Transdifferential and Transintegral Calculus

Tiago S. dos Reis and James A.D.W. Anderson

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Tiago S. dos Reis is with the Federal Institute of Education, Science and Technology of Rio de Janeiro, Brazil, 27215-350 and simultaneously with the Program of History of Science, Technique, and Epistemology, Federal University of Rio de Janeiro, Brazil, 21941-916 e-mail: tiago.reis@ifrj.edu.br

J.A.D.W. Anderson is with the School of Systems Engineering, Reading University, England, RG6 6AY e-mail: j.anderson@reading.ac.uk

Abstract

The set of transreal numbers is a superset of the real numbers. It totalises real arithmetic by defining division by zero in terms of three definite, non-finite numbers: positive infinity, negative infinity and nullity. Elsewhere, in this proceedings, we extended continuity and limits from the real domain to the transreal domain, here we extended the real derivative to the transreal derivative. This continues to demonstrate that transreal analysis contains real analysis and operates at singularities where real analysis fails. Hence computer programs that rely on computing derivatives – such as those used in scientific, engineering and financial applications – are extended to operate at singularities where they currently fail. This promises to make software, that computes derivatives, both more competent and more reliable.

We also extended the integration of absolutely convergent functions from the real domain to the transreal domain.

Keywords: transreal arithmetic, transreal analysis, transderivative.

1 Introduction

Transreal [5] and transcomplex arithmetic [2][6] are developments of Computer Science that are now being normalised in Mathematics [7]. They define division in terms of operations on the lexical reciprocal. This lexical definition contains the usual definition of division, as multiplication by the multiplicative inverse, but also defines division by zero. Consequently transreal and transcomplex arithmetic are supersets of, respectively, real and complex arithmetic. There is a machine proof [5] and a human proof [6] that transreal arithmetic is consistent if real arithmetic is. The hand proof also demonstrates that transreal arithmetic contains real arithmetic and establishes a similar relationship between transcomplex arithmetic and complex arithmetic.

Transreal arithmetic uses a subset of the algorithms of real arithmetic so the general reader will be able to follow any computation in transreal arithmetic but will have little chance of deriving a valid, non-finite, computation until the axioms [5] or algorithms [2] of transreal arithmetic have been properly learned. The reader is cautioned that the relational operators of transreal arithmetic, less-than ($<$), equal-to ($=$), greater-than ($>$), form a total set of independent operations, unlike their real counterparts. The general reader will not understand the transreal relations until the material in [3] has been properly learned. We are aware that this places a heavy burden on the reader but this is inevitable because transmathematics operates in a new paradigm. The reader must understand the paradigm before much progress can be made on any particular result.

We have already demonstrated that continuity and limits in transreal analysis contain all of their real counterparts and support continuity and limits at the exact, transreal singularities that arise on division by zero. We now do the same for the transreal derivative so that it contains the real derivative and operates at singularities where the real derivative is undefined. This establishes the foundation for a great deal of future work in which all of the results of real, differential calculus are extended to transreal differential calculus. We expect that in every case transreal analysis will contain its real counterpart.

We make a start to this further work, by extending the integral of absolutely convergent, real functions, to the transintegral of absolutely convergent, transreal functions. Thus the transintegral contains all of these real integrals and extends them to operate at singularities. However, this is a rather restricted set of functions. Since the preparation of this integral, a much wider extension of the real integral to the transreal integral has been developed. That material has been submitted for publication elsewhere.

2 Transreal Analysis

In this section we extend the concepts of derivative and integral from the domain of real numbers, $\mathbb{R}$, to the domain of transreal numbers, $\mathbb{R}^T$, largely replacing earlier work on this topic [1]. We draw heavily on the results in [3][4].

