, \|g(u_p,\mathrm{D}u_p)\|_{L^p(\Omega)} - \|g(u_\infty,\mathrm{D}u_\infty)\|_{L^p(\Omega)}\\ &=\, \|g(u_\infty,\mathrm{D}u_\infty)\|_{L^p(\Omega)} \,+\, \mathrm{O}\Big(\|g(u_p,\mathrm{D}u_p) - g(u_\infty,\mathrm{D}u_\infty)\|_{L^\infty(\Omega)}\Big)\\ &\longrightarrow \|g(u_\infty,\mathrm{D}u_\infty)\|_{L^\infty(\Omega)},\end{aligned}$$

as $p_j \to \infty$. Consequently, u_∞ satisfies the constraint, and therefore lies in the admissible class of (4.1). Since v was arbitrary in the energy bound, we conclude that u_∞ in fact solves (4.1). let us now define

$$\Lambda_\infty := \Big\| f(\mathrm{D}^2 u_\infty)\Big\|_{L^\infty(\Omega)}.$$

We now show that $\Lambda_\infty > 0$. Indeed, by our assumptions (4.3)-(4.4), there is only one map in $W^{2,\infty}(\Omega;\mathbb{R}^N)$ satisfying $\|f(\mathrm{D}^2 u_0)\|_{L^\infty(\Omega)} = 0$ and $|u_0| \equiv 0$ on $\partial\Omega$, namely the trivial map $u_0 \equiv 0$, but u_0 is not contained in the admissible class of (4.1) because $\|g(u_0,\mathrm{D}u_0)\|_{L^\infty(\Omega)} =$

0. We now show that $\Lambda_p \longrightarrow \Lambda_\infty$ as $p_j \to \infty$. By our earlier energy estimate, we have $L_p \longrightarrow \Lambda_\infty$ as $p_j \to \infty$. By Lemma 4.2.4, we have

$$0 < \lim_{p_j\to\infty} \left(\frac{C_1}{C_8}\right)^{\frac{1}{p}} L_p \leq \lim_{p_j\to\infty} \Lambda_p \leq \lim_{p_j\to\infty} \left(\frac{C_2}{C_7}\right)^{\frac{1}{p}} L_p,$$

and therefore $\Lambda_p \longrightarrow \Lambda_\infty$ as $p_j \to \infty$. The result ensues. $\square$

Lemma 4.2.6. *For any $p > (n/\alpha) + 2$, there exist measures $\nu_\infty \in \mathcal{M}(\overline{\Omega})$ and $\mathrm{M}_\infty \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^{N\times n^2}_s)$ such that, along perhaps a further sequence $(p_j)_1^\infty$ of exponents, we have*

$$\left\{\begin{array}{ll} \nu_p \xrightarrow{*} \nu_\infty, & \text{in } \mathcal{M}(\overline{\Omega}), \\ \mathrm{M}_p \xrightarrow{*} \mathrm{M}_\infty, & \text{in } \mathcal{M}(\overline{\Omega}; \mathbb{R}^{N\times n^2}_s), \end{array}\right.$$

as $j \to \infty$, where the approximating measures ν_p, M_p are given by (4.10).

Proof of Lemma 4.2.6. We begin by noting that since $g \geq 0$ and $\|g(u_p, \mathrm{D}u_p)\|_{L^p(\Omega)} = 1$, in view of (4.10) we have the bound

$$\|\nu_p\|(\overline{\Omega}) = \nu_p(\overline{\Omega}) = \fint_\Omega g(u_p, \mathrm{D}u_p)^{p-1}\, \mathrm{d}\mathcal{L}^n \leq \left(\fint_\Omega g(u_p, \mathrm{D}u_p)^p\, \mathrm{d}\mathcal{L}^n\right)^{\frac{p-1}{p}} = 1.$$

