Multiplicative Toeplitz matrices and the Riemann zeta function

Book or Report Section

Accepted Version

Hilberdink, T. (2015) Multiplicative Toeplitz matrices and the Riemann zeta function. In: Bringmann, K., Bugeaud, Y., Hilberdink, T. and Sander, J. (eds.) Four faces of number theory. European Mathematical Society, Zurich, pp. 77-122. ISBN 9783037191422 doi: https://doi.org/10.4171/142 Available at http://centaur.reading.ac.uk/50981/

To link to this article DOI: http://dx.doi.org/10.4171/142

Publisher: European Mathematical Society

Chapter 3

Multiplicative Toeplitz matrices and the Riemann zeta function

Titus Hilberdink

Contents

Introduction

In this short course, we aim to highlight connections between a certain class of matrices and Dirichlet series, in particular the Riemann zeta function. The matrices we study are of the form

$$\begin{pmatrix} f(1) & f(\frac{1}{2}) & f(\frac{1}{3}) & f(\frac{1}{4}) & \cdots \\ f(2) & f(1) & f(\frac{2}{3}) & f(\frac{1}{2}) & \cdots \\ f(3) & f(\frac{3}{2}) & f(1) & f(\frac{3}{4}) & \cdots \\ f(4) & f(2) & f(\frac{4}{3}) & f(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{$*$}$$

i.e., with entries $a_{ij} = f(i/j)$ for some function $f\colon \mathbb{Q}^+ \to \mathbb{C}$. They are a multiplicative version of Toeplitz matrices which have entries of the form $a_{ij} = a_{i-j}$. For this reason we call them *Multiplicative Toeplitz Matrices*.

Toeplitz matrices (and operators) have been studied in great detail by many authors. They are most naturally studied by associating with them a function (or 'symbol') whose Fourier coefficients make up the matrix. With $a_{ij} = a_{i-j}$, this 'symbol' is

$$a(t) = \sum_{n=-\infty}^{\infty} a_n t^n. \qquad t \in \mathbb{T}$$

Then properties of the matrix (or rather the operator induced by the matrix) imply properties of the symbol and vice versa. For example, the boundedness of the operator is essentially related to the boundedness of the symbol, while invertibility of the operator is closely related to $a(t)$ not vanishing on the unit circle.

For matrices of the form ($*$) we associate, by analogy, the (formal) series

$$\sum_{q\in\mathbb{Q}^+} f(q)q^{it},$$

where q ranges over the positive rationals. Note, in particular, that if f is supported on the natural numbers, this becomes the Dirichlet series

$$\sum_{n\in\mathbb{N}} f(n)n^{it}.$$

In the special case where $f(n) = n^{-\alpha}$, the symbol becomes $\zeta(\alpha - it)$. It is quite natural then to ask to what extent properties of these Multiplicative Toeplitz Matrices are related to properties of the associated symbol. Rather surprisingly perhaps, these type of matrices do not appear to have been studied much at all – at least not in this respect. Finite truncations of them have appeared on occasions, notably Redheffer's matrix [27], the determinant of which is related to the Riemann Hypothesis. Denoting by $A_n(f)$ the $n \times n$ matrix with entries $f(i/j)$ if $j|i$ and zero otherwise, it is easy to see that

$$A_n(f)A_n(g) = A_n(f * g), \qquad \text{where } f * g \text{ is Dirichlet convolution,}$$

since the ij^{th} entry on the left product is

$$\sum_{r=1}^{n} A_n(f)_{ir}A_n(g)_{rj} = \sum_{j|r|i} f\Big(\frac{i}{r}\Big)g\Big(\frac{r}{j}\Big) = \sum_{d|i/j} f\Big(\frac{i/i^{\text{TS3}}}{d}\Big)g(d)$$

if $j|i$ by putting $r = jd$, and zero otherwise. With 1 and μ denoting the constant 1 and the Möbius functions, respectively, it follows that $A_n(1)A_n(\mu) = I_n$ – the identity matrix. Note also that $\det A_n(1) = \det A_n(\mu) = 1$. Redheffer's matrix is

$$R_n = A_n(1) + \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = A_n(1) + E_n,$$

TS3 Is this intended to be i/j?

say, where the matrix E_n has only 1s on the topmost row from the 2nd column onwards. Then, with $M(n) = \sum_{r=1}^{n} \mu(r)$,

$$R_n A_n(\mu) = I_n + \begin{pmatrix} M(n)-1 & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} M(n) & * & \cdots & * \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

so that $\det R_n = M(n)$. The well-known connection between the Riemann Hypothesis (RH) and $M(n)$ therefore implies that RH holds if and only if $\det R_n = O(n^{\frac{1}{2}+\varepsilon})$ for every $\varepsilon > 0$. (See also [18] for estimates of the largest eigenvalue of R_n).