2.1 Transreal Derivative

Definition 1. *Let $A \subset \mathbb{R}^T$ and $x_0 \in A$. Here A' denotes the set of limit points of A.*

i) *If $x_0 \in \mathbb{R} \cap A'$, we say f is differentiable at x_0 on $\mathbb{R}^T$ if and only if f is differentiable at x_0 in the usual sense. And in this case, $f'(x_0)$ is called the derivative of f at x_0 on $\mathbb{R}^T$ and it is denoted as $f'_{\mathbb{R}^T}(x_0)$.*

ii) *If $x_0 \in \{-\infty, \infty\} \cap D'$ (where D denotes the set of points in A at which f is differentiable in the usual sense), we say f is differentiable at x_0 on $\mathbb{R}^T$ if and only if the following limit exists*

$$\lim_{x \to x_0} f'(x).$$

And if this limit exists then it is called the derivative of f at x_0 on $\mathbb{R}^T$ and it is denoted as $f'_{\mathbb{R}^T}(x_0)$.

iii) *If $x_0 \notin A'$ we define the derivative of f at x_0 on $\mathbb{R}^T$ as $f'_{\mathbb{R}^T}(x_0) := \Phi$.*

Observe that is not possible to define the derivative at $x_0 \notin A'$ by way of a limit because, as is known, if we try to apply the limit definition at $x_0 \notin A'$, any $L \in \mathbb{R}^T$ could be the limit $\lim\limits_{x \to x_0} f(x)$. In fact, since $x_0 \notin A'$, there is a neighbourhood U of x_0 such that $A \cap U = \emptyset$, hence for any neighbourhood V of L, $f(x) \in V$ for all $x \in A \cap U$. Because, vacuously, there is no $x \in A \cap U$ such that $f(x) \notin V$. Rather than accept the indeterminacy of the derivative at $x_0 \notin A'$, we choose to define $f'_{\mathbb{R}^T}(x_0) = \Phi$. This will presently lead us to the position where the exponential is identically its own derivative with $e'(x) = e(x)$ so that the usual properties of this important function hold when extended to $\mathbb{R}^T$.

Observation 2. *Note that differentiability on $\mathbb{R}^T$ does not imply continuity. For example let $f : \mathbb{R}^T \to \mathbb{R}^T$, where*

$$f(x) = \begin{cases} e^x & , \text{ if } x \neq \infty \\ 1 & , \text{ if } x = \infty \end{cases} .$$

Clearly f is not continuous at ∞ but $\lim\limits_{x \to \infty} f'(x) = \infty$, whence f is differentiable at ∞ on $\mathbb{R}^T$. For the definition of e^x in $\mathbb{R}^T$ see [1].

Example 3. *Let $f(x) = e^x$. It follows from Definition 1 that $f'_{\mathbb{R}^T}(x) = e^x$ for all $x \in \mathbb{R}^T$. Particularly, $f'_{\mathbb{R}^T}(-\infty) = 0$, $f'_{\mathbb{R}^T}(\infty) = \infty$ and $f'_{\mathbb{R}^T}(\Phi) = \Phi$.*

Definition 4. *Let $A \subset \mathbb{R}^T$, $f : A \times A \to \mathbb{R}^T$, $x_0 \in A'$ and $L \in \mathbb{R}^T$. We say that*

$$\lim_{\substack{x \to x_0 \\ y \to x_0}} f(x, y) = L$$

if and only if, given an arbitrary neighbourhood V of L there is a neighbourhood U of x_0 such that $f(x, y) \in V$ whenever $x \neq y$ and $x, y \in A \cap U \setminus \{x_0\}$.

Note that $\lim\limits_{\substack{x \to x_0 \\ y \to x_0}} f(x, y) \neq \lim\limits_{(x,y) \to (x_0, x_0)} f(x, y)$, where $\lim\limits_{(x,y) \to (x_0, x_0)} f(x, y)$ denotes the limit, in the usual sense, of a function of two variables. In other words, these are different limiting processes.

Proposition 5. *Let $a \in \mathbb{R}$ and $f : (a, \infty] \to \mathbb{R}^T$ such that f is differentiable in (a, ∞). It follows that f is differentiable at ∞ if and only if there exists $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{f(x)-f(y)}{x-y}$. And in this case,*

$$f'_{\mathbb{R}^T}(\infty) = \lim_{\substack{x\to\infty\\ y\to\infty}} \frac{f(x)-f(y)}{x-y}.$$

Proof. Let $a \in \mathbb{R}$ and $f : (a, \infty] \to \mathbb{R}^T$ such that f is differentiable in (a, ∞). Observe that f is continuous in (a, ∞).

