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On the Group-Velocity Property for Wave-Activity Conservation Laws

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ABSTRACT

The density and the flux of wave-activity conservation laws are generally required to satisfy the group-velocity property: under the WKB approximation (i.e., for nearly monochromatic small-amplitude waves in a slowly varying medium), the flux divided by the density equals the group velocity. It is shown that this property is automatically satisfied if, under the WKB approximation, the only source of rapid variations in the density and the flux lies in the wave phase. A particular form of the density, based on a self-adjoint operator, is proposed as a systematic choice for a density verifying this condition.

1. Introduction

A central feature of many studies of geophysical flows is the separation between a simple basic state and a disturbance. Conservation laws, that is, local relations of the form

$$\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{F} = 0 \tag{1}$$

then prove extremely useful for analyzing the evolution of the disturbance. Those which are quadratic (to the lowest order) in the disturbance amplitude are the most advantageous, notably because they can be calculated accurately using a perturbative approach. An important literature is devoted to these particular conserved quantities, which are called wave activities (e.g., McIntyre and Shepherd 1987; Haynes 1988), and several recent studies use them as diagnostic tools (e.g., Scinocca and Peltier 1994a,b; Brunet and Haynes 1996; Magnusdottir and Haynes 1996).

A difficulty of (1) lies in the intrinsic ambiguity in the definition of the density A and flux $\mathbf{F}$; A and $\mathbf{F}$ are indeed not uniquely determined, as the transformation

$$A \mapsto A + \nabla \cdot \mathbf{B}, \qquad \mathbf{F} \mapsto \mathbf{F} - \frac{\partial \mathbf{B}}{\partial t}$$

leaves (1) unchanged, for any vector $\mathbf{B}$.[1] In particular, for a given density, the flux is determined only up to an arbitrary nondivergent vector. The methods that are used to derive the conservation laws of a system using its Hamiltonian structure and its symmetries (see Shepherd 1990) do not favor a particular form of the pair (A, $\mathbf{F}$). Rather, they provide an algorithm for obtaining the integral

$$\mathcal{A} = \int_V A \, d\mathbf{x},$$

which is conserved if $\int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, ds = 0$. The density A is then calculated up to the arbitrary divergence $\nabla \cdot \mathbf{B}$, and the corresponding flux $\mathbf{F}$ is derived from (1) using the equations of motion. [Note that the flux can also be derived systematically by exploiting the Hamiltonian structure (Vanneste 1997).]

To remove part of the arbitrariness in the definition of A and $\mathbf{F}$, it is generally required that they satisfy the *group-velocity property*. This means that when the disturbance has a small amplitude (linear theory) and takes the form of a nearly monochromatic wave train in a slowly varying basic state (WKB approximation), the following relation holds at leading order:

$$\langle \mathbf{F} \rangle = \mathbf{C} \langle A \rangle. \tag{2}$$

Here, $\langle \cdot \rangle$ denotes the average over the phase of the wave and $\mathbf{C}$ is the group velocity of the wave. There is no general method for the construction of a density and a flux satisfying (2); the authors who have presented wave-activity conservation laws for particular models (e.g., McIntyre and Shepherd 1987; Scinocca and Shepherd 1992; Durran 1995) resorted to a trial-and-error procedure, explicitly checking the group-velocity property (2) for specific forms of A and $\mathbf{F}$. In itself, this latter point involves a large amount of cumbersome algebra, even for relatively simple models, as exemplified

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[1] More generally, the density and the flux can be modified by the addition of trivial conservation laws as discussed by Olver (1993, 264).

by appendix B of Scinocca and Shepherd (1992). There is clearly a need for generic results that facilitate the search for densities and fluxes satisfying the group-velocity property.