Briefly then, the course is designed as follows: In §1, we recall some basic aspects of the theory of Toeplitz operators, in particular their boundedness and invertibility. In §2, we study bounded multiplicative Toeplitz operators. This is partly based on some of Toeplitz's own work [32], [33] and recent results from [15] and [16], but we also present new results, mainly in §2. Thus Theorem 2.1 is new, generalising Theorem 1.1 of [16], which in turn is now contained in Corollary 2.2. Also Subsection 2.2 and parts of 2.3 are new.

Preliminaries and Notation

(a) The sequence spaces l^p $(1 \leq p < \infty)$ consist of sequences (a_n) for which $\sum_{n=1}^{\infty} |a_n|^p$ converges. They are Banach spaces with the norm

$$\|(a_n)\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}.$$

The space l^∞ is the space of all bounded sequences, equipped with the norm $\|(a_n)\|_\infty = \sup_{n\in\mathbb{N}} |a_n|$. We shall also use $l^p(\mathbb{Q}^+)$, which is the space of sequences a_q where q ranges over the positive rationals such that $\sum_q |a_q|^p < \infty$, with analogous norms and also for $p = \infty$.

l^2 and $l^2(\mathbb{Q}^+)$ are Hilbert spaces with the inner products

$$\langle a, b\rangle = \sum_{n=1}^{\infty} a_n \overline{b_n} \quad \text{and} \quad \langle a, b\rangle = \sum_{q\in\mathbb{Q}^+} a_q \overline{b_q},$$

respectively.

(b) Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ – the unit circle. We denote by $L^2(\mathbb{T})$ the space of square-integrable functions on $\mathbb{T}$. $L^2(\mathbb{T})$ is a Hilbert space with the inner product and corresponding norm given by

$$\langle f, g\rangle = \int_{\mathbb{T}} f\overline{g} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\theta, \qquad \|f\| = \sqrt{\int_{\mathbb{T}} |f|^2}.$$

The space $L^\infty(\mathbb{T})$ consists of the essentially bounded functions on $\mathbb{T}$ with norm $\|f\|_\infty$ denoting the essential supremum of f. (Strictly speaking, L^2 and L^∞ consist of *equivalence classes* of functions satisfying the appropriate conditions, with two functions belonging to the same class if they differ on a set of measure zero.)

Let $\chi_n(t) = t^n$ for $n \in \mathbb{Z}$. Then $(\chi_n)_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{T})$ and $L^2(\mathbb{T})$ is isometrically isomorphic to $l^2(\mathbb{Z})$ via the mapping $f \mapsto (f_n)_{n\in\mathbb{Z}}$, where f_n are the *Fourier coefficients* of f, i.e.,

$$f_n = \langle f, \chi_n\rangle = \int_{\mathbb{T}} f\overline{\chi_n}.$$

(c) A linear operator φ on a Banach space X is *bounded* if $\|\varphi x\| \le C\|x\|$ for all $x \in X$. In this case the *operator norm* of φ is defined to be

$$\|\varphi\| = \sup_{x\in X, x\neq 0} \frac{\|\varphi x\|}{\|x\|} = \sup_{\|x\|=1} \|\varphi x\|.$$

The algebra of bounded linear operators on X is denoted by $B(X)$.

(d) An infinite matrix $A = (a_{ij})$ induces a bounded operator on a Hilbert space H if there exists $\varphi \in B(H)$ such that

$$a_{ij} = \langle \varphi e_j, e_i\rangle,$$

where (e_i) is an orthonormal basis of H. Note that not every infinite matrix induces a bounded operator, and it may be difficult to tell when it does.

(e) For the later sections we require the usual $O, o, \sim, \ll$ notation. Given f, g defined on neighbourhods of ∞ with g eventually positive, we write $f(x) = o(g(x))$ (or simply $f = o(g)$) to mean $\lim_{x\to\infty} f(x)/g(x) = 0$, $f(x) = O(g(x))$ to mean $|f(x)| \le Ag(x)$ for some constant A and all x sufficiently large, and $f(x) \sim g(x)$ to mean $\lim_{x\to\infty} f(x)/g(x) = 1$.

The notation $f \ll g$ means the same as $f = O(g)$, while $f \lesssim g$ means $f(x) \le (1 + o(1))g(x)$.