First let us suppose that $f'_{\mathbb{R}^T}(\infty) = L \in \mathbb{R}^T$, that is $\lim\limits_{z\to\infty} f'_{\mathbb{R}^T}(z) = L$. Let V be an arbitrary neighbourhood of L. Then there is $M > a$ such that $f'_{\mathbb{R}^T}(z) \in V$ for all $z \in (M, \infty)$. Let $x, y \in (M, \infty)$ such that $x \neq y$. Say $x < y$. Since f is continuous in $[x, y]$ and differentiable in (x, y), by the Mean Value Theorem, there is $z \in (x, y)$ such that $\dfrac{f(x)-f(y)}{x-y} = f'_{\mathbb{R}^T}(z)$. Since $z \in (x, y) \subset (M, \infty)$ we have

$$\frac{f(x)-f(y)}{x-y} = f'_{\mathbb{R}^T}(z) \in V.$$

Thus $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{f(x)-f(y)}{x-y} = L$.

Now suppose that $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{f(x)-f(y)}{x-y} = L$. Note that $L \neq \Phi$ for $f'_{\mathbb{R}^T}(z) \in \mathbb{R}$ for all $z \in (a, \infty)$. If $L \in \mathbb{R}$, let there be an arbitrary $\varepsilon \in \mathbb{R}^+$. Then there is $M \geq a$ such that $-\dfrac{\varepsilon}{2} < \dfrac{f(x)-f(y)}{x-y} - L < \dfrac{\varepsilon}{2}$ whenever $x, y \in (M, \infty)$ and $x \neq y$. For each $x \in (M, \infty)$, taking the limit in the inequality with y tending to x, we obtain $-\varepsilon < -\dfrac{\varepsilon}{2} \leq \lim\limits_{y\to x} \dfrac{f(y)-f(x)}{y-x} - L \leq \dfrac{\varepsilon}{2} < \varepsilon$, whence $-\varepsilon < f'_{\mathbb{R}^T}(x) - L < \varepsilon$, therefore $\lim\limits_{x\to\infty} f'_{\mathbb{R}^T}(x) = L$. If $L = \infty$, let there be an arbitrary $N \in \mathbb{R}^+$. Then there is $M \geq a$ such that $2N < \dfrac{f(x)-f(y)}{x-y}$ whenever $x, y \in (M, \infty)$ and $x \neq y$. For each $x \in (M, \infty)$, taking the limit in the inequality with y tending to x, we obtain $N < 2N \leq \lim\limits_{y\to x} \dfrac{f(y)-f(x)}{y-x}$, whence $N < f'(x)$, therefore $\lim\limits_{x\to\infty} f'_{\mathbb{R}^T}(x) = \infty$. If $L = -\infty$ the result follows similarly. □

Proposition 6. *Let $a \in \mathbb{R}$ and $f : [-\infty, a) \to \mathbb{R}^T$ such that f is differentiable in $(-\infty, a)$ in the usual sense. It follows that f is differentiable at $-\infty$ if and only if there exists $\lim\limits_{\substack{x\to-\infty\\ y\to-\infty}} \dfrac{f(x)-f(y)}{x-y}$. And in this case,*

$$f'_{\mathbb{R}^T}(-\infty) = \lim_{\substack{x\to-\infty\\ y\to-\infty}} \frac{f(x)-f(y)}{x-y}.$$

Proof. The proof is similar to the proof of Proposition 5. $\square$

Proposition 7. *Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. If f is continuous at x_0 and there exists the limit $\lim\limits_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y}$ then f is differentiable at x_0 and*

$$f'_{\mathbb{R}^T}(x_0) = \lim_{\substack{x \to x_0 \\ y \to x_0}} \frac{f(x) - f(y)}{x - y}.$$

Proof. Let f be continuous at x_0 such that there exists a limit $\lim\limits_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y}$, say $\lim\limits_{\substack{x \to x_0 \\ y \to x_0}} \dfrac{f(x) - f(y)}{x - y} = a$. Since f is continuous at x_0, $\lim\limits_{y \to x_0} f(y) = f(x_0)$. Let there be an arbitrary $\varepsilon \in \mathbb{R}^+$. Then there is a $\delta \in \mathbb{R}^+$ such that for each $x \in A \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$, it follows that

$$-\frac{\varepsilon}{2} < \frac{f(x) - f(y)}{x - y} - a < \frac{\varepsilon}{2} \text{ for all } y \in A \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}.$$

