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## Energetics of a Symmetric Circulation Including Momentum Constraints

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### ABSTRACT

A theory of available potential energy (APE) for symmetric circulations, which includes momentum constraints, is presented. The theory is a generalization of the classical theory of APE, which includes only thermal constraints on the circulation. Physically, centrifugal potential energy is included along with gravitational potential energy. The generalization relies on the Hamiltonian structure of the conservative dynamics, although (as with classical APE) it still defines the energetics in a nonconservative framework. It follows that the theory is exact at finite amplitude, has a local form, and can be applied to a variety of fluid models. It is applied here to the  $f$ -plane Boussinesq equations. It is shown that, by including momentum constraints, the APE of a symmetrically stable flow is zero, while the energetics of a mechanically driven symmetric circulation properly reflect its causality.

### 1. Introduction

Lorenz (1955) proposed a diagnostic framework for the energy cycle based on the concept of available potential energy (APE). The APE is supposed to represent that part of the total potential energy (the sum of the gravitational potential and internal energy) of a system that is available for conversion into kinetic energy. The classical theory of APE accounts only for thermal (or mass) constraints, but not for momentum constraints. The natural question is whether there is any need to redefine APE to include the latter. Regarding this issue, Lorenz (1955) himself remarked, "There is no assurance in any individual case that all the available potential energy will be converted into kinetic energy. For example, if the flow is purely zonal, and the mass and momentum distributions are in dynamically stable equilibrium, no kinetic energy at all can be realized. It might seem desirable to redefine available potential energy, so that, in particular, it will be zero in the above example." Hence, one would conclude that, for reasons both aesthetic and practical, a theory accounting for both momentum and thermal constraints is required. The purpose of this paper is to provide just such a theory, and in particular, to show that the properly defined APE of a symmetrically stable circulation is zero, as Lorenz suggested it should be.

As examples of systems where the motion is constrained by momentum or angular momentum conservation one may consider quasi-steady symmetric cir-

culations, such as hurricanes, frontal systems, and the zonal-mean flow in the atmosphere. The addition of forcing and dissipation (including 3D effects) to these systems leads to symmetric cross-stream circulations (Eliassen 1951). The analysis of the energetics of such circulations with Lorenz's APE may then be misleading.

To exemplify this let us consider the mesosphere, where we know that the meridional circulation is mechanically forced by wave drag and thermally damped by radiation (see, e.g., Shepherd 2000). At solstice, there is rising motion and diabatic heating over the summer pole at the temperature  $T^{\text{SP}}$ , say, and sinking motion and diabatic cooling over the winter pole at the temperature  $T^{\text{WP}}$ . In the present mesosphere,  $T^{\text{WP}} > T^{\text{SP}}$  so we have heating where it is cold and cooling where it is warm. According to Lorenz's APE, this implies a thermally indirect circulation—which agrees with its mechanically driven nature. But let us now imagine a situation with less wave drag, such that  $T^{\text{WP}} < T^{\text{SP}}$ . The circulation is still mechanically forced and thermally damped, but Lorenz's theory now diagnoses it as being a thermally direct circulation. Hence the inferred causality is incorrect.

This paradox arises from the fact that in the mesosphere, in the absence of wave drag (and meridional circulation) the zonal flow is not at rest, but in radiative equilibrium, with zonal wind  $u_{\text{rad}} \neq 0$ . The appropriate reference state would then seem to be  $(u_{\text{rad}}, T_{\text{rad}})$ . Relative to this, the radiative cooling (or heating)  $R$  is always a thermal damping since  $R(T - T_{\text{rad}}) < 0$  (Andrews et al. 1987), which gives the correct causality. However, this requires generalizing APE to a nonresting reference state.

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It has been shown by Shepherd (1993) that Lorenz's APE arises generally within a Hamiltonian framework, as the nonkinetic part of a disturbance pseudoenergy. The system itself need not be conservative, but the conservative part determines the energetics (as with Lorenz's theory). Shepherd (1993) also suggested that the same Hamiltonian pseudoenergy approach could be used to generalize the concept of APE to a nonresting reference state, thereby incorporating momentum constraints. The physical requirement is that the reference state be symmetrically stable, a natural generalization of Lorenz's statically stable reference state.

In the present paper, we develop a theory for the energetics of a symmetric circulation, including momentum constraints. Eddy fluxes are treated as forcings. For simplicity, we consider parallel flows within the Boussinesq  $f$ -plane equations, to illustrate the approach. The approach can be easily extended to axisymmetric circulations, using angular momentum constraints.

The paper is organized as follows. In section 2 we present the equations to be used. We then briefly review the derivation of APE in section 3. In section 4 we develop the small-amplitude version of the energetics, taking into account the dynamical constraints on the flow. Section 5 illustrates the small-amplitude theory with an example. The theory is generalized to finite amplitude in section 6. An extension of the theory to include inertia terms describing the cross-stream circulation is presented in section 7. Our conclusions are presented in section 8.

## 2. Governing equations

The nonhydrostatic,  $f$ -plane Boussinesq equations (e.g., Holton 1992) are

$$\frac{Du}{Dt} - fv = -\frac{p_x}{\rho_0}, \quad (2.1)$$

$$\frac{Dv}{Dt} + fu = -\frac{p_y}{\rho_0}, \quad (2.2)$$

$$\frac{Dw}{Dt} = -\frac{p_z}{\rho_0} + g\frac{\theta}{\theta_0}, \quad (2.3)$$

$$\frac{D\theta}{Dt} = R, \quad (2.4)$$

$$u_x + v_y + w_z = 0. \quad (2.5)$$

In the above,  $R$  is the diabatic heating,  $\theta$  is the departure of potential temperature from a constant basic-state value  $\theta_0$ , and the other symbols have their usual meaning. We assume no explicit dissipation, since the equations will shortly be averaged and eddy flux terms will emerge. The subscripts stand for partial derivatives, and  $D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$ , with  $\mathbf{u} \equiv (u, v, w)$  and  $\nabla \equiv (\partial_x, \partial_y, \partial_z)^T$ . The Coriolis parameter is taken to be constant ( $f > 0$ ), as well as the basic-state density  $\rho_0$  and potential temperature  $\theta_0$ . Let us denote,

$$m \equiv u - fy, \quad p^* \equiv \frac{p}{\rho_0} + \frac{1}{2}f^2y^2, \quad (2.6)$$

and take the  $x$  average, denoted by an overbar, of (2.1)–(2.5). Of course, on the  $f$  plane there is no inherent distinction between the  $x$  and  $y$  directions; we choose to average over  $x$  without loss of generality.

