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A Paley–Wiener Theorem for the Mehler–Fock Transform

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Abstract

In this note, we prove a Paley–Wiener Theorem for the Mehler–Fock transform. In particular, we show that it induces an isometric isomorphism from the Hardy space $\mathcal{H}^2(\mathbb{C}^+)$ onto $L^2(\mathbb{R}^+, (2\pi)^{-1} t \sinh(\pi t)\, dt)$. The proof we provide here is very simple and is based on an old idea that seems to be due to G. R. Hardy. As a consequence of this Paley–Wiener theorem we also prove a Parseval's theorem. In the course of the proof, we find a formula for the Mehler–Fock transform of some particular functions.

Keywords Payley-Wiener theorem · Mehler-Fock transform · Hardy space · Parseval's theorem

Mathematics Subject Classification Primary 44A15 · Secondary 30H10

Peter Duren was a wonderful mathematician and a great math lover who did not hesitate to support any mathematician as much as he could. Mathematics has lost one of his best supporters.

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1 Introduction

Given a measurable function $f : [1, \infty) \to \mathbb{C}$, its Mehler–Fock transform is defined by

$$\hat{f}(t) = \mathcal{P} f(t) = \int_1^\infty f(x) P_{it-1/2}(x)\, dx, \quad t \geq 0,$$

whenever the integral exists. Here P_ν is the Legendre function of the first kind of order ν. A sufficient condition for the integral above to exist is that $f(x)/\sqrt{x}$ belongs to $L^1[1, \infty)$, see [2, p.108]. Under suitable conditions, the Mehler–Fock transform has an inverse. For instance, if $\sqrt{t}\hat{f}(t)$ is in $L^1(\mathbb{R}^+)$, then

$$f(x) = \mathcal{P}^{-1} \hat{f}(x) = \int_0^\infty t \tanh(\pi t) P_{it-1/2}(x) \hat{f}(t)\, dt, \quad x \geq 1,$$

see [2, Thm. 1.9.53]. There is an extensive development of the properties of the Mehler–Fock transform, see for instance [2], where further references can be found. It is also well known that the Mehler–Fock transform has applications in Mathematical Physics, where it is used to solve Dirichlet problems on the sphere and conical surfaces, see the book by Lebedev [5].

Let $\mathbb{C}^+ = \{z = x + iy \in \mathbb{C} : x > 0\}$. The Hardy space $\mathcal{H}^2(\mathbb{C}^+)$ consists of those analytic functions on $\mathbb{C}^+$ for which the norm

$$\|f\|^2_{\mathcal{H}^2(\mathbb{C}^+)} = \sup_{x>0} \frac{1}{2\pi} \int_{\mathbb{R}} |f(x+iy)|^2 dy$$

is finite. It is well known that the functions in $\mathcal{H}^2(\mathbb{C}^+)$ have boundary values almost everywhere, see [7], and indeed $\mathcal{H}^2(\mathbb{C}^+)$ is a Hilbert space. In fact, the norm is given by

$$\|f\|^2_{\mathcal{H}^2(\mathbb{C}^+)} = \frac{1}{2\pi} \int_{\mathbb{R}} |f(iy)|^2\, dy.$$

One of the main interests when dealing with an integral transform is Parseval's theorem in the Hilbert space setting, see for instance [7], in which it is proved that the Fourier transform is an isometric isomorphism from $L^2(\mathbb{R}, dx/\sqrt{2\pi})$ onto itself. In the same vein, the Mehler–Fock transform is an isometric isomorphism from $L^2([1, \infty), dx)$ onto $L^2(\mathbb{R}^+, t \tanh(\pi t)\, dt)$, see [1, Thm. 1.9.54] for instance.