The purpose of this article is twofold. First, we point out that condition (2) need not be checked explicitly: using a result of Hayes (1977), it is shown that (2) is automatically satisfied if under the WKB approximation both the density and the flux have no rapid variations other than those resulting from the wave phase. This is the case if, when expressed in terms of disturbance variables suited to the WKB approximation, the density and the flux are defined by slowly varying coefficients. Second, we propose a unique choice of a ''well-behaved'' density, that is, a density that satisfies the property just described, so that it can be used to build a conservation law satisfying the group-velocity property. These two points should prove useful for the derivation of new wave-activity conservation laws.

2. Condition on the density and flux

Since the group-velocity property concerns small-amplitude disturbances, we restrict our study to linearized systems. The wave-activity density is then exactly quadratic and takes the general form

$$A = \mathsf{L}u \cdot \mathsf{L}'u, \tag{3}$$

where L and L' are two linear operators that depend on the basic state, and u is the vector of the dynamical variables for the disturbance. The components F_i of $\mathbf{F}$ have a similar form, with operators N_i and N_i'.

When the basic state is slowly varying in some sense, a WKB solution can be sought for the equations governing the evolution of the disturbance. Such a solution often requires the introduction of a change of variables

$$u = \mathsf{M}\upsilon, \tag{4}$$

where the matrix M depends on the basic state, so that the coefficients of the evolution equations of υ (constituted of combinations of the basic-state variables) vary slowly over the scale of variation of υ. For instance, in the anelastic model considered by Scinocca and Shepherd (1992), the equations for the original variables $u = (\psi', \theta')$ (disturbance streamfunction and potential temperature) have rapidly varying coefficients because of the (possibly) rapid vertical variation of the basic density ρ_0, but the equations for the new variables $\upsilon = (\phi', \theta') = (\rho_0^{-1/2}\psi', \theta')$ have only slowly varying coefficients. It is worth emphasizing here that a WKB approximation generally requires several assumptions about the spatial and temporal dependence of the coefficients in the equations for u, and that these assumptions dictate the form of M. Different types of WKB approximations, with different assumptions and matrices M, are often possible; the verification of the group-velocity property should then be specific to each type of approximation.

Slow coordinates can be introduced as

$$\mathbf{X} = \epsilon\mathbf{x}, \qquad T = \epsilon t,$$

where $\epsilon \ll 1$ characterizes the scale separation between the disturbance and the basic state,[2] and the WKB solution can be written in the form

$$\upsilon = \hat{\upsilon}(\mathbf{X}, T)\exp[i\epsilon^{-1}\vartheta(\mathbf{X}, T)] + \text{c.c.} \tag{5}$$

where c.c. denotes the complex conjugate. The frequency and the wave vector are defined by

$$\Omega = -\frac{\partial\vartheta}{\partial T}, \qquad \mathbf{k} = \frac{\partial\vartheta}{\partial\mathbf{X}},$$

and satisfy a dispersion relation,

$$\Omega = \Omega(\mathbf{k}, \mathbf{X}, T). \tag{6}$$

To verify the group-velocity property (2) explicitly, one must introduce the ansatz (5) in the governing equations, derive the dispersion relation and the polarization relations constraining $\hat{\upsilon}$, and finally calculate $\langle A\rangle$, $\langle\mathbf{F}\rangle$, and $\mathbf{C} = \partial\Omega/\partial\mathbf{k}$.

Here, we establish that a wave-activity density and the corresponding flux automatically satisfy the group-velocity property, provided that their expressions in the WKB approximation vary rapidly through the wave phase ϑ/ϵ only. This condition can be reformulated by stating that, in the proper variables υ, the coefficients defining A and $\mathbf{F}$ depend only on the slow coordinates $\mathbf{X}$ and T; that is,