Taking the limit in the above inequality with y tending to x_0, we obtain $-\dfrac{\varepsilon}{2} \leq \dfrac{f(x) - f(x_0)}{x - x_0} - a \leq \dfrac{\varepsilon}{2}$. Thus

$$-\varepsilon < -\frac{\varepsilon}{2} \leq \frac{f(x) - f(x_0)}{x - x_0} - a \leq \frac{\varepsilon}{2} < \varepsilon$$

for all $x \in A \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$, whence $f'_{\mathbb{R}^T}(x_0) = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = a$. $\square$

Observation 8. *Notice that in Proposition 7, the hypothesis of the continuity of f is, in fact, needed. For instance let $f : \mathbb{R} \to \mathbb{R}$, where*

$$f(x) = \begin{cases} x & , \text{ if } x \neq 0 \\ 1 & , \text{ if } x = 0 \end{cases}.$$

Clearly f is not differentiable at 0, but $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{f(x) - f(y)}{x - y} = 1$.

Proposition 9. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$. If f is continuously differentiable in I (which means f is differentiable in I and $f'_{\mathbb{R}^T}$ is continuous in I), then there exists $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{f(x) - f(y)}{x - y}$ and*

$$\lim_{\substack{x \to x_0 \\ y \to x_0}} \frac{f(x) - f(y)}{x - y} = f'_{\mathbb{R}^T}(x_0)$$

for all $x_0 \in I$.

Proof. Let $f : I \to \mathbb{R}$ be a continuously differentiable function and let $x_0 \in I$. Let us denote as a the derivative of f at x_0, that is, $f'_{\mathbb{R}^T}(x_0) = a$. Let there be an arbitrary $\varepsilon \in \mathbb{R}^+$. Since $f'_{\mathbb{R}^T}$ is continuous at x_0, there is a $\delta \in \mathbb{R}^+$ such that $f'_{\mathbb{R}^T}(z) \in (a-\varepsilon, a+\varepsilon)$ whenever $z \in I \cap (x_0 - \delta, x_0 + \delta)$. Now let $x, y \in I \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ such that $x \neq y$. Say $x < y$. Since f is continuous in $[x, y]$ and differentiable in (x, y), by the Mean Value Theorem, there is $z \in (x, y)$ such that $\dfrac{f(x) - f(y)}{x - y} = f'_{\mathbb{R}^T}(z)$. Since $z \in (x, y) \subset I \cap (x_0 - \delta, x_0 + \delta)$, we have

$$\frac{f(x) - f(y)}{x - y} = f'_{\mathbb{R}^T}(z) \in (a - \varepsilon, a + \varepsilon).$$

Thus $\displaystyle\lim_{\substack{x \to x_0 \\ y \to x_0}} \frac{f(x) - f(y)}{x - y} = a$. $\square$

Observation 10. *Notice that in Proposition 9, the hypothesis of the continuity of $f'_{\mathbb{R}^T}$ is, in fact, needed. Let $f : \mathbb{R} \to \mathbb{R}$, where*

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases} .$$

Note that $f'_{\mathbb{R}^T}(0) = 0$ but $\displaystyle\lim_{\substack{x \to x_0 \\ y \to x_0}} \frac{f(x) - f(y)}{x - y}$ does not exist. Indeed given an arbitrary $\delta \in \mathbb{R}^+$, let us take a positive, even integer, n, that is sufficiently large that $\dfrac{1}{n\pi} \in (-\delta, \delta)$. Denoting $x = \dfrac{1}{n\pi}$, $y = \dfrac{1}{n\pi + \frac{\pi}{2}}$ and $z = \dfrac{1}{(n+1)\pi + \frac{\pi}{2}}$, we have $x, y, z \in (-\delta, \delta)$ and $\dfrac{f(x) - f(y)}{x - y} = -\dfrac{4n}{2n\pi + \pi}$ and $\dfrac{f(x) - f(z)}{x - z} = \dfrac{4n}{6n\pi + 9\pi}$.

If we make some changes to the definition of $\displaystyle\lim_{\substack{x \to x_0 \\ y \to x_0}} \frac{f(x) - f(y)}{x - y}$ then, under suitable conditions, we can withdraw the hypothesis of the continuity of $f'_{\mathbb{R}^T}$ in Proposition 9. This is explained in the following proposition.