By the sequential weak* compactness of the space of Radon measures we can conclude that $\nu_p \xrightarrow{*} \nu_\infty$, in $\mathcal{M}(\overline{\Omega})$ up to the passage to a further subsequence. Now we establish appropriate total variation bounds for the measure M_p. Since $f \geq 0$, by the bounds of Lemma 4.2.4 and assumption (4.3), we estimate (for sufficiently large p)

$$\begin{aligned} \|\mathrm{M}_p\|(\overline{\Omega}) &= \fint_\Omega \left(\frac{f(\mathrm{D}^2 u_p)}{\Lambda_p}\right)^{p-1} |\partial f(\mathrm{D}^2 u_p)|\, \mathrm{d}\mathcal{L}^n \\ &\leq \frac{1}{\Lambda_p^{p-1}} \fint_\Omega f(\mathrm{D}^2 u_p)^{p-1}\Big(C_5 f(\mathrm{D}^2 u_p)^\beta + C_6\Big)\, \mathrm{d}\mathcal{L}^n \\ &= \frac{C_5}{\Lambda_p^{p-1}} \fint_\Omega f(\mathrm{D}^2 u_p)^{p-1+\beta}\, \mathrm{d}\mathcal{L}^n + \frac{C_6}{\Lambda_p^{p-1}} \fint_\Omega f(\mathrm{D}^2 u_p)^{p-1}\, \mathrm{d}\mathcal{L}^n. \end{aligned}$$

Hence,

$$
\begin{aligned}
\|\mathrm{M}_p\|(\overline{\Omega}) \ &\leq\ \frac{C_5}{\Lambda_p^{p-1}}\bigg(-\!\!\!\!\!\!\int_\Omega f(\mathrm{D}^2u_p)^p\,\mathrm{d}\mathcal{L}^n\bigg)^{\frac{p-1+\beta}{p}}\ +\ \frac{C_6}{\Lambda_p^{p-1}}\bigg(-\!\!\!\!\!\!\int_\Omega f(\mathrm{D}^2u_p)^p\,\mathrm{d}\mathcal{L}^n\bigg)^{\frac{p-1}{p}}\\
&=\ C_5\frac{(L_p)^{p-1+\beta}}{\Lambda_p^{p-1}}\ +\ C_6\frac{(L_p)^{p-1}}{\Lambda_p^{p-1}}\\
&=\ \bigg(\frac{L_p}{\Lambda_p}\bigg)^{p-1}\Big(C_5L_p^\beta+C_6\Big)\\
&\leq\ \bigg(\frac{C_8}{C_1}\bigg)^{1-\frac{1}{p}}\Big(C_5(\Lambda_\infty+1)^\beta+C_6\Big).
\end{aligned}
$$

The above bound allows to conclude that $\mathrm{M}_p \overset{*}{\rightharpoonup} \mathrm{M}_\infty$ in $\mathcal{M}(\overline{\Omega};\mathbb{R}_s^{N\times n^2})$, along perhaps a further subsequence of indices $(p_j)_1^\infty$ as $j\to\infty$. $\square$

To conclude the proof of Theorem 4.1.1 we must ensure the PDE system (4.5) is indeed satisfied by the quadruple $(u_\infty,\Lambda_\infty,\mathrm{M}_\infty,\nu_\infty)$.

Lemma 4.2.7. *If $\mathrm{M}_\infty\in\mathcal{M}(\overline{\Omega};\mathbb{R}_s^{N\times n^2})$ and $\nu_\infty\in\mathcal{M}(\overline{\Omega})$ are the measures obtained in Lemma 4.2.6, then the pair $(u_\infty,\Lambda_\infty)$ satisfies (4.5) for all $\phi\in C^2_{\mathrm{B}}(\overline{\Omega};\mathbb{R}^N)$.*

Proof of Lemma 4.2.7. Fix a test function $\phi\in C^2_{\mathrm{B}}(\overline{\Omega};\mathbb{R}^N)$ and $p>n/\alpha+2$ by (4.10) we may rewrite the PDE system in (4.9) as follows

$$
\int_\Omega \mathrm{D}^2\phi:\mathrm{dM}_p=\ \Lambda_p\int_\Omega\Big(\partial_\eta g(u_p,\mathrm{D}u_p)\cdot\phi+\partial_P g(u_p,\mathrm{D}u_p):\mathrm{D}\phi\Big)\,\mathrm{d}\nu_p,
$$

Recall that, by Proposition 4.2.5, we have $\Lambda_p\longrightarrow\Lambda_\infty$ and also $(u_p,\mathrm{D}u_p)\longrightarrow(u_\infty,\mathrm{D}u_\infty)$ uniformly on $\overline{\Omega}$, as $p_j\to\infty$. By assumption (4.4)(a), we have that $\partial_\eta g(u_p,\mathrm{D}u_p)\longrightarrow\partial_\eta g(u_\infty,\mathrm{D}u_\infty)$ and also $\partial_P g(u_p,\mathrm{D}u_p)\longrightarrow\partial_P g(u_\infty,\mathrm{D}u_\infty)$, both uniformly on $\overline{\Omega}$, as $p_j\to\infty$. The result ensues by invoking Lemma 4.2.6, in conjunction with weak*-strong continuity of the duality pairing $\mathcal{M}(\overline{\Omega})\times C(\overline{\Omega})\longrightarrow\mathbb{R}$. $\square$