This yields

$$\frac{D\bar{m}}{Dt} = X, \quad (2.7)$$

$$\frac{D\bar{v}}{Dt} = -\bar{p}_y^* - f\bar{m} + Y, \quad (2.8)$$

$$\frac{D\bar{w}}{Dt} = -\bar{p}_z^* + g\frac{\bar{\theta}}{\theta_0} + Z, \quad (2.9)$$

$$\frac{D\bar{\theta}}{Dt} = \bar{R} + T, \quad (2.10)$$

$$\bar{v}_y + \bar{w}_z = 0, \quad (2.11)$$

where  $X$ ,  $Y$ ,  $Z$ , and  $T$  are eddy flux terms, and  $\bar{D}/Dt \equiv \partial/\partial t + \bar{\nabla} \cdot \bar{\nabla}$ , with  $\bar{\nabla} \equiv (\bar{v}, \bar{w})$  and  $\bar{\nabla} \equiv (\partial_y, \partial_z)^T$ . By the nondivergence property (2.11) we can define a streamfunction  $\psi$  in the meridional  $y$ – $z$  plane such that

$$\bar{v} = -\psi_z, \quad \bar{w} = \psi_y. \quad (2.12)$$

The advection term can now be written  $\bar{D}/Dt \equiv \partial/\partial t + \partial(\psi, \cdot)$ , where  $\partial(g, h) \equiv g_y h_z - g_z h_y$  is the two-dimensional Jacobian operator.

We now make the semigeostrophic approximation of a nearly symmetric balanced flow, where (2.8) is replaced by geostrophic balance and (2.9) by hydrostatic balance. These conditions will be relaxed in section 7. This leads to a condition of thermal-wind balance

$$\bar{m}_z = -\frac{g}{f\theta_0}\bar{\theta}_y. \quad (2.13)$$

Our system then consists of (2.7), (2.10), and (2.13).

In many applications it is useful to rewrite this system in the transformed Eulerian mean (TEM) format (Andrews et al. 1987), where the right-hand side (rhs) of the thermodynamic equation consists only of the diabatic heating  $\bar{R}$ . A motivation for such a transformation is that the eddy flux terms in the thermodynamic equation can in any case be eliminated by working in isentropic coordinates (no analogue of this is possible for the momentum equations, since  $m$  is not materially conserved in 3D motion). Moreover, the rhs of the  $x$  momentum equation then becomes the convergence of pseudomomentum flux, and is thereby connected to a wave-activity conservation law.

In the quasigeostrophic (QG) case—where  $T = -(\bar{v}'\theta')$ , and  $X = -(\bar{u}'v')$ , with the primes denoting departures from the zonal mean, and where, in the advection terms of (2.7) and (2.10),  $\bar{m}$  is replaced by  $-fy$  and  $\bar{\theta}$  by a specified  $\Theta_0(z)$ —this transformation can be made by replacing  $\psi$  with  $\psi^* \equiv \psi + (\bar{v}'\theta'/\Theta_{0z})$ . Then

$T$  disappears from the rhs of (2.10), while  $X$  in (2.7) is replaced by  $X^* \equiv X + (\overline{v'\theta'}/\overline{\theta})_z$ , which is the divergence of the QG Eliassen–Palm (EP) flux. More generally, taking  $\psi^* \equiv \psi + (\overline{v'\theta'}/\overline{\theta})$  implies (Andrews et al. 1987) that  $T = -(\overline{v'\theta'})_y - (\overline{w'\theta'})_z$  is replaced by

$$T^* = -\left(\frac{\overline{v'\theta'}\overline{\theta}_y + \overline{w'\theta'}\overline{\theta}_z}{\overline{\theta}_z}\right). \quad (2.14)$$

As long as the motion is along the mean isentropic surfaces (this is true for quasi-hydrostatic dynamics), then  $\overline{v'\theta'}\overline{\theta}_y + \overline{w'\theta'}\overline{\theta}_z = 0$  and  $T^*$  vanishes. In what follows, we set  $T = 0$  in (2.10) and regard  $\psi$  as representing the TEM circulation and  $X$  the EP flux divergence. However, by reinterpreting  $X$  and  $R$ , other physical interpretations are possible, and may be more appropriate depending on the context (Plumb 1983). The overbar is henceforth dropped from all symbols.

### 3. Available energy

We first consider the conservative form of the system (2.7), (2.10), (2.13), with  $X = 0 = R$ . We follow Cho et al. (1993), except that, because of the suppression of the inertia terms in the  $y$  and  $z$  directions, the energy consists only of centrifugal and gravitational potential energy given by

$$\mathcal{H} = \iint_D \left( mfy - \frac{\theta gz}{\theta_0} \right) dy dz. \quad (3.1)$$

As a boundary condition we impose (for convenience)  $\psi = 0$  on  $\partial D$ , in which case  $\mathcal{H}$  is conserved in time. Considering  $\mathcal{H}$  as a functional of  $m$  and  $\theta$ , we have the functional (variational) derivatives (see, e.g., Shepherd 1990 for a definition)

$$\frac{\delta \mathcal{H}}{\delta m} = fy, \quad \frac{\delta \mathcal{H}}{\delta \theta} = -\frac{gz}{\theta_0}. \quad (3.2)$$

Besides the energy itself there is also a class of Casimir invariants of the form

$$C = \iint_D C(m, \theta) dy dz, \quad (3.3)$$

for arbitrary functions  $C(\cdot, \cdot)$ . Their conservation follows from the material conservation (for  $X = 0 = R$ ) of  $m$  and  $\theta$  expressed by (2.7) and (2.10). Evidently,

$$\frac{\delta C}{\delta m} = C_m, \quad \frac{\delta C}{\delta \theta} = C_\theta. \quad (3.4)$$

We next introduce a reference state (RS) with

$$\psi \equiv 0, \quad m = M(y, z), \quad \theta = \Theta(y, z), \quad (3.5)$$

satisfying thermal-wind balance (2.13). The goal is now to choose the arbitrary function  $C$  in such a way that the conserved quantity  $\mathcal{H} + C$  defines a positive definite measure of disturbance energy relative to this RS. This quantity is called the pseudoenergy (Shepherd 1990). In order for  $\mathcal{H} + C$  to be positive definite, the RS must be a conditional extremum for  $\mathcal{H} + C$ , which requires

$$\frac{\delta \mathcal{H}}{\delta m} = -\frac{\delta C}{\delta m}, \quad \frac{\delta \mathcal{H}}{\delta \theta} = -\frac{\delta C}{\delta \theta} \quad (3.6)$$

when evaluated at the RS. From (3.2), (3.4), and (3.6) we obtain

$$C_m(M, \Theta) = -fy, \quad C_\theta(M, \Theta) = \frac{gz}{\theta_0}. \quad (3.7)$$