Next we have Paley–Wiener theorems. For instance, the Fourier transform is an isometric isomorphism between $\mathcal{H}^2(i\mathbb{C}^+)$ and $L^2(\mathbb{R}^+, dt/\sqrt{2\pi})$, see [7, Thm. 19.2]. Paley–Wiener theorem versions for the Fourier transform for weighted Bergman spaces and weighted Dirichlet spaces can be found in [1]. In this note, we will prove a Paley–Wiener Theorem for the Mehler–Fock transform, whose statement is similar to the one for the Fourier transform. In particular, we will show that the Mehler–Fock transform $\mathcal{P}$ defines an isometric isomorphism from $\mathcal{H}^2(\mathbb{C}^+)$ onto

$L^2(\mathbb{R}^+, (2\pi)^{-1} t \sinh t\, dt)$. The idea of the proof, as in [1], is based on the fact that if a linear map takes a complete orthogonal system onto another complete orthogonal system, then the map extends to an isometric isomorphism. This idea seems to be due to G. R. Hardy, but we have been unable to find a precise reference.

2 Main Theorem

In this section, we will prove the following Paley–Wiener theorem for the Mehler–Fock transform.

Theorem 2.1 *The Mehler–Fock transform $\mathcal{P}$ extends to an isometric isomorphism from $\mathcal{H}^2(\mathbb{C}^+)$ onto*

$$L^2(\mathbb{R}^+, (2\pi)^{-1} t \sinh(\pi t)\, dt).$$

Indeed, for $f \in \mathcal{H}(\mathbb{C}^+)$, we have

$$\int_0^\infty |\mathcal{P} f(t)|^2 t \sinh(\pi t)\, dt = \sup_{x>0} \int_{-\infty}^\infty |f(x+iy)|^2\, dy.$$

What Theorem 2.1 states is different from Parseval's theorem for the Mehler–Fock transform stated in the introduction. It claims that for functions in $L^2([1, \infty), dx)$ that extend to functions in $\mathcal{H}^2(\mathbb{C}^+)$, then it is again an isometric isomorphism too, but with different norms, see also Corollary 2.5 in this connection.

Remark As a consequence of Theorem 2.1, for $f \in L^2(\mathbb{R}^+, (2\pi)^{-1} t \sinh(\pi t))$, the inverse Mehler–Fock transform

$$\mathcal{P}^{-1} f(z) = \int_0^\infty t \tanh(\pi t) P_{it-1/2}(z)\, dt, \qquad z \in \mathbb{C}^+,$$

is well defined and belongs to $\mathcal{H}^2(\mathbb{C}^+)$.

2.1 The Hardy Space of the Unit Disk $\mathbb{D}$

To prove Theorem 2.1, we need to deal with the Hardy space of the unit disk $\mathbb{D}$ of the complex plane. Recall that the Hardy space $\mathcal{H}^2(\mathbb{D})$ is the space of functions

$$f(z) = \sum_{n=0}^\infty a_n z^n$$

analytic on $\mathbb{D}$ for which the norm

$$\|f\|^2_{\mathcal{H}^2(\mathbb{D})} = \sum_{n=0}^\infty |a_n|^2$$

is finite. It is obvious that $\mathcal{H}^2(\mathbb{D})$ is a Hilbert space. It is also well known that the functions in the Hardy space have boundary values almost everywhere, see [7] for instance; in fact, the norm above has the integral representation

$$\|f\|^2_{\mathcal{H}^2(\mathbb{D})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2\, d\theta = \frac{1}{2\pi}\int_{|z|=1} |f(z)|^2 |dz|.$$

We recall that, given two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, their inner product is defined by

$$\langle f, g\rangle_{\mathcal{H}^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \bar{b}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\theta = \frac{1}{2\pi}\int_{|z|=1} f(z)\overline{g(z)}\, |dz|.$$

In addition, the Gelfand transform

$$(\mathcal{G}f)(w) = \frac{2}{w+1} f\left(\frac{w-1}{w+1}\right)$$

is an isometric isomorphism from $\mathcal{H}^2(\mathbb{D})$ onto $\mathcal{H}^2(\mathbb{C}^+)$, see [3, p. 106].