$$\left.\begin{aligned} \frac{\partial}{\partial\mathbf{x}}(\mathsf{LM})\upsilon\cdot\mathsf{L}'\mathsf{M}\upsilon + \mathsf{LM}\upsilon\cdot\frac{\partial}{\partial\mathbf{x}}(\mathsf{L}'\mathsf{M})\upsilon = O(\epsilon) \\ \frac{\partial}{\partial t}(\mathsf{LM})\upsilon\cdot\mathsf{L}'\mathsf{M}\upsilon + \mathsf{LM}\upsilon\cdot\frac{\partial}{\partial t}(\mathsf{L}'\mathsf{M})\upsilon = O(\epsilon) \end{aligned}\right\}, \tag{7}$$

for any $\upsilon(\mathbf{x}, t)$, with similar expressions for N_i and N_i'. [For any linear operator O, $\partial(\mathsf{O})/\partial x$ denotes the operator obtained by taking the x-derivative of the coefficients defining O.] Note that condition (7) is usually very easy to check. In what follows, we often refer to (7) as the condition ensuring that the coefficients defining (A, $\mathbf{F}$) are slowly varying; it must be kept in mind that the variables υ suited to the WKB approximation must be used to verify this condition.

The demonstration parallels the proof of Hayes (1977) that (in a medium at rest) the wave energy travels at the group velocity. Indeed, when (7) is satisfied, his arguments apply directly to the wave-activity problem. We nevertheless describe them for completeness.

Consider a wave with phase ϑ, wave vector $\mathbf{k}$, and frequency Ω. At leading order in ϵ, the corresponding wave-activity density takes the form

[2] We assume that the basic state is slowly varying in time, but the most common wave activities (pseudoenergy and pseudomomentum) are conserved only when the basic state is stationary.

$$A = a + \tilde{a}\exp(2i\epsilon^{-1}\vartheta) + \tilde{a}^*\exp(-2i\epsilon^{-1}\vartheta), \quad (8)$$

where

$$a = \mathsf{P}\hat{\upsilon}\cdot(\mathsf{P}')^*\hat{\upsilon}^* + (\mathsf{P})^*\hat{\upsilon}^*\mathsf{P}'\hat{\upsilon},$$
$$\tilde{a} = \mathsf{P}\hat{\upsilon}\cdot\mathsf{P}'\hat{\upsilon}.$$

Here, we have introduced the matrices P and P', which are obtained by making the substitutions $\partial_{\mathbf{x}} \mapsto i\mathbf{k}$ and $\partial_t \mapsto -i\Omega$ in LM and $\mathsf{L}'\mathsf{M}$, respectively. The flux takes a similar form,

$$\mathbf{F} = \mathbf{f} + \tilde{\mathbf{f}}\exp(2i\epsilon^{-1}\vartheta) + \tilde{\mathbf{f}}^*\exp(-2i\epsilon^{-1}\vartheta). \quad (9)$$

When applied to the wave (5), condition (7) shows that a, $\tilde{a}$, $\mathbf{f}$, and $\tilde{\mathbf{f}}$ are slowly varying. Averaging (8) and (9) thus yields to leading order

$$\langle A\rangle = a, \qquad \langle \mathbf{F}\rangle = \mathbf{f}. \quad (10)$$

Consider now another wave, with a slightly disturbed phase $\vartheta + i\delta\vartheta$, with $|\delta\vartheta| \ll |\vartheta|$. The wave vector and frequency become $\mathbf{k} + i\delta\mathbf{k}$ and $\Omega + i\delta\Omega$, with

$$\delta\Omega = \delta\mathbf{k}\cdot\frac{\partial\Omega}{\partial\mathbf{k}} = \delta\mathbf{k}\cdot\mathbf{C}, \quad (11)$$

as prescribed by the dispersion relation. The structure of this wave can be written as

$$\upsilon = (\hat{\upsilon} + \delta\hat{\upsilon})\exp(i\epsilon^{-1}\vartheta)\exp(-\epsilon^{-1}\delta\vartheta) + \text{c.c.},$$

and the pair $(A, \mathbf{F})$ take the form

$$A = [a + \delta a + (\tilde{a} + \delta\tilde{a})\exp(2i\epsilon^{-1}\vartheta) + (\tilde{a}^* + \delta\tilde{a}^*)\exp(-2i\epsilon^{-1}\vartheta)]\exp[-2\epsilon^{-1}\delta\vartheta],$$