Proposition 11. *Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'_- \cap A'_+$. If f is differentiable at x_0 then, given an arbitrary neighbourhood V of $f'_{\mathbb{R}^T}(x_0)$, there is a neighbourhood U of x_0 such that $\dfrac{f(x) - f(y)}{x - y} \in V$, whenever $x, y \in A \cap U$ and $x < x_0 < y$.*

Proof. Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'_- \cap A'_+$ such that f is differentiable at x_0. Let us denote as a the derivative of f at x_0, that is $f'_{\mathbb{R}^T}(x_0) = a$.

Let $V = (a - \varepsilon, a + \varepsilon)$ for some $\varepsilon \in \mathbb{R}^+$. Then there is a $\delta \in \mathbb{R}^+$ such that $\left| \dfrac{f(x) - f(x_0)}{x - x_0} - a \right| < \dfrac{\varepsilon}{3}$ whenever $x \in A \cap (x_0 - \delta, x_0)$ and $\left| \dfrac{f(y) - f(x_0)}{y - x_0} - a \right| <$

$\dfrac{\varepsilon}{3}$ whenever $y \in A \cap (x_0, x_0+\delta)$. Now let $x, y \in A \cap (x_0-\delta, x_0+\delta)$ such that $x < x_0 < y$. Observe that $\dfrac{f(x)-f(y)}{x-y} - a = \dfrac{y-x_0}{y-x}\left(\dfrac{f(y)-f(x_0)}{y-x_0} - a\right) - \dfrac{y-x_0}{y-x}\left(\dfrac{f(x)-f(x_0)}{x-x_0} - a\right) + \left(\dfrac{f(x)-f(x_0)}{x-x_0} - a\right)$ and that $\left|\dfrac{y-x_0}{y-x}\right| < 1$. Hence $\left|\dfrac{f(x)-f(y)}{x-y} - a\right| \leq \left|\dfrac{y-x_0}{y-x}\right| \left|\dfrac{f(y)-f(x_0)}{y-x_0} - a\right| + \left|\dfrac{y-x_0}{y-x}\right| \left|\dfrac{f(x)-f(x_0)}{x-x_0} - a\right| + \left|\dfrac{f(x)-f(x_0)}{x-x_0} - a\right| < \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} = \varepsilon$. Thus $\dfrac{f(x)-f(y)}{x-y} \in V$. $\square$

2.2 Transreal Integral

Definition 12. *Let $a, b \in \mathbb{R}^T$. We define:*

a) $(a,b) := \{x \in \mathbb{R}^T;\ a < x < b\}$, $(a,b] := (a,b) \cup \{b\}$, $[a,b) := \{a\} \cup (a,b)$ and $[a,b] := \{a\} \cup (a,b) \cup \{b\}$. We say that A, with $A \subset \mathbb{R}^T$, is an interval if and only if A is one of these four types of sets.

Notice that $(a,\Phi) = \emptyset = (\Phi,a)$, $(a,\Phi] = \{\Phi\} = [\Phi,a)$, $[a,\Phi) = \{a\} = (\Phi,a]$ and $[a,\Phi] = \{\Phi,a\} = [\Phi,a]$ for all $a \in \mathbb{R}^T$.

b) If $I \in \{(a,b), (a,b], [a,b), [a,b]\}$, we define the length of I as

$$|I| := \begin{cases} 0 & , \text{ if } I = \emptyset \\ k-k & , \text{ if } I = \{k\} \text{ for some } k \in \mathbb{R}^T \\ b-a & , \text{ otherwise} \end{cases} .$$

c) Let $A \subset \mathbb{R}^T$. We say that $\mathcal{X}_A$ is the characteristic function of A if and only if

$$\mathcal{X}_A(x) = \begin{cases} 1 & , \text{ if } x \in A \\ 0 & , \text{ if } x \notin A \end{cases} .$$

d) Let $[a,b]$ be an interval. A set P is said to be a partition of $[a,b]$ if and only if there are $n \in \mathbb{N}$, $x_0, \ldots, x_n \in [a,b]$ such that $P = (x_0, \ldots, x_n)$ where $x_0 = a$, $x_n = b$ and, furthermore, if $n = 2$, $x_0 \leq x_1$ and if $n > 2$, $x_0 < x_1 < \cdots < x_{n-1} < x_n$.