In (3.7),  $y$  and  $z$  on the rhs are to be regarded as functions of  $M$  and  $\Theta$ , namely  $y(M, \Theta)$  and  $z(M, \Theta)$ . In order for these functions to be well defined, the transformation  $(M, \Theta) \rightarrow (y, z)$  must be invertible, which requires

$$Q \equiv \frac{\partial(\Theta, M)}{\partial(y, z)} \neq 0. \quad (3.8)$$

Here,  $Q$  is the potential vorticity. One may show that

$$\frac{\partial z}{\partial \Theta} \Big|_M = \frac{\partial(z, M)}{\partial(\Theta, M)} = \frac{\partial(z, M)}{\partial(z, y)} \frac{\partial(z, y)}{\partial(\Theta, M)} = \frac{\partial M}{\partial y} \Big|_z \left( -\frac{1}{Q} \right), \quad (3.9)$$

and the corresponding relations when replacing  $z \leftrightarrow y$ ,  $\Theta \leftrightarrow M$ . The second derivatives of  $C$  are given by

$$C_{mm} = -f \frac{\partial y}{\partial M} = \frac{f}{Q} \frac{\partial \Theta}{\partial z}, \quad (3.10)$$

$$C_{\theta\theta} = \frac{g}{\theta_0} \frac{\partial z}{\partial \Theta} = -\frac{g}{\theta_0 Q} \frac{\partial M}{\partial y}, \quad (3.11)$$

$$C_{m\theta} = -f \frac{\partial y}{\partial \Theta} = -\frac{f}{Q} \frac{\partial M}{\partial z}, \quad (3.12)$$

$$= \frac{g}{\theta_0} \frac{\partial z}{\partial M} = \frac{g}{\theta_0 Q} \frac{\partial \Theta}{\partial y}. \quad (3.13)$$

The pseudoenergy is then given by

$$\mathcal{A} = \mathcal{H} + C - \mathcal{H}^{\text{RS}} - C^{\text{RS}}, \quad (3.14)$$

with  $\mathcal{H}^{\text{RS}}$  and  $C^{\text{RS}}$  the energy and Casimir, respectively, evaluated at the RS. Using (3.1) and (3.7), the pseudoenergy may be written as

$$\mathcal{A} = \iint_D \underbrace{[C(m, \theta) - C(M, \Theta) - C_m(M, \Theta)(m - M) - C_\theta(M, \Theta)(\theta - \Theta)]}_{\text{APE}} dy dz. \quad (3.15)$$

Here, the integrand has been labeled APE—the density of available potential energy. With our choice of Hamiltonian (3.1), there is no kinetic energy component in the pseudoenergy, hence the above notation. In contrast, Cho et al.'s (1993) Eq. (3.3) includes a kinetic energy term. By construction,  $\mathcal{A}$  is conserved and is quadratic in the disturbance  $(m - M, \theta - \Theta)$ . To ensure convexity of  $\mathcal{A}$  around the RS it is evidently sufficient that

$$C_{mm} > 0, \quad C_{\theta\theta} > 0, \quad C_{mm}C_{\theta\theta} - (C_{m\theta})^2 > 0, \quad (3.16)$$

with the derivatives evaluated at the RS. These are also the conditions for symmetric stability (Cho et al. 1993; Mu et al. 1996). They moreover imply  $Q > 0$ , so (3.8) is satisfied.

Thus, while Lorenz's APE is defined relative to a stably stratified resting RS, our APE is defined relative to a symmetrically stable nonresting RS. It would not seem to make physical sense to define an APE relative to a symmetrically unstable RS. In addition, while Lorenz's RS is uniquely specified as the minimum potential energy state obtained by adiabatic rearrangement of mass, there are no such constraints on the RS here. In a conservative problem, it is natural to choose an RS that provides the minimum value of  $\mathcal{A}$ , for a given state of the system. However, this is not so clearly relevant in the forced-dissipative context to be considered next.

#### 4. Small-amplitude theory

One may write perturbations around the RS (3.5) as

$$m = M + m', \quad \theta = \Theta + \theta', \quad \psi = \psi'. \quad (4.1)$$

These primes are not to be confused with the departure from the zonal average associated with eddies, mentioned in section 2. The disturbance equations (2.7), (2.10), and (2.13) then become, after linearization,

$$\frac{\partial m'}{\partial t} + \partial(\psi', M) = X, \quad (4.2)$$

$$\frac{\partial \theta'}{\partial t} + \partial(\psi', \Theta) = R, \quad (4.3)$$

$$m'_z = -\frac{g}{f\theta_0}\theta'_y. \quad (4.4)$$

The quadratic approximation to the integrand of (3.15) is given by

$$\text{APE} = \frac{1}{2}[C_{mm}(m')^2 + 2C_{m\theta}m'\theta' + C_{\theta\theta}(\theta')^2]. \quad (4.5)$$

For the linearized conservative disturbance equations (i.e., with  $X = 0 = R$ ), it can be verified that the integral of the quadratic APE (4.5) is an exact invariant. With  $X \neq 0$  and  $R \neq 0$ , we find instead,

$$\frac{d}{dt} \iint_D \text{APE} \, dy \, dz = \iint_D [C_{mm}m'X + C_{m\theta}(m'R + \theta'X) + C_{\theta\theta}\theta'R] \, dy \, dz \equiv S, \quad (4.6)$$

where  $S$  is the net source (or sink) of  $\mathcal{A}$ . In deriving (4.6), we used (4.2), (4.3), and (4.5), and assumed the  $C$ s not to be explicitly dependent on time. The local form of (4.6) is (see appendix)

$$\frac{\partial}{\partial t}(\text{APE}) + \frac{\partial}{\partial y} \left( \frac{g}{\theta_0} \theta' \psi' \right) + \frac{\partial}{\partial z} (f m' \psi') = S, \quad (4.7)$$

where  $S$  is the density of  $S$  defined in (4.6).

In the source–sink term for  $\mathcal{A}$ , we make the partition  $S = S_X + S_R$ , where

$$S_X = \iint_D (C_{mm}m' + C_{m\theta}\theta')X \, dy \, dz, \quad (4.8)$$

$$S_R = \iint_D (C_{m\theta}m' + C_{\theta\theta}\theta')R \, dy \, dz. \quad (4.9)$$

Because  $S = 0$  under steady conditions, we have either  $S_X > 0$  and  $S_R < 0$  (mechanically driven circulation), or  $S_X < 0$  and  $S_R > 0$  (thermally driven circulation). In the former case let us define the thermodynamic efficiency as

$$\eta \equiv -\frac{S_R}{S_X}. \quad (4.10)$$

Clearly,  $\eta$  so defined equals unity for a steady flow. But there may be viscous damping in a system, or one may wish to regard certain parts of  $X$  as the “forcing” (e.g., planetary wave drag in the stratosphere) and other parts as the “response” (e.g., gravity wave drag). Similarly, one might only wish to regard part of  $R$  as the output (e.g., departure from radiative equilibrium) and other parts as losses (e.g., thermal diffusion). In such cases, one may anticipate  $\eta < 1$ .