2.2 The Legendre Function as Hypergeometric Functions

Given complex parameters a, b, c, where $c \neq -1, -2, \ldots$, the hypergeometric function ${}_2F_1(a, b; c, z)$ is defined by

$${}_2F_1(a, b; c, z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)n!} z^n,$$

see [6]. Consider now

$$f_\nu(z) = (1-z)^{\nu-1}\, {}_2F_1(\nu, \nu-1; 1; z).$$

The Legendre functions are related to the above hypergeometric function. Using Kummer's first formula, see [6, Thm. 20,p. 60], and the definition of the Legendre functions, see [5, p. 165], we have

$$f_\nu(z) = \frac{1}{1-z}\, {}_2F_1\left(\nu, 1-\nu, 1; \frac{z}{z-1}\right) = \frac{1}{1-z} P_{-\nu}\left(\frac{1+z}{1-z}\right). \tag{2.1}$$

Since $P_\nu = P_{-\nu-1}$, see [5, p. 167], for $\nu = it + 1/2$, $t \geq 0$, we have $P_{-it-1/2} = P_{it+1/2-1} = P_{it-1/2}$. Therefore, it follows from (2.1) that

$$f_{it+1/2}(z) = \frac{1}{1-z} P_{it-1/2}\left(\frac{1+z}{1-z}\right). \tag{2.2}$$

2.3 An Integral Representation for the Legendre Function

What follows relies on the formula

$$P_{it-1/2}(y) = \frac{\cosh(\pi t)}{\pi} \int_1^\infty \frac{P_{it-1/2}(x)}{x+y}\, dx, \quad \text{for } y \geq 1, \tag{2.3}$$

see [5, p. 202, Ex. 12].

For each $z \in \mathbb{D}$, we set

$$\varphi_z(x) = \frac{1}{x(1-z)+1+z}, \quad \text{for } x \geq 1.$$

We are now ready to state the connection between the functions $f_{it+1/2}$ and the Mehler–Fock transform.

Theorem 2.2 *For $\nu = it + 1/2$, $t \geq 0$, we have the the following representation via the Mehler–Fock transform*

$$f_{it+1/2}(z) = \frac{\cosh(\pi t)}{\pi} \mathcal{P}\varphi_z(t), \quad z \in \mathbb{D}.$$

Proof First of all, for each $z \in \mathbb{D}$, we have $|\varphi_z(x)| = O(x^{-1})$ as $x \to +\infty$ and of course $\varphi_z(x)$ is locally integrable on $[1, \infty)$. Thus $\mathcal{P}\varphi_z$ is a well defined function.

Next assume that $0 \leq z < 1$. Then $(1+z)/(1-z) \geq 1$. Thus using (2.2) in the first equality below and (2.3) in the second, we find that

$$\begin{aligned} f_{it+1/2}(z) &= \frac{1}{1-z} P_{it-1/2}\left(\frac{1+z}{1-z}\right) \\ &= \frac{1}{1-z} \frac{\cosh(\pi t)}{\pi} \int_1^\infty \frac{P_{it-1/2}(x)}{x + \frac{1+z}{1-z}}\, dx \\ &= \frac{\cosh(\pi t)}{\pi} \int_1^\infty \frac{P_{it-1/2}(x)}{x(1-z)+1+z}\, dx. \end{aligned}$$

Since both sides are analytic functions of z on $\mathbb{D}$, by the identity principle, it follows that the equality above holds true for each $z \in \mathbb{D}$. The proof is complete. □