e) We say that $\varphi : [a,b] \to \mathbb{R}^T$ is a step function if and only if there is a partition $P = (x_0, \ldots, x_n)$ of $[a,b]$ and $c_1, \ldots, c_n \in \mathbb{R}^T$ such that

$$\varphi = \sum_{j=1}^{n} c_j \mathcal{X}_{I_j},$$

where $I_j = (x_{j-1}, x_j]$ for all $j \in \{1, \ldots, n\}$.

We denote as $\mathcal{S}([a,b])$ the set of step functions on $[a,b]$ and note that the description of a step function is not unique.

Definition 13. *Let $a, b \in \mathbb{R}^T$ and $\varphi = \sum_{j=1}^{n} c_j \mathcal{X}_{I_j}$ be a step function on $[a, b]$. We define the integral in $\mathbb{R}^T$ of φ on $[a, b]$ as*

$$\int_{a}^{b}\limits_{\mathbb{R}^T} \varphi(x)\ dx := \sum_{\substack{j=1 \\ j;\ c_j \neq 0}}^{n} c_j \left| I_j \right| .$$

Notice that the integral of a step function is independent of the particular step function used.

If $x, y \in \mathbb{R}^T$, we write $x \not< y$, if and only if $x < y$ does not hold and we write $x \not> y$, if and only if $x > y$ does not hold. Notice that $\not<$ is not equivalent to $\geq$. For example $\Phi \not< 0$ but $\Phi \geq 0$ does not hold. See [3].

Definition 14. *Let there be a non-empty set $A \subset \mathbb{R}^T$. We say that $u \in \mathbb{R}^T$ is the supremum of A and we write $u = \sup A$ if and only if one of the following conditions occurs:*

i) $A = \{\Phi\}$ and $u = \Phi$ or

ii) $u \neq \Phi$ and $u \not< x$ for all $x \in A$ and if $w \in \mathbb{R}^T$, such that $w \not< x$ for all $x \in A$, then $w \not< u$.

And we say that $v \in \mathbb{R}^T$ is the infimum of A and we write $v = \inf A$ if and only if one of the following conditions occurs:

iii) $A = \{\Phi\}$ and $v = \Phi$ or

iv) $v \neq \Phi$ and $x \not< v$ for all $x \in A$ and if $w \in \mathbb{R}^T$, such that $x \not< w$ for all $x \in A$, then $v \not< w$.

Definition 15. *Let $a, b \in \mathbb{R}^T$ and let there be a function $f : [a, b] \to \mathbb{R}^T$. We say that f is integrable in $\mathbb{R}^T$ on $[a, b]$ if and only if*

$$\inf \left\{ \int_{a}^{b}\limits_{\mathbb{R}^T} \varphi(x)\ dx;\ \varphi \in \mathcal{S}([a, b]) \text{ and } \varphi \not< f \right\} =$$

$$\sup \left\{ \int_{a}^{b}\limits_{\mathbb{R}^T} \sigma(x)\ dx;\ \sigma \in \mathcal{S}([a, b]) \text{ and } f \not< \sigma \right\}.$$

And in this case the integral of f in $\mathbb{R}^T$ on $[a, b]$ is defined as

$$\int_{a}^{b}\limits_{\mathbb{R}^T} f(x)\ dx :=$$

$$\inf \left\{ \int_{a}^{b}\limits_{\mathbb{R}^T} \varphi(x)\ dx;\ \varphi \in \mathcal{S}([a, b]) \text{ and } \varphi \not< f \right\}.$$

Notice that if φ is a step function on $[a, b]$ then definitions 13 and 15 give the same result.