Chapter 5

Conclusion and Future Work

In this chapter we discuss the main conclusions drawn from Chapters 2, 3, 4 and how these relate to the aim of our thesis, mentioned in Chapter 1. Several directions for further work are outlined, also how we can surpass some of the limitations within our current work.

5.1 Conclusions

In conclusion, this thesis is a collection of papers, presented as chapters, that are comprised of original research. This work consists of current progress in the field of vectorial Calculus of Variations in L^∞. These contemporary results are varied in nature and include the contemplation of new problems and the generalisation of previously existing theory. For example, Chapter 2 is a novel consideration, whilst Chapters 3 and 4 are extensions of previous publications.

The main results throughout this thesis are concerned with establishing conditions, that constrained supremal functionals must satisfy. Specifically, the results are Theorems 2.1.3, 3.1.3 and 4.1.1. These results are built on the methodology of L^p approximations, where we have explored sophisticated contrasting limiting processes. Given the anatomy of the vectorial environment, we could not employ the intrinsic characterisation that exists for scalar problems. The technique of L^p approximations was the only means we had available to us, to tackle the problem of finding such conditions in the L^∞ setting.

In each chapter we have pursued the same goal, whilst varying the nature of the investigation. We have noted how minor adaptations to a constraint can have far reaching implications, in the derivation and manifestation of PDE conditions.

Once we achieved our main intentions in Chapter 2, we started to investigate more comprehensive problems, beyond the specificity of the Navier-Stokes equations. This led to the outputs of Chapters 3 and 4, where we addressed more abstract questions concerning constrained minimisation problems. Overall, we have achieved our objective and made a significant contribution to the area. We have furthered the development of the theoretical framework required for establishing PDE conditions in the field of constrained supremal functionals.

One could continue this investigation and even contemplate some of the alternative proposals in the next section.

5.2 Future work

This is in an extremely fascinating field of mathematics and there are still many open problems one could examine.

We could contemplate the premise of constrained vectorial absolute minimisers. It is not apparent if you have a constraint how you define an absolute minimiser. We do not review it here, as the problems are already considerably complicated. Arronson and Barron have already noted that this is a rather challenging question in [11]. In the specific case of Chapter 2 this will be a demanding task, we have multiple components with different levels of regularity. It is a rather thought provoking task how to define variations with respect to the Navier-Stokes equations, whilst acknowledging the structure of the three components in the problem. Typically the method of L^p approximations gives us limits that are eligible to be the best we might hope for, but this does need to be rigorously justified.

One could attempt to achieve optimality in our existence results. This could be done by considering the L^∞ problem directly, without using L^p approximations. This would require the notion of "BJW-quasiconvexity" of Barron-Jensen-Wang in [17], as opposed to Morrey quasiconvexity used in this thesis. For us the L^p problem was a stepping stone to the L^∞ one. We can not differentiate the L^∞ norm. Hence, we can not directly implement the generalised Kuhn-Tucker theory (method of generalised Lagrange multipliers) in the L^∞ setting, without an estimation technique.

In this thesis, we have investigated constrained vectorial supremal functionals over Euclidean domains. One could potentially broaden this approach, to minimise functionals over more complicated domains, such as manifolds.

This area of research has only developed in last few years, with all results being of analytical interest. Currently, no numerical considerations have been made. Consequently, one could investigate the construction of numerical methods. Note that our results for finite p should not be disregarded and could potentially support the discretisation process of the problems presented in this thesis.

A possible extension to Chapter 2 would be to consider the same motivational notion of variational data assimilation, but constrain the process by a different equation. The structure of the equation will certainly modify the techniques required in the limiting process. Depending on the choice of equation, this research could produce outcomes that are of theoretical and applications based interests. For instance, variational data assimilation can also be used to model traffic flow. We could constrain the minimisation process by relevant conservation laws.