1 Toeplitz matrices and operators – a brief overview

Toeplitz matrices are matrices of the form

$$T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.1}$$

i.e., $T = (t_{ij})$, where $t_{ij} = a_{i-j}$. They are characterised by being constant on diagonals.

For a Toeplitz matrix, the question of boundedness of T was solved by Toeplitz.

Theorem 1.1 (Toeplitz [32]). *The matrix T induces a bounded operator on l^2 if and only if there exists $a \in L^\infty(\mathbb{T})$ whose Fourier coefficients are a_n $(n \in \mathbb{Z})$. If this is the case, then $\|T\| = \|a\|_\infty$.*

We refer to the function a as the 'symbol' of the matrix T, and we write $T(a)$.

Sketch of Proof. For $a \in L^2(\mathbb{T})$, the multiplication operator

$$M(a)\colon L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T}),\ f \longmapsto af$$

is bounded if and only if $a \in L^\infty(\mathbb{T})$. If bounded, then $\|M(a)\| = \|a\|_\infty$. The matrix representation of $M(a)$ with respect to $(\chi_n)_{n\in\mathbb{Z}}$ is given by

$$\langle M(a)\chi_j, \chi_i\rangle = \langle a\chi_j, \chi_i\rangle = \int_{\mathbb{T}} a\chi_j\overline{\chi_i} = \int_{\mathbb{T}} a\overline{\chi_{i-j}} = a_{i-j},$$

i.e., by the so-called *Laurent matrix*

$$L(a) := \left(\begin{array}{ccc|cccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ \cdots & a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ \hline \cdots & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \cdots & a_3 & a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right). \tag{3.2}$$

The matrix for T is just the lower right quarter of $L(a)$. We can therefore think of T as the compression $PL(a)P$, where P is the projection of $l^2(\mathbb{Z})$ onto $l^2 = l^2(\mathbb{N})$. An easy argument shows that T is bounded if and only if $a \in L^\infty$, and then $\|T\| = \|L(a)\| = \|a\|_\infty$. □

Hardy space. Let $H^2(\mathbb{T})$ denote the subspace of $L^2(\mathbb{T})$ of functions f whose Fourier coefficients f_n vanish for $n < 0$. Let P be the orthogonal projection of L^2 onto H^2, i.e., $P(\sum_{n\in\mathbb{Z}} f_n\chi_n) = \sum_{n\geq 0} f_n\chi_n$. The operator $f \mapsto P(af)$ has matrix representation (3.1). For, with $j \geq 0$ (so that $\chi_j \in H^2(\mathbb{T})$),

$$\langle T(a)\chi_j, \chi_i\rangle = \int_{\mathbb{T}} P(a\chi_j)\overline{\chi_i} = \int_{\mathbb{T}} P\Big(\sum_{n\in\mathbb{Z}} a_n\chi_{n+j}\Big)\chi_{-i} = \sum_{n\geq 0} a_{n-j}\int_{\mathbb{T}}\chi_{n-i} = a_{i-j}$$

if $i \geq 0$, and zero otherwise. Hence, we can equivalently view $T(a)$ as the operator

$$T(a)\colon H^2(\mathbb{T}) \longrightarrow H^2(\mathbb{T}),\ f \longmapsto P(af).$$

1.1 $C(\mathbb{T})$, $W(\mathbb{T})$, and winding number Let $C(\mathbb{T})$ denote the space of continuous functions on $\mathbb{T}$. For $a \in C(\mathbb{T})$ such that $a(t) \neq 0$ for all $t \in \mathbb{T}$, we denote by wind$(a, 0)$ the *winding number* of a with respect to zero. More generally, wind(a, λ) =wind$(a - \lambda, 0)$ denotes the winding number with respect $\lambda \in \mathbb{C}$. For example, wind$(\chi_n, 0) = n$.

The *Wiener Algebra* is the set of absolutely convergent Fourier series:

$$W(\mathbb{T}) = \left\{ \sum_{-\infty}^{\infty} a_n \chi_n : \sum_{-\infty}^{\infty} |a_n| < \infty \right\}.$$

Some properties:

(i) $W(\mathbb{T})$ forms a Banach algebra under pointwise multiplication, with norm

$$\|a\|_W := \sum_{-\infty}^{\infty} |a_n|.$$

(ii) (Wiener's Theorem) If $a \in W$ and $a(t) \neq 0$ for all $t \in \mathbb{T}$, then $a^{-1} \in W$.

(iii) If $a \in W(\mathbb{T})$ has no zeros and wind$(a, 0) = 0$, then $a = e^b$ for some $b \in W(\mathbb{T})$.