and

$$\mathbf{F} = [\mathbf{f} + \delta\mathbf{f} + (\tilde{\mathbf{f}} + \delta\tilde{\mathbf{f}})\exp(2i\epsilon^{-1}\vartheta) + (\tilde{\mathbf{f}}^* + \delta\tilde{\mathbf{f}}^*)\exp(-2i\epsilon^{-1}\vartheta)]\exp[-2\epsilon^{-1}\delta\vartheta],$$

when the δ quantities are small disturbances. Introducing these expressions in the local conservation law (1) yields at leading order in ϵ

$$\delta\Omega(a + \delta a) - \delta\mathbf{k}\cdot(\mathbf{f} + \delta\mathbf{f}) = 0$$

for the term independent of the phase ϑ. Taking (11) into account, one gets (at first order in $\delta\mathbf{k}$)

$$\mathbf{C}a - \mathbf{f} = 0,$$

since $\delta\mathbf{k}$ is arbitrary. The group-velocity property (2) is finally derived from (10).

It is interesting to discuss the wave-activity conservation laws of some standard models in light of the condition (7) of slow variation of the coefficients defining the density and the flux. For the barotropic model considered by McIntyre and Shepherd (1987), the basic state is defined by the streamfunction Ψ and the potential vorticity Q, but only the gradients of these two quantities, $\boldsymbol{\nabla}\Psi$ and $\boldsymbol{\nabla}Q$, are present in the evolution equation of the disturbance. Hence, it is only required that $\boldsymbol{\nabla}\Psi$ and $\boldsymbol{\nabla}Q$ vary slowly for the WKB approach to be applicable; Q can vary rapidly, and in fact does vary rapidly when the β-plane approximation is used. The pseudoenergy density obtained by McIntyre and Shepherd (1987), given by

$$A = \frac{|\boldsymbol{\nabla}\psi|^2}{2} + \Psi'(Q)\frac{q^2}{2},$$

where ψ and q are the disturbance streamfunction and vorticity, and $\Psi'(Q) := d\Psi/dQ = \boldsymbol{\nabla}\Psi/\boldsymbol{\nabla}Q$, clearly has slowly varying coefficients only. As for the flux, they provide three expressions. The first two depend only on the slowly varying quantities $\boldsymbol{\nabla}\Psi$ and $\boldsymbol{\nabla}Q$. Therefore, it does not come as a surprise that both satisfy the group-velocity property. The third one, however, differs from the previous ones by the nondivergent vector $\mathbf{z} \times \boldsymbol{\nabla}(\tfrac{1}{2}Q\psi^2)$ and thus involves Q itself; it does not satisfy the group-velocity property.

Following Pedlosky (1987, section 6.10), it can be argued that the presence of Q in the flux is artificial, since it does not appear in the equation of the disturbance. In general, if the derivation of a conservation law relies on (and only on) equations with slowly varying coefficients, the density and flux should also have slowly varying coefficients only, and hence the group-velocity property should be satisfied. But, often, the equations for the disturbance do contain coefficients that are rapidly varying, and a WKB approach is possible only after the change of variables (4) is made. To derive a density and a flux that directly satisfy the group-velocity property, the procedure leading to the wave-activity conservation law must be entirely formulated in terms of the transformed variable υ. This implies tedious calculations (notably to reformulate the Hamiltonian structure) that should be avoided, especially if the nonlinear equations are considered.