A further augmentation of Chapter 2 would be to look at the same problem but relax some of our assumptions. For instance, strong solutions can be quite restrictive, hence we could limit our attention to only weak solutions of the Navier-Stokes equations. This is a completely different investigation. We would need to reestablish coercivity for the functional, to deduce relevant bounds, to substantiate any form of compactness. This would involve deriving a new bound for the solutions of Navier-Stokes equations, under less regularity than (2.15).

As mentioned in Chapter 3, we could further explore the PDE conditions required for null Lagrangian and determinant constraints in the isosupremic problem. Given the complexity of such a nonlinear constraint, we would require an alternative method to the standard approach employed throughout this thesis.

A more general consideration would be to contemplate higher order problems. For instance, we could extend Chapter 4, to examine the third order generalised vectorial ∞-eigenvalue problem. Given the nature of this problem, we could most likely allow the constraint g to depend upon the Hessian, specifically $g(u, \mathrm{D}u, \mathrm{D}^2u)$. Some potential assumptions for the functions f and g are as follows:

$$
\begin{cases}
(a)\ f \in C^1(\mathbb{R}_s^{N\times n^3}).\\
(b)\ f \text{ is (Morrey) quasiconvex.}\\
(c)\ \text{There exist } 0 < C_1 \leq C_2 \text{ such that, for all } Y \in \mathbb{R}_s^{N\times n^3} \setminus \{0\},\\
\qquad\qquad 0 < C_1 f(Y) \ \leq\ \partial f(Y) : Y \ \leq\ C_2 f(Y).\\
(d)\ \text{There exist } C_3, ..., C_6 > 0, \alpha > 1, \beta \leq 1 \text{ such that, for all } Y \in \mathbb{R}_s^{N\times n^3},\\
\qquad\qquad -C_3 + C_4|Y|^\alpha \ \leq\ f(Y) \leq C_5|Y|^\alpha + C_6,\\
\qquad\qquad |\partial f(Y)| \ \leq\ C_5 f(Y)^\beta + C_6.
\end{cases}
\tag{5.1}
$$

$$
\begin{cases}
(a)\ g \in C^1(\mathbb{R}^N \times \mathbb{R}^{N\times n} \times \mathbb{R}_s^{N\times n^2}).\\
(b)\ g \text{ is coercive, i.e for any } (\eta, P, X) \in \left(\mathbb{R}^N \times \mathbb{R}^{N\times n} \times \mathbb{R}_s^{N\times n^2}\right) \setminus \{(0,0,0)\},\\
\qquad\qquad \lim_{t\to\infty} g(t\eta, tP, tX) = \infty.\\
(c)\ \text{There exist } 0 < C_7 \leq C_8 \text{ such that, for all}\\
\qquad\qquad (\eta, P, X) \in \left(\mathbb{R}^N \times \mathbb{R}^{N\times n} \times \mathbb{R}_s^{N\times n^2}\right) \setminus \{(0,0,0)\},\\
0 < C_7\, g \ \leq\ \partial_\eta g \cdot \eta + \partial_P g : P + \partial_X g : X \ \leq\ C_8\, g.
\end{cases}
\tag{5.2}
$$

An addition to Chapter 4 would be to produce uniform bounds for the eigenvalue, as in [67]. In fact, we have already initiated this avenue of inquisition. Note that the lower bound can be ascertained using similar ideas to the first order problem through application of Poincaré and Poincaré-Wirtinger inequalities. Due to the nature of the second order problem, the derivation of the upper bound is substantially more difficult than the first order case. This involves sophisticated geometric concepts such as the second fundamental form, when selecting appropriate mollified test functions.

We have only started to scratch the surface of this vast and elegant area of mathematics.

Appendix A

Additional Bound for the Operator $\mathfrak{M}_p$

Here we provide a proof for the $L^{p'}$ bound mentioned in Chapter 2 (page 15). Recall that for any $M \in \mathbb{N}$ and $p \in (1, \infty)$, we define the operator

$$\mathfrak{M}_p \;:\; L^p(\Omega_T; \mathbb{R}^M) \longrightarrow L^{p'}(\Omega_T; \mathbb{R}^M),$$

where $p' := p/(p-1)$, by setting

$$\mathfrak{M}_p(V) \;:=\; \frac{|V|_{(p)}^{p-2}\, V}{\Big(\|V\|_{\dot{L}^p(\Omega_T)}\Big)^{p-1}}.$$

Here $|\cdot|_{(p)}$ is the regularisation of the Euclidean norm of $\mathbb{R}^M$.