#### 5. An example

In order to make the above theory concrete, let us consider a symmetric zonal flow with a meridional circulation. The situation is as depicted in Fig. 1a, with a negative zonal force driving a positive meridional flow in the upper part of the domain, and a compensating positive zonal force allowing a return flow in the lower part of the domain. The scenario models (from the point of view of causality) a mechanically driven circulation



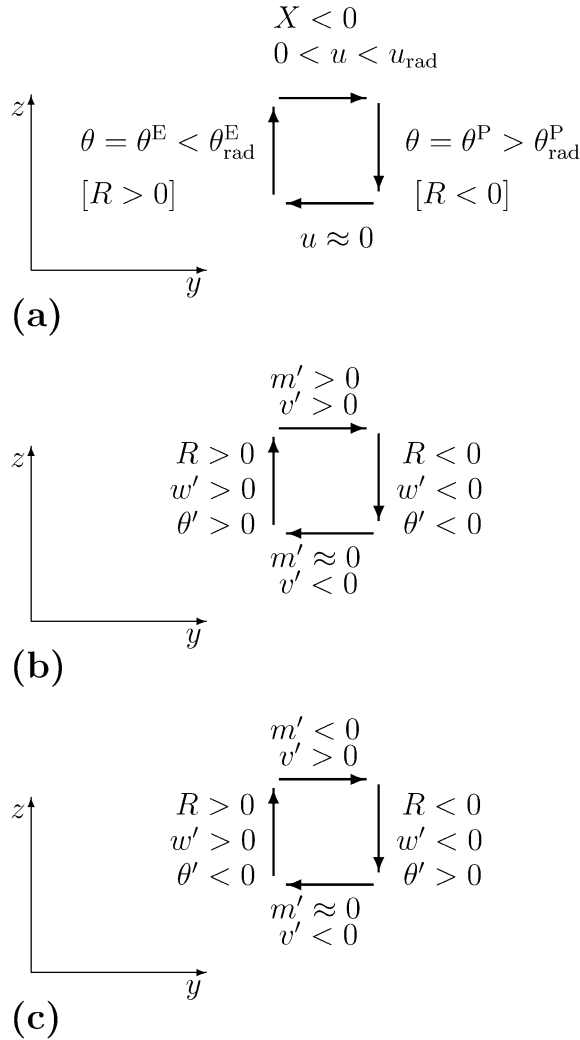


FIG. 1. (a) A mechanically driven meridional circulation, diagnosed with respect to (b) a resting RS and with respect to (c) a nonresting RS. Here,  $\theta_{\text{rad}}$  is the radiative equilibrium potential temperature distribution, and  $u_{\text{rad}}$  is the zonal wind in thermal-wind balance with  $\theta_{\text{rad}}$ . The arrows denote the sense of the circulation. It is assumed that  $\theta^P < \theta^E$ , so the meridional temperature gradient is weakened by the circulation, but not actually reversed.

with thermal damping, but with the forcing not strong enough to reverse the meridional temperature gradient. It is thus a demonstration of the thought experiment for the mesosphere considered in the introduction (albeit, here, on the Boussinesq  $f$  plane), interpreting the lower part of the domain as the troposphere. We analyze this circulation with respect to two different reference states.

For a resting RS (the Lorenz picture), the circulation is described by Fig. 1b. We have  $U = 0$  so  $M = -fy$ , and  $m' = u' > 0$  for  $z > 0$ . The RS has  $\Theta = \text{const}$  with latitude ( $\theta^P < \Theta < \theta^E$ ) so  $\theta'$  has the signs shown on the diagram. With a nonresting RS in which  $\Theta = \theta_{\text{rad}}$ ,  $M = m_{\text{rad}}$ , the circulation is described by Fig. 1c. We now quantitatively analyse the above-described cases under the small-amplitude approximation.

Let us consider the domain defined by  $0 \leq z \leq H$ ,  $-L/2 \leq y \leq L/2$ . We assume that  $X$  is given, and that  $R$  is described by the Newtonian cooling approximation

$$R = -r(\theta - \theta_{\text{rad}}) = -r(\theta' + \Theta - \theta_{\text{rad}}). \quad (5.1)$$

We assume a linear vertical variation of the radiative equilibrium potential temperature  $\theta_{\text{rad}}$  (constant static stability) and a negative meridional gradient:

$$\theta_{\text{rad}}(y, z) = -\frac{f\theta_0\lambda}{g}y + \frac{N^2\theta_0}{g}z, \quad (5.2)$$

for some  $\lambda > 0$ . The circulation in this system is clearly mechanically forced; if  $X = 0$ , then  $v = 0 = w$ ,  $R = 0$ , and  $\theta = \theta_{\text{rad}}$ .

#### a. Nonresting reference state

In the case of the nonresting RS, we choose  $\Theta = \theta_{\text{rad}}$ , and by using (2.13), the RS is then defined as

$$\Theta = \theta_{\text{rad}}, \quad M(y, z) = \lambda z - fy, \quad (5.3)$$

$$U(y, z) = U(z) = \lambda z. \quad (5.3)$$

Here we choose  $U(z)$  to satisfy  $U(0) = 0$ . In this case the potential vorticity  $Q$ , defined by (3.8), is given by the expression

$$Q = \frac{f\theta_0}{g}(N^2 - \lambda^2); \quad (5.4)$$

for a symmetrically stable flow we need  $fQ > 0$ —that is,  $N^2/\lambda^2 = N^2/U_z^2 \equiv \text{Ri} > 1$ —where  $\text{Ri}$  is the Richardson number (Stone 1966). Then, by using (3.10)–(3.13) and (5.4),

$$C_{mm} = \frac{N^2}{N^2 - \lambda^2}, \quad C_{m\theta} = -\frac{g}{\theta_0} \frac{\lambda}{N^2 - \lambda^2}, \quad (5.5)$$

$$C_{\theta\theta} = \frac{g^2}{\theta_0^2} \frac{1}{N^2 - \lambda^2}.$$

We assume a steady circulation, so  $\partial/\partial t = 0$ . If we impose  $X$ , then everything else should follow. The momentum and thermodynamic equations (4.2) and (4.3) now become

$$w'M_z + v'M_y = X, \quad (5.6)$$

$$w'\Theta_z + v'\Theta_y = R. \quad (5.7)$$

For the case of a steady circulation we have the following global constraints expressing the fact that we cannot have a net mass transport across any vertical or horizontal surface:

$$\int_{-L/2}^{L/2} w' dy = 0, \quad \forall z; \quad \int_0^H v' dz = 0, \quad \forall y. \quad (5.8)$$