As mentioned before, the anelastic model is an example of a system requiring such a change of variables. It is instructive to examine Scinocca and Shepherd's (1992) construction of the pseudoenergy conservation law. A first proposed density is given by their equation (5.10); that is,

$$A = \frac{|\boldsymbol{\nabla}\psi'|^2}{2\rho_0} + \left[\frac{\overline{\omega}}{2}\Psi''(\overline{\theta}) - \frac{\rho_0}{2}Z_0'(\overline{\theta})\right](\theta')^2 + \Psi'(\overline{\theta})\omega'\theta', \quad (12)$$

where $\overline{\omega}$ (ω') and $\overline{\theta}$ (θ') are the basic-state (disturbance) vorticity and potential temperature, Ψ is the basic-state streamfunction, and Z_0 is analogous to Long's function. In the WKB approximation discussed by these authors, the basic-state velocity varies slowly, but ρ_0 can vary rapidly with the altitude z; the change of variables leading to evolution equations with slowly-varying coefficients is $(\phi', \theta') = (\rho_0^{-1/2}\psi', \theta')$. In the WKB approximation, ϕ' and θ' vary rapidly through the phase only. When these results are substituted into (12), it is found that

$$A = \frac{|\nabla \phi'|^2}{2} - \frac{1}{2H}\phi'\phi'_z + \frac{1}{8H^2}(\phi')^2 + \left[\frac{\overline{\omega}}{2}\Psi''(\overline{\theta}) - \frac{\rho_0}{2}Z'_0(\overline{\theta})\right](\theta')^2 + \rho_0^{-1/2}\Psi'(\overline{\theta})(\gamma\phi' + \nabla^2\phi')\theta', \quad (13)$$

where the scale height H is defined by

$$\frac{1}{H} = -\frac{1}{\rho_0}\frac{d\rho_0}{dz}$$

and

$$\gamma = -\frac{3}{4\rho_0^2}\left(\frac{d\rho_0}{dz}\right)^2 + \frac{1}{2\rho_0}\frac{d^2\rho_0}{dz^2}$$

[see Scinocca and Shepherd's Eqs. (B2)–(B3)]. To satisfy the group-velocity property, Scinocca and Shepherd add the divergence of $\mathbf{B} = -\psi'\nabla\psi'/2\rho_0$ to A; they derive a new density given by their equation (5.20b) and an associated flux for which they explicitly verify the group-velocity property. In terms of the variables (ϕ', θ'), the new density, A', say, takes the form

$$A' = -\frac{1}{2}\left[\gamma(\phi')^2 + \phi'\nabla^2\phi'\right] + \left[\frac{\overline{\omega}}{2}\Psi''(\overline{\theta}) - \frac{\rho_0}{2}Z'_0(\overline{\theta})\right](\theta')^2 + \rho_0^{-1/2}\Psi'(\overline{\theta})(\gamma\phi' + \nabla^2\phi')\theta'. \quad (14)$$

Using Scinocca and Shepherd's (1992) assumptions on the basic state, it can be established that $\frac{1}{2}\overline{\omega}\Psi''(\overline{\theta}) - \frac{1}{2}\rho_0 Z'_0(\overline{\theta})$ and $\rho_0^{-1/2}\Psi'(\overline{\theta})$ vary slowly. It is then clear that the density (14) satisfies our condition of rapid variation through the phase only provided that γ is slowly varying, that is, $\gamma = \gamma(Z)$. This latter requirement is in fact necessary for the consistency of the WKB approximation itself (notably because γ appears in the dispersion relation). Similarly, it can be verified that the flux associated with (14) also satisfies our condition. Therefore, we can conclude that the group-velocity property is satisfied, as explicitly shown by Scinocca and Shepherd (1992). In many applications, the scale height H can be considered as slowly varying. In this situation, it can be seen that (13) varies rapidly through the wave phase only [i.e., that (7) is satisfied], and one can thus expect the density A to be as good a choice as A' with regard to the group-velocity property. This can be easily confirmed by noting that when H is slowly varying, both $\nabla\cdot\mathbf{B}$ and $\partial\mathbf{B}/\partial t$ have a vanishing phase average and thus do not affect the group-velocity property (2). The modification introduced to A is therefore not necessary when H is slowly varying.