Proposition 16. *a) Let $a, b \in \mathbb{R}$ and let there be a bounded function $f : [a, b] \to \mathbb{R}$. It follows that f is Riemann integrable in $\mathbb{R}$ if and only if f is integrable in $\mathbb{R}^T$. And in this case, $\displaystyle\int_a^b f(x)\ dx = \underset{\mathbb{R}^T}{\int_a^b} f(x)\ dx$.*

b) Let $a \in \mathbb{R}$ and let $f : [a, \infty] \to \mathbb{R}$ be a function that is Riemann integrable on every closed subinterval of $[a, \infty)$. The improper Riemann integral $\displaystyle\int_a^\infty |f|(x)\ dx$ exists if and only if f is integrable in $\mathbb{R}^T$. And in this case, $\displaystyle\int_a^\infty f(x)\ dx = \underset{\mathbb{R}^T}{\int_a^\infty} f(x)\ dx$.

c) Let $b \in \mathbb{R}$ and let $f : [-\infty, b] \to \mathbb{R}$ be a function that is Riemann integrable on every closed subinterval of $(-\infty, b]$. The improper Riemann integral $\displaystyle\int_{-\infty}^b |f|(x)\ dx$ exists if and only if f is integrable in $\mathbb{R}^T$. And in this case, $\displaystyle\int_{-\infty}^b f(x)\ dx = \underset{\mathbb{R}^T}{\int_{-\infty}^b} f(x)\ dx$.

d) Let $f : [-\infty, \infty] \to \mathbb{R}$ be a function that is Riemann integrable on every closed subinterval of $(-\infty, \infty)$. The improper Riemann integral $\displaystyle\int_{-\infty}^\infty |f|(x)\ dx$ exists if and only if f is integrable in $\mathbb{R}^T$. And in this case, $\displaystyle\int_{-\infty}^\infty f(x)\ dx = \underset{\mathbb{R}^T}{\int_{-\infty}^\infty} f(x)\ dx$.

e) Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}^T$ be a function such that $f((a, b]) \subset \mathbb{R}$, $f(a) = \infty$ and f is Riemann integrable on any subinterval in $(a, b]$. The improper Riemann integral $\displaystyle\int_a^b |f|(x)\ dx$ exists if and only if f is integrable in $\mathbb{R}^T$. And in this case $\displaystyle\int_a^b f(x)\ dx = \underset{\mathbb{R}^T}{\int_a^b} f(x)\ dx$.

f) Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}^T$ be a function such that $f([a, b)) \subset \mathbb{R}$, $f(b) = \infty$ and f is Riemann integrable on any subinterval in $[a, b)$. The improper Riemann integral $\displaystyle\int_a^b |f|(x)\ dx$ exists if and only if f is integrable in $\mathbb{R}^T$. And in this case $\displaystyle\int_a^b f(x)\ dx = \underset{\mathbb{R}^T}{\int_a^b} f(x)\ dx$.

Proof. a) It is sufficient to observe that, since $[a, b] \subset \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$, the integral $\displaystyle\underset{\mathbb{R}^T}{\int_a^b} f(x)\ dx$ is precisely the Darboux integral, which is known to be equivalent to the Riemann integral.

b), c), d), e) and f) It is sufficient to note that if $[a, b] \subset [-\infty, \infty]$ and $f : [a, b] \to [-\infty, \infty]$ is a non-negative function that is Lebesgue integrable then the integral $\underset{\mathbb{R}^T}{\int_a^b} f(x)\ dx$ is equal the Lebesgue integral of f on (a, b). See [9], Section 2.1 and use the Theorems 37, 38, 45 and 46 in [8]. $\square$

Example 17. *Let $f : \mathbb{R}^T \to \mathbb{R}^T$ and let there be an aribtrary $a \in \mathbb{R}^T$. It follows that:*

a) If $a \in \mathbb{R}$ and $f(a) \in \mathbb{R}$ then $\underset{\mathbb{R}^T}{\int_a^a} f(x)\ dx = 0$. Because $\underset{\mathbb{R}^T}{\int_a^a} f(x)\ dx = f(a)\,|[a, a]| = f(a) \times 0 = 0$;

b) If $a \in \{-\infty, \infty, \Phi\}$ then $\underset{\mathbb{R}^T}{\int_a^a} f(x)\ dx = \Phi$. Because $\underset{\mathbb{R}^T}{\int_a^a} f(x)\ dx = f(a)|[a, a]| = f(a) \times \Phi = \Phi$;

c) $\underset{\mathbb{R}^T}{\int_a^\Phi} f(x)\ dx = \underset{\mathbb{R}^T}{\int_\Phi^a} f(x)\ dx = \Phi$. In order to see this, let $\varphi \in \mathcal{S}([a, \Phi])$. Whence $\underset{\mathbb{R}^T}{\int_a^\Phi} \varphi(x)\ dx = \varphi(a)|[a, a]| + \varphi(\Phi)|[\Phi, \Phi]| = \varphi(a)|[a, a]| + \varphi(\Phi)\Phi = \varphi(a)|[a, a]| + \Phi = \Phi$. Thus $\underset{\mathbb{R}^T}{\int_a^\Phi} f(x)\ dx = \Phi$.