Substituting  $w'$ ,  $v'$  from (5.6) and (5.7), the global constraints (5.8) become

$$\int_{-L/2}^{L/2} X \left( \frac{1}{Q} \Theta_y \right) dy = \int_{-L/2}^{L/2} R \left( \frac{1}{Q} M_y \right) dy, \quad \forall z; \quad (5.9)$$

$$\int_0^H X \left( \frac{1}{Q} \Theta_z \right) dz = \int_0^H R \left( \frac{1}{Q} M_z \right) dz, \quad \forall y. \quad (5.10)$$

With our particular choice of RS, (5.9) and (5.10) take the form

$$\frac{\lambda \theta_0}{g} \int_{-L/2}^{L/2} X dy = \int_{-L/2}^{L/2} R dy, \quad \forall z; \quad (5.11)$$

$$\frac{N^2 \theta_0}{g} \int_0^H X dz = \lambda \int_0^H R dz, \quad \forall y. \quad (5.12)$$

Let us now take the forcing as

$$X = \alpha \left[ f \frac{k}{n} \cos(kz) \cos(ny) - \lambda \sin(kz) \sin(ny) \right]. \quad (5.13)$$

Here, we have denoted for the sake of conciseness  $k \equiv \pi/H$  and  $n \equiv \pi/L$ , and  $\alpha$  represents the magnitude of the forcing. The choice of  $X$  is motivated by Fig. 1, as well as the desire to obtain an analytical solution. For this RS, (5.6) gives

$$\lambda w' - f v' = X. \quad (5.14)$$

We impose as boundary conditions  $w' = 0$  at  $z = 0, H$  and  $v' = 0$  at  $y = \pm L/2$ . Using (2.11), the solution of (5.14) is

$$\begin{aligned} v' &= -\alpha \frac{k}{n} \cos(kz) \cos(ny), \\ w' &= -\alpha \sin(kz) \sin(ny). \end{aligned} \quad (5.15)$$

From the thermodynamic balance (5.7), it follows that

$$\begin{aligned} R &= \frac{\alpha \theta_0}{g} \left[ -N^2 \sin(kz) \sin(ny) \right. \\ &\quad \left. + f \lambda \frac{k}{n} \cos(kz) \cos(ny) \right]. \end{aligned} \quad (5.16)$$

We observe that the global constraints (5.11) and (5.12) are fulfilled. Now, since  $\Theta = \theta_{\text{rad}}$ , we have  $R = -r\theta'$ , and hence,

$$\begin{aligned} \theta' &= -\frac{R}{r} = -\frac{\alpha \theta_0}{gr} \left[ -N^2 \sin(kz) \sin(ny) \right. \\ &\quad \left. + f \lambda \frac{k}{n} \cos(kz) \cos(ny) \right]. \end{aligned} \quad (5.17)$$

Finally, from (4.4) and (5.17) the disturbance momentum is found to be

$$\begin{aligned} m' = u' &= -\frac{\alpha n}{rfk} \left\{ N^2 [1 - \cos(kz)] \cos(ny) \right. \\ &\quad \left. + f \lambda \frac{k}{n} \sin(kz) \sin(ny) \right\}, \end{aligned} \quad (5.18)$$

where we have imposed the lower boundary condition  $u' |_{z=0} = 0$ . It then follows that the expressions for the source–sink terms (4.8) and (4.9) are

$$S_R = -\frac{\alpha^2 HL}{4r(N^2 - \lambda^2)} \left( N^4 + 2N^2 \lambda^2 + f^2 \lambda^2 \frac{L^2}{H^2} \right) < 0, \quad (5.19)$$

$$S_X = \frac{\alpha^2 HL}{4r(N^2 - \lambda^2)} \left( N^4 + 2N^2 \lambda^2 + f^2 \lambda^2 \frac{L^2}{H^2} \right) > 0. \quad (5.20)$$

We see that the circulation is always diagnosed as mechanically driven and thermally damped.

### b. Resting reference state

For the resting RS, we choose  $\Theta$  to be independent of latitude and follow the same steps, but with a few changes.

We let the forcing  $X$  be different from the case of a nonresting RS but keep the velocities the same in both cases. The purpose of doing so is to show that, for a given circulation [i.e., a given  $(v', w')$ ], the diagnostic for the resting RS may be opposite to what causality would suggest.

The resting RS has

$$\Theta = \frac{N^2 \theta_0}{g} z, \quad U = 0. \quad (5.21)$$

The disturbance velocities are given by (5.15). Hence, from (5.6), we obtain

$$X = \alpha f \frac{k}{n} \cos(kz) \cos(ny), \quad (5.22)$$

while from (5.7), we obtain

$$R = -\frac{\alpha N^2 \theta_0}{g} \sin(kz) \sin(ny), \quad (5.23)$$

which, with the Newtonian cooling approximation (5.1), implies

$$\theta' = \frac{\alpha N^2 \theta_0}{rg} \sin(kz) \sin(ny) - \frac{f \lambda \theta_0}{g} y. \quad (5.24)$$

By using (4.4) the momentum disturbance is then

$$m' = u' = \frac{\alpha N^2 n}{rfk} [\cos(kz) - 1] \cos(ny) + \lambda z, \quad (5.25)$$

again setting  $u' |_{z=0} = 0$ . With the RS defined by (5.21) we obtain

$$C_{mm} = 1, \quad C_{m\theta} = 0, \quad C_{\theta\theta} = \frac{g^2}{\theta_0^2 N^2}. \quad (5.26)$$

It then follows that



$$S_R = -HL \left( \frac{N^2 \alpha^2}{4r} - \frac{4\alpha f \lambda L}{\pi^3} \right), \quad (5.27)$$

$$S_X = HL \left( \frac{N^2 \alpha^2}{4r} - \frac{4\alpha f \lambda L}{\pi^3} \right). \quad (5.28)$$

We see that the circulation is diagnosed as thermally driven and mechanically damped for sufficiently small  $\alpha$  (i.e., for sufficiently weak forcing), which contradicts causality. As  $\alpha$  increases the temperature profile is reversed and the circulation is diagnosed as being mechanically driven and thermally damped.