3. Choice of the density

The difficulty in the search for a pair (A, $\mathbf{F}$) satisfying the group-velocity property mainly lies in the choice of a suitable density, that is, one defined by slowly varying coefficients only. If such a density is found, one can avoid using rapidly varying quantities in the derivation of the flux (possibly by using the equations for υ rather than u), and so obtain a consistent pair (A, $\mathbf{F}$) whose only rapid variations in the WKB approximation result from the wave phase.

Here, we propose a systematic choice for the density that guarantees (under certain conditions) that the density is defined by slowly varying coefficients according to (7), and hence can satisfy the group-velocity property. This particular density A_S, which equals $A + \nabla\cdot\mathbf{B}$ for some $\mathbf{B}$, is defined by the symmetric form

$$A_S = \frac{1}{2}u\cdot\mathsf{A}_S u, \quad (15)$$

where A_S is the self-adjoint operator $\mathsf{L}^\dagger\mathsf{L}' + (\mathsf{L}')^\dagger\mathsf{L}$ († denotes the adjoint).

It can be shown that the self-adjoint form (15) is unique; therefore A_S can be used for an unambiguous definition of the wave-activity density. However, this definition is interesting in the present context only if A_S has slowly varying coefficients in the sense of (7). This cannot be proven in general, for the existence of both a conservation law (1) and a WKB solution to the equations of motion does not ensure that a pair (A, $\mathbf{F}$) can be written with slowly varying coefficients only.[3] The following can nevertheless be established: if one can find a density A defined by slowly varying coefficients only, then the symmetric form A_S is also defined by slowly varying coefficients. To show this, we consider the derivative with respect to one (fast) coordinate, x, say, of the relation $A_S = A + \nabla\cdot\mathbf{B}$:

$$\frac{\partial}{\partial x}\left(\frac{1}{2}\mathsf{M}\upsilon\cdot\mathsf{A}_S\mathsf{M}\upsilon\right) = \frac{\partial}{\partial x}(\mathsf{L}\mathsf{M}\upsilon\cdot\mathsf{L}'\mathsf{M}\upsilon) + \nabla\cdot\left(\frac{\partial\mathbf{B}}{\partial x}\right).$$

Expanding this expression, we obtain

$$\frac{1}{2}\left[\upsilon\cdot\frac{\partial}{\partial x}(\mathsf{M}^T\mathsf{A}_S\mathsf{M})\upsilon + \frac{\partial\upsilon}{\partial x}\cdot\mathsf{M}^T\mathsf{A}_S\mathsf{M}\upsilon + \upsilon\cdot\mathsf{M}^T\mathsf{A}_S\mathsf{M}\frac{\partial\upsilon}{\partial x}\right] = \frac{\partial}{\partial x}(\mathsf{L}\mathsf{M})\upsilon\cdot\mathsf{L}'\mathsf{M}\upsilon + \mathsf{L}\mathsf{M}\upsilon\cdot\frac{\partial}{\partial x}(\mathsf{L}'\mathsf{M})\upsilon + \nabla\cdot\mathbf{D} + \frac{\partial\upsilon}{\partial x}\cdot\mathsf{M}^T\mathsf{L}^\dagger\mathsf{L}'\mathsf{M}\upsilon + \upsilon\cdot\mathsf{M}^T\mathsf{L}^\dagger\mathsf{L}'\mathsf{M}\frac{\partial\upsilon}{\partial x},$$

where M^T is the transpose of M and $\mathbf{D}$ is a vector whose details are unimportant. Using the definition of A_S, it can be seen that the last two terms of the left-hand side of the above expression combine with the last two terms of the right-hand side to give a divergence. The expression thus simplifies as

[3] In particular, one can find examples of systems with constant coefficients that admit invariants depending explicitly on the coordinates.