We have

$$W(\mathbb{T}) \subset C(\mathbb{T}) \subset L^\infty(\mathbb{T}) \subset L^2(\mathbb{T}).$$

1.2 Invertibility and fredholmness Let A be a bounded operator on a Banach space X. TS2

(i) A is *invertible* if there exists a bounded operator B on X such that $AB = BA = I$. As such, B is the unique *inverse* of A, and we write $B = A^{-1}$. The *spectrum* of A is the set

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible in } X\}.$$

The *kernel* and *image* of A are defined by

$$\text{Ker } A = \{x \in X : Ax = 0\}, \qquad \text{Im } A = \{Ax : x \in X\}.$$

(ii) The operator A is *Fredholm* if ImA is a closed subspace of X and both Ker$A$ and $X/$ImA are finite-dimensional. As such, the *index* of A is defined to be

$$\text{Ind } A = \dim \text{Ker } A - \dim (X/\text{Im } A).$$

For example, $T(\chi_n)$ is Fredholm with Ind $T(\chi_n) = -n$.

Equivalently, A is Fredholm if it is invertible modulo compact operators; that is, there exists bounded operator B on X such that $AB - I$ and $BA - I$ are both compact.

The *essential spectrum* of A is the set

$$\sigma_{\text{ess}}(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not Fredholm in } X\}.$$

Clearly $\sigma_{\text{ess}}(A) \subset \sigma(A)$. Note that A invertible implies A is Fredholm of index zero. For Toeplitz operators, the converse actually holds (see [3], p. 12).

TS2 Please check the range of the following enumeration.

1.3 Hankel matrices These are matrices of the form

$$H = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ a_4 & a_5 & a_6 & a_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.3}$$

i.e., $H = (h_{ij})$, where $h_{ij} = a_{i+j-1}$. They are characterised by being constant on cross diagonals. The boundedness of H was solved by Nehari, and the compactness of H by Hartman.

Theorem 1.2 ([26], [12]). *The matrix H generates a bounded operator on l^2 if and only if there exists $b \in L^\infty(\mathbb{T})$ (with Fourier coefficients b_n) such that $b_n = a_n$ for $n \geq 1$. Furthermore, the operator H is compact if and only if $b \in C(\mathbb{T})$.*

We refer to the function a as the 'symbol' of the matrix H, and we write $H(a)$. Given a function a defined on $\mathbb{T}$, let $\tilde{a}$ be the function

$$\tilde{a}(t) = a(1/t) \quad (t \in \mathbb{T}).$$

Proposition 1.3. *For $a, b \in L^\infty(\mathbb{T})$,*

$$T(ab) = T(a)T(b) + H(a)H(\tilde{b})$$
$$H(ab) = H(a)T(\tilde{b}) + T(a)H(b).$$

Proof. The matrix $L(a)$ in (3.2) is of the form

$$L(a) = \left(\begin{array}{c|c} T(a) & H(\tilde{a}) \\ \hline H(a) & T(a) \end{array} \right).$$

Since $L(ab) = L(a)L(b)$, the result follows by multiplying the 2×2 matrices. $\square$

As a special case, we see that $T(abc) = T(a)T(b)T(c)$ for $a \in \overline{H^\infty}, b \in L^\infty, c \in H^\infty$. The space H^∞ is defined analogously to L^∞. ($T(a)$ is upper-triangular and $T(c)$ is lower-triangular.)

By Theorem 1.2, if $a, b \in C(\mathbb{T})$, then $H(a)H(\tilde{b})$ is compact, so that $T(ab) - T(a)T(b)$ is compact. In particular, if a has no zeros on $\mathbb{T}$, we can take $b = a^{-1} \in C(\mathbb{T})$. Then $T(ab) = T(1) = I$, so $T(a)$ is invertible modulo compact operators (i.e., Fredholm) with 'inverse' $T(a^{-1})$. This type of reasoning leads to:

Theorem 1.4 (Gohberg [7]). *Let $a \in C(\mathbb{T})$. Then $T(a)$ is Fredholm if and only if a has no zeros on $\mathbb{T}$, in which case*

$$\operatorname{Ind} T(a) = -\operatorname{wind}(a, 0).$$

Hence $T(a)$ is invertible if and only if a has no zeros on $\mathbb{T}$ and $\mathrm{wind}(a,0) = 0$. *Equivalently, since $T(\lambda - a) = \lambda I - T(a)$ for $\lambda \in \mathbb{C}$, we have*

$$\begin{aligned}\sigma_{\mathrm{ess}}(T(a)) &= a(\mathbb{T}),\\ \sigma(T(a)) &= a(\mathbb{T}) \cup \{\lambda \in \mathbb{C} \setminus a(\mathbb{T}) : \mathrm{wind}(a,\lambda) \neq 0\}.\end{aligned}$$