### c. Inclusion of Rayleigh drag

In order to examine the effects of explicit mechanical damping, let us now assume that there is Rayleigh drag acting on the zonal flow perturbation. We shall show that in this case  $\eta < 1$  for a steady flow, as was argued heuristically in section 4. We work with the nonresting RS (5.3). To simplify the calculation, we keep the meridional circulation as in (5.15); in this way, the global constraints (5.8) are automatically satisfied. Consequently, (5.14) becomes  $\lambda w' - f v' = X - \mu u'$ , with Rayleigh drag ( $\mu > 0$ ) acting in addition to the EP flux divergence. Then  $R$ ,  $\theta'$  and  $u'$  are given by (5.16)–(5.18) as before,  $S_R$  is again given by (5.19), but  $S_X$  is now given by

$$S_X = S_{X_0} + \iint_D (C_{mm} m' + C_{m\theta} \theta') \mu u' dy dz, \quad (5.29)$$

where  $S_{X_0}$  equals the  $S_X$  of (5.20), and  $C_{mm}$  and  $C_{m\theta}$  are given by (5.5) as before. Since  $S_R = -S_{X_0}$ , the thermodynamic efficiency becomes

$$\begin{aligned} \eta &= \frac{1}{1 + \frac{\iint_D (C_{mm} m' + C_{m\theta} \theta') \mu u' dy dz}{S_{X_0}}} \\ &= \frac{1}{1 + \frac{3\mu n^2 N^2 (n^2 N^4 + f^2 k^2 \lambda^2)}{f^2 k^2 r [f^2 k^2 \lambda^2 + n^2 N^2 (N^2 + 2\lambda^2)]}} < 1. \end{aligned} \quad (5.30)$$

We may note that  $\eta$  is independent of  $\alpha$  (the forcing amplitude) and is a monotonically decreasing function of  $\mu$ , as expected.

If the Rayleigh drag is instead assumed to act on the total zonal flow  $u = U + u'$ , then  $\eta$  may be negative or may exceed unity. These unphysical results arise because the drag on  $U$  is nonzero and drives a circulation even when  $\alpha = 0$ ; that is, there is a nonzero response for zero forcing.

## 6. Finite-amplitude theory

The expressions derived in sections 4 and 5 were obtained under a small-amplitude approximation. The

small-amplitude approximation works well if the actual temperature profile  $\theta$  is close to  $\Theta$ . One may have such a situation for both the resting and nonresting RS (e.g., in the limit of weak mechanical forcing and with the  $\theta_{\text{rad}}$  profile almost flat), but this is not the case in general. Therefore, it is important to be able to generalize the theory to finite amplitude. The possibility of doing so is ensured by the underlying Hamiltonian structure, as is clear from section 3.

The finite amplitude analogue of (4.2)–(4.4) is

$$\frac{\partial m'}{\partial t} + \partial(\psi', M + m') = X, \quad (6.1)$$

$$\frac{\partial \theta'}{\partial t} + \partial(\psi', \Theta + \theta') = R, \quad (6.2)$$

$$m'_z = -\frac{g}{f\theta_0} \theta'_y. \quad (6.3)$$

The criterion for the small-amplitude theory to be valid is  $|\partial(\psi', m')| \ll |\partial(\psi', M)|$  and  $|\partial(\psi', \theta')| \ll |\partial(\psi', \Theta)|$ . The first condition is true for  $u' \ll M \sim fl$ , or  $u'/fl \equiv \text{Ro} \ll 1$ , where  $l$  is a characteristic horizontal length scale and  $\text{Ro}$  is the Rossby number. The second condition is true if  $\theta' \ll (d\Theta/dz)h \sim \theta_0 N^2 h/g$ , where  $h$  is a characteristic vertical length scale. But thermal-wind balance implies  $u'/h \sim (g/\theta_0 f)(\theta'/l)$ , so

$$\frac{u'}{fl} = \frac{g\theta'}{\theta_0 N^2 h} \frac{h^2 N^2}{f^2 l^2}. \quad (6.4)$$

Hence, the criteria are essentially the same, because, in (6.4), the second factor on the rhs is the Burger number, which is assumed to be  $O(1)$  in QG scaling. Thus, for the small-amplitude theory to be valid, one needs  $\text{Ro} \ll 1$ , but with  $\text{Ro}$  defined by  $u'$ , not by  $u$  itself. In particular the nonresting RS can have large  $\text{Ro}$ , so long as the forcing is sufficiently weak that the induced circulation does not cause the zonal flow to depart strongly from the RS.

It is important to note that the small-amplitude expressions in section 4 are exact at finite amplitude, provided the RS is linear in  $y$  and  $z$  (Mu et al. 1996). For the general case, though, the quadratic pseudoenergy density (4.5) is replaced by the integrand of (3.15), namely

$$\text{APE} = C - C^{\text{RS}} - C_m^{\text{RS}} m' - C_\theta^{\text{RS}} \theta'. \quad (6.5)$$

Here, for conciseness we have denoted  $C \equiv C(m, \theta)$  and  $C^{\text{RS}} \equiv C(M, \Theta)$ ;  $C_m^{\text{RS}}$ ,  $C_\theta^{\text{RS}}$  stand for the derivatives of  $C$  taken at the RS. The RS and  $C^{\text{RS}}$  are assumed not to be explicitly dependent on time.

As in the small-amplitude case, if  $X = 0 = R$  then  $\mathcal{A}$  is an exact invariant. With  $X \neq 0$  and  $R \neq 0$ , we find instead,

$$\frac{d\mathcal{A}}{dt} = \iint_D [(C_m - C_m^{\text{RS}})X + (C_\theta - C_\theta^{\text{RS}})R] dy dz \equiv \mathcal{S}, \quad (6.6)$$

with the derivatives of  $C^{RS}$  given by (3.7). The local form of (6.6) is (see appendix)

$$\begin{aligned} \frac{\partial}{\partial t}(\text{APE}) + \partial(\psi', \text{APE}) + \frac{\partial}{\partial y} \left( \frac{g}{\theta_0} \theta' \psi' \right) \\ + \frac{\partial}{\partial z} (fm' \psi') = S, \end{aligned} \tag{6.7}$$

where  $S$  is the density of  $S$  defined in (6.6). One may perform the partition of  $S$  as in section 4 and obtain the analogue of (4.8) and (4.9):

$$S_X = \iint_D (C_m - C_m^{RS}) X \, dy \, dz, \tag{6.8}$$

$$S_R = \iint_D (C_\theta - C_\theta^{RS}) R \, dy \, dz. \tag{6.9}$$

### 7. Inclusion of inertia terms

We now return to (2.7)–(2.11) to consider the effect of retaining the inertia terms in the meridional and vertical momentum equations. The conservative form of this system is still Hamiltonian, with the Casimirs (3.3), but the energy now includes the kinetic energy of the meridional vertical cross-stream motion; it is given by

$$\mathcal{H} = \iint_D \left( \frac{1}{2} |\nabla \psi|^2 + mfy - \frac{\theta gz}{\theta_0} \right) dy \, dz. \tag{7.1}$$