$$\frac{1}{2}v\cdot\frac{\partial}{\partial x}(\mathsf{M}^{\mathrm{T}}\mathsf{A}_{\mathrm{S}}\mathsf{M})v$$
$$= \frac{\partial}{\partial x}(\mathsf{LM})v\cdot\mathsf{L}'\mathsf{M}v + \mathsf{LM}v\cdot\frac{\partial}{\partial x}(\mathsf{L}'\mathsf{M})v + \nabla\cdot\mathbf{G}.$$

The first two terms on the right-hand side precisely correspond to the condition (7) that A is slowly varying, so we get the relation

$$\frac{1}{2}v\cdot\frac{\partial}{\partial x}(\mathsf{M}^{\mathrm{T}}\mathsf{A}_{\mathrm{S}}\mathsf{M})v = \nabla\cdot\mathbf{G} + O(\epsilon). \tag{16}$$

The fact that $\partial(\mathsf{M}^{\mathrm{T}}\mathsf{A}_{\mathrm{S}}\mathsf{M})/\partial x$ is self-adjoint (since $\mathsf{M}^{\mathrm{T}}\mathsf{A}_{\mathrm{S}}\mathsf{M}$ is) can now be exploited. Introducing $v = r + s$, we obtain from (16) that

$$\int_V r\cdot\frac{\partial}{\partial x}(\mathsf{M}^{\mathrm{T}}\mathsf{A}_{\mathrm{S}}\mathsf{M})s\, d\mathbf{x} = O(\epsilon).$$

Since r and s are arbitrary this holds only if

$$\frac{\partial}{\partial x}(\mathsf{M}^{\mathrm{T}}\mathsf{A}_{\mathrm{S}}\mathsf{M}) = O(\epsilon).$$

From this, the condition (7) that A_{S} is slowly varying, that is,

$$\frac{\partial}{\partial x}(\mathsf{M})v\cdot\mathsf{A}_{\mathrm{S}}\mathsf{M}v + \mathsf{M}v\cdot\frac{\partial}{\partial x}(\mathsf{MA}_{\mathrm{S}})v = O(\epsilon)$$

is readily derived.

Note that Brunet (1994) also proposed the use of a self-adjoint form for wave-activity densities, arguing that this form naturally emerges in the theory of empirical normal modes.

Returning to the example of the anelastic model, it can be seen that the slowly varying pseudoenergy density (14) proposed by Scinocca and Shepherd (1992) has the symmetric form (15). Indeed, with $u = (\omega', \theta')$, one can write their Eq. (5.20b) as (15) with

$$\mathsf{A}_{\mathrm{S}} = \begin{pmatrix} -\Pi^{-1} & \Psi'(\overline{\theta}) \\ \Psi'(\overline{\theta}) & \overline{\omega}\Psi''(\overline{\theta}) - \rho_0 Z_0'(\overline{\theta}) \end{pmatrix},$$

where the operator

$$\Pi(\cdot) = \frac{1}{\rho_0}\nabla^2(\cdot) - \frac{1}{\rho_0^2}\frac{d\rho_0}{dz}\frac{\partial(\cdot)}{\partial z} = \nabla\cdot\left[\frac{1}{\rho_0}\nabla(\cdot)\right]$$

is self-adjoint. Clearly, A_{S} is self-adjoint, as required. Similarly, the pseudomomentum density of Scinocca and Shepherd takes a symmetric form [see their Eq. (6.15)]. The fact that both densities finally lead to local conservation laws satisfying the group-velocity property is thus well explained by the arguments developed above.

4. Summary

The definition of the local quantities (A, $\mathbf{F}$) corresponding to a given global conservation law contains a certain degree of arbitrariness. Therefore, the group-velocity property (2) is often imposed as a constraint, because it clarifies the physical interpretation of the local conservation law. Technically, the verification of the group-velocity property, and a fortiori the search for a pair (A, $\mathbf{F}$) satisfying this property, turns out to be quite complicated: the explicit calculation of the terms involved in (2) requires a large amount of algebra, and there is no constructive procedure leading to a correct pair (A, $\mathbf{F}$).