The reader will appreciate that it would be possible to define the integral in $\mathbb{R}^T$ in a more general way, for example by defining it in a manner analogous to the Lebesgue integral. However, in this paper, we had the more modest aim of giving the first, detailed definition of the integral in $\mathbb{R}^T$, replacing the earlier proposal in [1]. We choose a definition that extends the concept of the integral to $\mathbb{R}^T$ in a simple way. We then found that it is totally coincident with the Riemann integral when the domain and codomain of a function are subsets of the real numbers: $\mathrm{Dm}(f) \subset \mathbb{R}$ and $\mathrm{CDm}(f) \subset \mathbb{R}$.

3 Discussion

The transintegral, as introduced above, is the first mathematical structure that has been defined for which the trans version is less general than the usual one. (We now know that there is a more general definition of the transintegral that contains the real integral. A paper on that subject has been submitted for publication elsewhere but we press on, here, with a notational device that admits all of the results of real analysis to transreal analysis. We mention it because this notational approach may be of more widespread use.)

One possibility for defining the transintegral, so that it contains the usual integral, is that we should define the transintegral asymptotically toward the infinities, as usual, and then observe that the the infinities are singleton points which make no additional contribution to the transintegral. The resulting transintegral differing from the usual one only in that it is defined over functions of transnumbers.

While a difference in integrals remains, we may handle the difference notationally. Consider the symbols for the usual integral: $\int_a^b f(x)\,\mathrm{d}x$. We introduce a notation to indicate whether a limit of integration, say a, is exact, $x = a$, or asymptotic, $x \to a$, for transreal a, x. We specify the reading of an isolated symbol, a, so that a is a shorthand for $x = a$ when $a \in \mathbb{R} \cup \{\Phi\}$ and a is a shorthand for $x \to a$ when $a \in \{-\infty, \infty\}$. When the shorthand does not apply we write the limit explicitly. For example the fragment $\int_0^\infty$ indicates the integral from exactly zero, asymptotically toward infinity, as usual, and the new fragment $\int_{x\to 0}^{x=\infty}$ indicates the integral asymptotically from zero, exactly to infinity. This notation preserves the whole of the usual notation for integrals, preserves all of the results of real integration and introduces new, non-finite results.

We believe it is important to examine many possible definitions of the transintegral and their uses before coming to a judgement on what the standard definition should be. This is entirely normal in a new area of mathematics, as recapitulated in the various revisions of the transmathematical structures developed to date.

The transreal derivative is and, in future, the transreal integral will be, supersets of their real counterparts. They differ from their real counterparts only in being more powerful: they give solutions at singularities where real analysis fails. Hence software that implements transreal analysis is more competent than software that implements real analysis.

However, both kinds of analysis and software are partial. There are occasions when both a real limit and a transreal limit fail to exist, say where the function oscillates, unboundedly, toward both positive and negative infinity. In these cases a solution can be had mathematically by operating on solution sets. Where the limit, derivative, integral, or whatever does not exist the solution is the empty set. In general it is impractical for a computer to operate on arbitrary sets but it may be feasible simply to return a flag to say that the limit, etc. does not exist.

It is already known that the methods just developed are sufficient to extend Newtonian Physics to a Trans-Newtonian Physics that operates at singularities. We hope the present series of paper will build confidence in transmathematics to the point where such results are accepted for publication.

4 Conclusion

In this paper and its companion [1], we extend real analysis to transreal analysis which allows division by zero. We do this by adding the usual topology of measure theory and integration theory to the transreal numbers and then use this topology to extend continuity, limits, derivatives and integrals so that they hold over functions of transreal numbers. This gives us a transmathematics which operates at mathematical singularities.

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