Lemma A.0.1. . *We have*

$$\Big\|\mathfrak{M}_p(V)\Big\|_{L^{p'}(\Omega_T)} \;\leq\; 1,$$

and therefore $\mathfrak{M}_p$ is valued in the unit ball of $L^{p'}(\Omega_T; \mathbb{R}^M)$.

Proof of Lemma A.0.1.

$$
\begin{aligned}
\left\|\mathfrak{M}_p(V)\right\|_{L^{p'}(\Omega_T)} &= \left(\fint_{\Omega_T} \left| \frac{|V|_{(p)}^{p-2}\, V}{\left(\|V\|_{\dot{L}^p(\Omega_T)}\right)^{p-1}} \right|^{\frac{p}{p-1}} \mathrm{d}\mathcal{L}^{n+1} \right)^{\frac{p-1}{p}} \\
&= \left(\fint_{\Omega_T} \left(\frac{|V|_{(p)}^{p-2}\, |V|}{\left(\|V\|_{\dot{L}^p(\Omega_T)}\right)^{p-1}} \right)^{\frac{p}{p-1}} \mathrm{d}\mathcal{L}^{n+1} \right)^{\frac{p-1}{p}} \\
&\leq \left(\fint_{\Omega_T} \left(\frac{|V|_{(p)}^{p-2}\, |V|_{(p)}}{\left(\|V\|_{\dot{L}^p(\Omega_T)}\right)^{p-1}} \right)^{\frac{p}{p-1}} \mathrm{d}\mathcal{L}^{n+1} \right)^{\frac{p-1}{p}} \\
&= \left(\fint_{\Omega_T} \left(\frac{|V|_{(p)}^{p-1}}{\left(\|V\|_{\dot{L}^p(\Omega_T)}\right)^{p-1}} \right)^{\frac{p}{p-1}} \mathrm{d}\mathcal{L}^{n+1} \right)^{\frac{p-1}{p}} \\
&= \left(\fint_{\Omega_T} \left(\frac{|V|_{(p)}^{p}}{\left(\|V\|_{\dot{L}^p(\Omega_T)}\right)^{p}} \right) \mathrm{d}\mathcal{L}^{n+1} \right)^{\frac{p-1}{p}} \\
&= 1
\end{aligned}
$$

□

Appendix B

The Modified Hölder Inequality

Here we establish the proof for the modified Hölder inequality used in Chapter 2 (page 18, 23).

Lemma B.0.1. *For any* $1 \leq q \leq p < \infty$ *and* $h \in L^p(X)$, *we have the inequality*

$$\|h\|_{\dot{L}^q(X)} \leq \|h\|_{\dot{L}^p(X)} + \sqrt{q^{-2} - p^{-2}}.$$

Proof of Lemma B.0.1 Set

$$|f|_{(p)} := \left(|f|^2 + p^{-2}\right)^{\frac{1}{2}}, \text{ where } \|f\|_{\dot{L}^p(X)} = \left(\fint_X \left(|f|_{(p)}\right)^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}.$$

Then,

$$\begin{aligned}
\|f\|_{\dot{L}^q(X)} = \left(\fint_X \left(|f|_{(q)}\right)^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} &= \left(\fint_X \left(|f|^2 + q^{-2} + p^{-2} - p^{-2}\right)^{\frac{q}{2}} \,\mathrm{d}\mu\right)^{\frac{1}{q}} \\
&= \left(\fint_X \left(|f|^2 + p^{-2} + \left(q^{-2} - p^{-2}\right)\right)^{\frac{q}{2}} \,\mathrm{d}\mu\right)^{\frac{2}{q}\cdot\frac{1}{2}} \\
&\leq \sqrt{\left(\fint_X \left(|f|^2 + p^{-2}\right)^{\frac{q}{2}} \,\mathrm{d}\mu\right)^{\frac{2}{q}} + q^{-2} - p^{-2}} \\
&\leq \sqrt{\left(\fint_X \left(|f|^2 + p^{-2}\right)^{\frac{p}{2}} \,\mathrm{d}\mu\right)^{\frac{2}{p}}} + \sqrt{q^{-2} - p^{-2}} \\
&= \|f\|_{\dot{L}^p(X)} + \sqrt{q^{-2} - p^{-2}}.
\end{aligned}$$

□

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