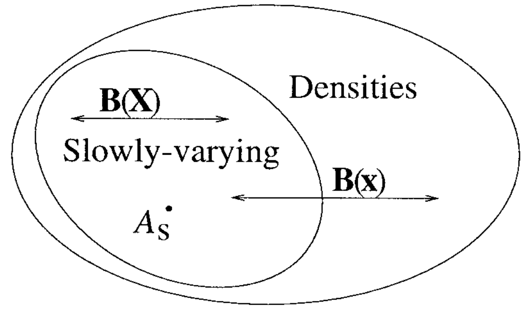

FIG. 1. The set of the equivalent densities, differing by the divergence of arbitrary vectors **B**. A subset is constituted by the densities defined by slowly varying coefficients, which differ by the divergence of vectors with slowly varying coefficients **B**(**X**). The symmetric density A_{S} belongs to this subset.

In this note, we make two remarks that should simplify the derivation of the local form of a wave-activity conservation law. First, we show that if the density A and the flux $\mathbf{F}$ are defined by slowly varying coefficients in the sense that, in the WKB approximation, their only rapid variations result from the wave phase, then the fact that they are quadratic and satisfy the conservation equation (1) ensures that they also satisfy the group-velocity property. The verification of (2) is thus reduced to the verification that the coefficients defining A and $\mathbf{F}$ (in terms of the variables suited to the WKB approximation) are slowly varying. Second, we propose a particular symmetric form, A_{S}, for the density, which is based on a self-adjoint operator. This form is unique and is shown to be defined by slowly varying coefficients under the assumption that at least one density satisfying the latter condition exists. The situation is summarized in Fig. 1. The set of all the equivalent densities is represented; these densities differ by divergences $\nabla\cdot\mathbf{B}$, and in general $\mathbf{B}$ can have rapidly varying coefficients, even in the WKB approximation. A subset contains the densities defined by slowly varying coefficients, which differ from each other by the divergence of a vector, $\mathbf{B}(\mathbf{X})$, with slowly varying coefficients [in the sense of (7)]. What we have shown is that the symmetric density A_{S} belongs to this subset if it is not empty. Note that in the WKB approximation the phase average of all the densities with slowly varying coefficients is equal at leading order. [This is because the components of the slowly varying vectors $\mathbf{B}(\mathbf{X})$ have a form similar to (8); when the divergence is taken, only the oscillatory terms—whose phase average vanishes—remain.] This also holds for the associated slowly varying fluxes. Our discussion highlights the fact that a large set of pairs (*A*, **F**) satisfy the group-velocity property, which thus does not strongly reduce the arbitrariness in the definition of the density and flux of wave activities.

When a globally conserved quantity, $\mathcal{A}$, is known, it is not difficult to find (merely through integration by parts) the symmetric form of the density A_S, that is, the self-adjoint operator A_S. It is then sufficient to check that this density has slowly varying coefficients in the WKB limit to be certain that a corresponding flux can be found such that the group-velocity property is satisfied. This flux can be derived from (1) provided that no rapidly varying quantities are used in the course of the derivation. Following this procedure should considerably limit the algebra required to find a density and a flux satisfying the group-velocity property. In particular, note that when a dispersion relation has multiple branches, the condition of slow variations of the coefficients defining *A* and **F** guarantees that the group-velocity property is satisfied for all branches [see the remarks in Durran (1995) and in Brunet and Haynes (1996)].

We conclude by remarking that the group-velocity property is not the only property that one may impose to reduce the arbitrariness in the definition of *A* and **F**. In a study of stationary waves, Plumb (1985) manipulated the expressions of *A* and **F** to obtain forms that are phase invariant in the WKB limit; this facilitates the interpretation of the density and the flux when they are not averaged. Although similar forms have been employed in a more general context (Brunet and Haynes 1996), they rely on a heuristic derivation and it is not clear whether they can be obtained in a general framework.

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