2.4 The Continuous Dual Hahn Polynomials and an Isometric Isomorphism

The functions $f_{it+1/2}$, $t \geq 0$, are the key in the development that follows. The point is that these function are the generating functions of the continuous dual Hahn polynomials see [4, (9.3.12) p. 199], that is,

$$f_{it+1/2}(z) = (1-z)^{-1/2+it}\,{}_2F_1\left(\frac{1}{2}+it, \frac{1}{2}+it; 1; z\right)$$
$$= \sum_{n=0}^{\infty} \frac{S_n\left(t^2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{(n!)^2} z^n,$$

where $S_n(t^2) = S_n(t^2, 1/2, 1/2, 1/2)$ are the continuous dual Hahn polynomials with parameters 1/2, 1/2, 1/2. Using the facts that $|\Gamma(ix)|^2 = \pi/(x\sinh(\pi x))$ and $|\Gamma(1/2+ix)|^2 = \pi/\cosh(\pi x)$ for x real, we see that these polynomials satisfy the following orthogonal relations, see [3, (9.3.2) on p. 196],

$$\int_0^\infty S_n(t^2)S_m(t^2)\frac{2\pi t\tanh(\pi t)}{\cosh(\pi t)}\,dt = (n!)^4\delta_{nm}, \tag{2.4}$$

where $\delta_{nm} = 1$ if $n = m$ and 0 otherwise. It is well known that the polynomials $S_n(t^2)$ form a complete orthogonal system for

$$L^2(\mathbb{R}^+, w(t)\,dt),$$

where $w(t) = 2\pi\tanh(\pi t)/\cosh(\pi t)$.

Consider the map Φ defined on $\mathcal{H}^2(\mathbb{D})$ by

$$(\Phi f)(t) = \langle f, f_{it+1/2}\rangle_{\mathcal{H}^2(\mathbb{D})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{f_{it+1/2}(e^{i\theta})}\,d\theta. \tag{2.5}$$

It is not difficult to see that the Taylor coefficients of $f_{1/2+it}$ behave asymptotically as $O(n^{-1/2})$, if $t \neq 0$, and as $O(n^{-1/2}\ln(n))$ if $t = 0$. Therefore, $f_{it+1/2}$ is not in $\mathcal{H}^2(\mathbb{D})$. Thus, in principle Φ is not well defined for general $f \in \mathcal{H}^2(\mathbb{D})$. However, we have the following theorem.

Theorem 2.3 *The map Φ defines an isometric isomorphism between $\mathcal{H}^2(\mathbb{D})$ and $L^2(\mathbb{R}^+, w(t)\,dt)$. In particular, for each $f \in \mathcal{H}^2(\mathbb{D})$, the function Φf is well defined almost everywhere with respect to $w(t)\,dt$.*

Proof Consider $u_n(z) = z^n$, $n \geq 0$, which form a complete orthonormal system of $\mathcal{H}^2(\mathbb{D})$. Then by definition of Φ, see (2.5), we have

$$(\Phi u_n)(t) = \frac{S_n(t^2)}{(n!)^2}$$

and because of (2.4), we find that

$$\|\Phi u_n\|_{L^2(R^+, w\,dt)} = 1 = \|u_n\|_{\mathcal{H}^2(\mathbb{D})}.$$

The result now follows from the linearity of Φ and the fact that Φ takes the complete orthonormal system $\{u_n\}$ of $\mathcal{H}^2(\mathbb{D})$ onto a complete orthonormal system of $L^2(\mathbb{R}^+, w\,dt)$. In particular, for each $f \in \mathcal{H}^2(\mathbb{D})$, the function Φf is defined almost everywhere with respect to $w(t)\,dt$. The proof is complete. □

2.5 Proof of Theorem 2.1

Proof Let $f \in \mathcal{H}(\mathbb{C}^+)$ such that $g = \mathcal{G}^{-1} f \in \mathcal{H}^2(\mathbb{D})$ is of integrable modulus on the unit circle. Having in mind that $P_{it-1/2}(x)$ is real for $x \geq 1$ and $t \geq 0$, using Fubini's Theorem in the fourth equality below and Theorem 2.3 in the sixth equality below, we have