We again take the RS to be a steady baroclinic flow defined by (3.5); this remains a steady solution of the unforced system (2.7)–(2.11) and must obey thermal-wind balance, although now the disturbances are not in thermal-wind balance. The equations (2.7)–(2.11) then become (neglecting for now any forcing terms in the  $v'$  and  $w'$  equations)

$$\frac{\partial m'}{\partial t} + \partial(\psi', M + m') = X, \tag{7.2}$$

$$\frac{\partial v'}{\partial t} + \partial(\psi', v') = -p'_y - fm', \tag{7.3}$$

$$\frac{\partial w'}{\partial t} + \partial(\psi', w') = -p'_z + g \frac{\theta'}{\theta_0}, \tag{7.4}$$

$$\frac{\partial \theta'}{\partial t} + \partial(\psi', \Theta + \theta') = R, \tag{7.5}$$

$$v'_y + w'_z = 0. \tag{7.6}$$

Here,  $p' \equiv p^* - p_{RS}^*$ , and the theory is developed at finite amplitude.

It is of interest to derive a criterion for the inertia terms to be negligible. The most stringent constraint is expected to come from (7.3) rather than (7.4); namely, that  $\partial v'/\partial t \sim (v')^2/l \ll fu'$ . From (5.15) and (5.18), one has  $v' \sim \alpha l/h$ ,  $u' \sim \alpha h N^2 / fl$ , and so we require

$$\begin{aligned} \frac{(v')^2}{fu'l} \sim \frac{\alpha l^2 r}{N^2 h^3} \sim \frac{\alpha h N^2}{fl} \cdot \frac{r^2 l^3 f}{N^4 h^4} \\ \sim \left( \frac{u'}{fl} \right) \left( \frac{f^2 l^2}{N^2 h^2} \right)^2 \left( \frac{r}{f} \right)^2 \ll 1. \end{aligned} \tag{7.7}$$

One recognizes in the last expression the Ro factor and the square of the inverse of the Burger number, as defined in section 6. For relatively weak thermal damping, we expect also  $r \ll f$  (certainly, this holds in the middle atmosphere).

With our choice of Hamiltonian (7.1), the pseudoeenergy density becomes

$$\begin{aligned} A &= \text{KE} + \text{APE} \\ &= \frac{1}{2} |\nabla \psi'|^2 + C - C^{RS} - C_m^{RS} m' - C_\theta^{RS} \theta', \end{aligned} \tag{7.8}$$

where KE and APE are the kinetic energy and available potential energy densities, respectively. The APE is the same as before, (6.5). We note that, in the conservative case (when  $X = 0 = R$ ), we have

$$\frac{d}{dt} \iint_D A \, dy \, dz = 0, \tag{7.9}$$

as follows from the construction of  $A$ , but [from (7.3) and (7.4)]

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$$\begin{aligned} \frac{d}{dt} \iint_D \text{KE} \, dy \, dz &= - \underbrace{\iint_D [v' \partial(\psi', v') + w' \partial(\psi', w')] \, dy \, dz}_{=0} - \underbrace{\iint_D (p'_y v' + p'_z w') \, dy \, dz}_{=0} \\ &+ \iint_D \left\{ \underbrace{\frac{g}{\theta_0} \theta' w'}_{C_T} - \underbrace{fm' v'}_{C_M} \right\} dy \, dz, \end{aligned} \tag{7.10}$$

where  $C_T$  and  $C_M$  are the densities of the thermal and mechanical energy conversion terms. The first and the second integrals on the rhs of (7.10) vanish due to the boundary conditions and the continuity equation (7.6). From (7.9) and (7.10), one obtains

$$\begin{aligned} \frac{d}{dt} \iint_D \text{APE} \, dy \, dz &= -\frac{d}{dt} \iint_D \text{KE} \, dy \, dz \\ &= -C_T - C_M, \end{aligned} \quad (7.11)$$

where  $C_T$ ,  $C_M$  stand for the integrals over the domain of the corresponding conversion densities. With  $X \neq 0$  and  $R \neq 0$ , (7.10) and the conversion terms are unchanged, but (7.11) is replaced by

$$\frac{d}{dt} \iint_D \text{APE} \, dy \, dz = -C_T - C_M + S, \quad (7.12)$$

where  $S$  is the net source (or sink) of the integral of APE and is given by (6.6). Since there are (by hypothesis) no sources or sinks of KE,  $S$  is also the source (or sink) of  $\mathcal{A}$  itself. For steady conditions, both  $C_T + C_M$  and  $S$  must separately vanish. Thus, the inclusion of the inertia terms does not change the source–sink terms in the energetics, but introduces a kinetic energy component (of the flow in the  $y$ – $z$  plane) with thermal and mechanical conversion terms between the kinetic and available potential energy.

#### *Inclusion of mechanical damping*

We now consider mechanical damping of the meridional circulation, denoted by  $Y$  and  $Z$ , as terms to be added on the rhs of (7.3) and (7.4), respectively. In this case, the tendency equation for KE becomes

$$\begin{aligned} \frac{d}{dt} \iint_D \text{KE} \, dy \, dz \\ = C_T + C_M + \underbrace{\iint_D (v'Y + w'Z) \, dy \, dz}_{\mathcal{F}}, \end{aligned} \quad (7.13)$$

while (7.12) is unchanged. Under the assumption that  $Y$  and  $Z$  act as friction on  $v'$  and  $w'$ , it follows that  $\mathcal{F} < 0$ .

For a steady flow without friction,  $S = 0$  and the thermodynamic efficiency defined by (4.10) is unity. But if friction as defined in (7.13) is present, then from (7.12) and (7.13), we have  $S = C_M + C_T = -\mathcal{F} > 0$  for a steady flow, whence

$$\eta = 1 + \frac{\mathcal{F}}{S_X} < 1, \quad (7.14)$$

assuming  $S_X > 0$  (i.e., a mechanically driven circulation).

The local forms of (7.12) and (7.13) are (see appendix)

$$\frac{\partial}{\partial t}(\text{APE}) + \partial(\psi', \text{APE}) = -C_T - C_M + S, \quad (7.15)$$

$$\frac{\partial}{\partial t}(\text{KE}) + \partial(\psi', \text{KE} + p') = C_T + C_M + F, \quad (7.16)$$

where  $F$  is the density of  $\mathcal{F}$  defined in (7.13).