$$\begin{aligned}
(\mathcal{P} f)(t) &= (\mathcal{P}\mathcal{G} g)(t)\\
&= \int_1^\infty P_{it-1/2}(x)\frac{2}{x+1} g\left(\frac{x-1}{x+1}\right) dx\\
&= 2\int_1^\infty P_{it-1/2}(x)\frac{1}{2\pi}\int_{|z|=1}\frac{g(z)}{1+x-(x-1)\bar{z}}|dz|dx\\
&= \frac{1}{\pi}\int_{|z|=1} g(z)\overline{\int_1^\infty \frac{P_{it-1/2}(x)}{1+x-(x-1)z}\,dx}\,|dz|\\
&= 2\,\langle g, (\mathcal{P}\varphi_z)(t)\rangle_{\mathcal{H}^2(\mathbb{D})}\\
&= 2\left\langle g, \frac{\pi}{\cosh(\pi t)} f_{it+1/2}\right\rangle_{\mathcal{H}^2(\mathbb{D})}\\
&= \frac{2\pi}{\cosh(\pi t)}\langle g, f_{it+1/2}\rangle_{\mathcal{H}^2(\mathbb{D})}.
\end{aligned}$$

Therefore, a standard density argument shows that $\mathcal{P} = 2M_\varphi \Phi \mathcal{G}^{-1}$. Since

$$2M_\varphi : L^2(\mathbb{R}^+, w(t)dt) \mapsto L^2(\mathbb{R}^+, (2\pi)^{-1} t \sinh(\pi t)\,dt)$$

is an isometric isomorphism, we obtain the statement of the theorem. The proof is complete. □

Remark 1 For $f \in \mathcal{H}^2(\mathbb{C}^+)$, one must compute in the following way

$$\mathcal{P} f(t) = \lim_{b\to\infty}\int_1^b f(x) P_{it-1/2}(x)\,dx,$$

which is only defined almost everywhere with respect to $(2\pi)^{-1} t \sinh(\pi t)\,dt$.

Remark 2 The proof of Theorem 2.1 shows that the following identity is true

$$\int_1^\infty \frac{1}{1+x}\left(\frac{1-x}{1+x}\right)^n P_{it-1/2}(x)\,dx = \frac{2\pi}{\cosh(\pi t)} S_n\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, t^2\right), \quad t \geq 0.$$

We have been unable to find the preceding formula in any table of Mehler–Fock transforms.

The next corollary provides an alternative formula for the Mehler–Fock transform of a function in $\mathcal{H}^2(\mathbb{C}^+)$.

Corollary 2.4 *If* $f \in \mathcal{H}^2(\mathbb{C}^+)$, *then*

$$\int_1^\infty f(x) P_{it-1/2}(x)\, dx = \frac{1}{\cosh(\pi t)} \int_{\mathbb{R}} f(ix) P_{it-1/2}(ix)\, dx.$$

Proof We take $g = \mathcal{G}^{-1} f \in \mathcal{H}^2(\mathbb{D})$. Then the definition of Φ and moving to the right-half plane yields

$$\begin{aligned}\Phi g(t) &= \frac{1}{2\pi} \int_{|z|=1} g(z) \overline{\left(\frac{1}{1-z} P_{it-1/2}\left(\frac{1+z}{1-z}\right)\right)} |dz| \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(iy) P_{it-1/2}(iy)\, dy,\end{aligned}$$

which along with the fact that $\mathcal{P} = 2M_\varphi \Phi \mathcal{G}^{-1}$ proves the result. The proof is complete. □

Consider now the Hilbert space $\overline{\mathcal{H}}^2(\mathbb{D})$ consisting of the conjugate functions of $\mathcal{H}^2(\mathbb{D})$ and let $\mathcal{H}_0^2(\mathbb{D}) = z\mathcal{H}^2(\mathbb{D})$ and $\overline{\mathcal{H}_0^2}(\mathbb{D}) = \bar{z}\overline{\mathcal{H}^2}(\mathbb{D})$. Then

$$L^2(\mathbb{R}, dx/2\pi) = \mathcal{H}_0^2(\mathbb{D}) \oplus [1] \oplus \overline{\mathcal{H}_0^2}(\mathbb{D}).$$

In this way, it is possible to go to the right half plane and easily prove a Parseval theorem for $L^2(i\mathbb{R})$.

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