## 8. Conclusions

We have shown that one can construct an APE for symmetric circulations incorporating momentum constraints, by considering the pseudoenergy relative to a nonresting symmetrically stable reference state. The underlying Hamiltonian structure of the conservative dynamics guarantees that this APE is defined at finite amplitude. The analysis was limited to symmetric circulations (with 3D eddy fluxes treated as forcings), because momentum is not a Lagrangian invariant for 3D motion, and therefore, one cannot have a fully 3D theory (like Lorenz's theory).

In this paper, for simplicity, we derived the APE diagnostics for the Boussinesq  $f$ -plane equations in both the small- and finite-amplitude cases, but the theory clearly generalizes to other systems. In the case of a mechanically driven circulation with thermal relaxation to a (stable) ‘‘radiative equilibrium’’ state, by choosing the reference state to equal the equilibrium state we always diagnose the correct causality of the circulation. This is in contrast to the traditional APE diagnostics (relative to a resting reference state), which might incorrectly diagnose such a circulation as being thermally forced and mechanically damped. A worked example was provided. The effects of certain viscous terms were discussed, as well as the inclusion of the inertia terms for the meridional circulation.

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## APPENDIX

### Mathematical Derivations

We derive here the local forms of the conservation laws given by (4.7), (6.7), (7.15), and (7.16).

In order to get (4.7), we differentiate (4.5) with respect to time and use (4.2), (4.3). We obtain

$$\begin{aligned} \frac{\partial}{\partial t}(\text{APE}) &= \overbrace{C_{mm}m'X + C_{m\theta}(m'R + \theta'X) + C_{\theta\theta}\theta'R}^{= S} \\ &\quad - m'[C_{mm}\partial(\psi', M) + C_{m\theta}\partial(\psi', \Theta)] \\ &\quad - \theta'[C_{m\theta}\partial(\psi', M) + C_{\theta\theta}\partial(\psi', \Theta)]. \end{aligned} \quad (\text{A.1})$$

Expanding the  $\partial(\psi', \cdot)$  terms, and using (3.8) and (3.10)–(3.13), we get

$$\begin{aligned} \frac{\partial}{\partial t}(\text{APE}) &= S - fm'\psi'_z - \frac{g}{\theta_0}\theta'\psi'_y \\ &= S - \frac{\partial}{\partial z}(fm'\psi') + fm'_z\psi' \\ &\quad - \frac{\partial}{\partial y}\left(\frac{g}{\theta_0}\theta'\psi'\right) + \frac{g}{\theta_0}\theta'_y\psi'. \end{aligned} \quad (\text{A.2})$$

By using thermal-wind balance (4.4), the last equation reduces to (4.7).

In order to get (6.7), we differentiate (6.5) with respect to time and use (6.1), (6.2). We obtain

$$\begin{aligned} \frac{\partial}{\partial t}(\text{APE}) &= S \\ &= \overbrace{(C_m - C_m^{\text{RS}})X + (C_\theta - C_\theta^{\text{RS}})R} \\ &\quad - (C_m - C_m^{\text{RS}})\partial(\psi', M + m') \\ &\quad - (C_\theta - C_\theta^{\text{RS}})\partial(\psi', \Theta + \theta') \\ &= S - C_m\partial(\psi', m) - C_\theta\partial(\psi', \theta) + C_m^{\text{RS}}\partial(\psi', M) \\ &\quad + C_\theta^{\text{RS}}\partial(\psi', \Theta) + C_m^{\text{RS}}\partial(\psi', m') + C_\theta^{\text{RS}}\partial(\psi', \theta') \\ &= S - \partial(\psi', C) + \partial(\psi', C^{\text{RS}}) + \partial(\psi', C_m^{\text{RS}}m') \\ &\quad + \partial(\psi', C_\theta^{\text{RS}}\theta') - m'\partial(\psi', C_m^{\text{RS}}) - \theta'\partial(\psi', C_\theta^{\text{RS}}) \\ &= S - \partial(\psi', \text{APE}) \\ &\quad - m'[C_{mm}^{\text{RS}}\partial(\psi', M) + C_{m\theta}^{\text{RS}}\partial(\psi', \Theta)] \\ &\quad - \theta'[C_{m\theta}^{\text{RS}}\partial(\psi', M) + C_{\theta\theta}^{\text{RS}}\partial(\psi', \Theta)]. \end{aligned} \quad (\text{A.3})$$

Apart from the  $\partial(\psi', \text{APE})$  term, the last equation is just the small-amplitude equation (A.1) but with the finite-amplitude  $S$ . Thus, the finite-amplitude form of (4.7) is seen to be (6.7).

The derivation of (7.15) follows as in the finite-amplitude case up to (A.3). However, there is no longer thermal-wind balance between  $\theta'$  and  $m'$ . Thus, the final step of (A.2) cannot be used to transform the quadratic terms. This then yields (7.15).

The derivation of (7.16) is straightforward, because the integrand of the first term of (7.10) is just  $-\partial(\psi', \text{KE})$ , while the integrand of the second term is  $-\partial(\psi', p')$ .

#### REFERENCES

- Andrews, D. G., J. R. Holton, and C. B. Leovy, 1987: *Middle Atmosphere Dynamics*. Academic Press, 489 pp.
- Cho, H.-R., T. G. Shepherd, and V. A. Vladimirov, 1993: Application of the direct Liapunov method to the problem of symmetric stability in the atmosphere. *J. Atmos. Sci.*, **50**, 822–836.
- Eliassen, A., 1951: Slow thermally or frictionally controlled meridional circulation in a circular vortex. *Astrophys. Norv.*, **5** (2), 19–60.
- Holton, J. R., 1992: *An Introduction to Dynamic Meteorology*. Academic Press, 511 pp.
- Lorenz, E. N., 1955: Available potential energy and the maintenance of the general circulation. *Tellus*, **7**, 157–167.
- Mu, M., T. G. Shepherd, and K. Swanson, 1996: On nonlinear symmetric stability and the nonlinear saturation of symmetric instability. *J. Atmos. Sci.*, **53**, 2918–2923.
- Plumb, R. A., 1983: A new look at the energy cycle. *J. Atmos. Sci.*, **40**, 1669–1688.
- Shepherd, T. G., 1990: Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics. *Advances in Geophysics*, Vol. 32, Academic Press, 287–338.
- , 1993: A unified theory of available potential energy. *Atmos.–Ocean*, **31**, 1–26.
- , 2000: The middle atmosphere. *J. Atmos. Solar-Terr. Phys.*, **62**, 1587–1601.
- Stone, P. H., 1966: On non-geostrophic baroclinic instability. *J. Atmos. Sci.*, **23**, 390–400.