Sketch of Proof. We have seen that $a \neq 0$ on $\mathbb{T}$ implies $T(a)$ is Fredholm. In this case, let $\mathrm{wind}(a,0) = k$. Then a is homotopic to χ_k, and (since the index varies continuously)

$$\mathrm{Ind}\, T(a) = \mathrm{Ind}\, T(\chi_k) = -k = -\mathrm{wind}\,(a,0).$$

For the converse, suppose $T(a)$ is Fredholm with index k, but a has zeros on $\mathbb{T}$. Then a can be slightly perturbed to produce two functions $b, c \in W(\mathbb{T})$ without zeros such that $\|a - b\|_\infty$ and $\|a - c\|_\infty$ are as small as we please, but $\mathrm{wind}(b,0)$ and $\mathrm{wind}(c,0)$ differ by one. As the index is stable under small perturbations, $T(b)$ and $T(c)$ are Fredholm with equal index. But $\mathrm{Ind}\, T(b) = -\mathrm{wind}(b,0)$ and $\mathrm{Ind}\, T(c) = -\mathrm{wind}(c,0)$ (by above), so $\mathrm{wind}(b,0) - \mathrm{wind}(c,0) = 0$ — a contradiction. □

1.4 Wiener–Hopf factorization Since $W(\mathbb{T}) \subset C(\mathbb{T})$, Theorem 1.4 applies to $W(\mathbb{T})$. However, for Wiener symbols we can obtain a quite explicit form for the inverse when it exists. This is because Wiener functions can be factorized.

Denote by W_+ and W_- the subspaces of W consisting of functions

$$\sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad \sum_{n=0}^{\infty} a_n t^{-n} \qquad t \in \mathbb{T}$$

respectively, where $\sum |a_n| < \infty$.

Theorem 1.5 (Wiener–Hopf factorization). *Let $a \in W(\mathbb{T})$ such that a has no zeros, and let* $\mathrm{wind}(a,0) = k$. *Then there exist $a_- \in W_-$ and $a_+ \in W_+$ such that*

$$a = \chi_k a_- a_+.$$

Proof. We have $\mathrm{wind}(a\chi_{-k},0) = 0$. So $a\chi_{-k} = e^b$ for some $b \in W$. But $b = b_- + b_+$, where $b_- \in W_-$ and $b_+ \in W_+$. Hence, writing $a_- = e^{b_-}$ and $a_+ = e^{b_+}$ gives

$$a\chi_{-k} = e^{b_-}e^{b_+} = a_- a_+. \qquad \square$$

Theorem 1.6 (Krein [21]). *Let $a \in W(\mathbb{T})$. Then $T(a)$ is Fredholm if and only if a has no zeros on $\mathbb{T}$, in which case*

$$\mathrm{Ind}\, T(a) = -\mathrm{wind}(a,0).$$

In particular, $T(a)$ is invertible if and only if a has no zeros on $\mathbb{T}$ and $\mathrm{wind}(a,0) = 0$. *In this case*

$$T(a)^{-1} = T(a_+^{-1})T(a_-^{-1}),$$

where $a = a_+a_-$ is the Wiener–Hopf factorization of a.

Proof of second part. Note that if $a \in W_-$, then $H(a) = 0$, while if $a \in W_+$, then $H(\tilde{a}) = 0$. Suppose a has no zeros on $\mathbb{T}$ and $\mathrm{wind}(a, 0) = 0$. Then a factorizes as $a = a_- a_+$ with $a_\pm \in W_\pm$. Applying Proposition 1.3 with a_- and a_+ in turn gives

$$T(a_-^{-1})T(a_-) = T(a_-^{-1}a_-) = I = T(a_-a_-^{-1}) = T(a_-)T(a_-^{-1}),$$
$$T(a_+^{-1})T(a_+) = T(a_+^{-1}a_+) = I = T(a_+a_+^{-1}) = T(a_+)T(a_+^{-1}),$$

so that $T(a_\pm)$ are invertible with $T(a_\pm)^{-1} = T(a_\pm^{-1})$. But also $T(a) = T(a_-a_+) = T(a_-)T(a_+)$ (by Proposition 1.3). Hence $T(a)^{-1} = T(a_+)^{-1}T(a_-)^{-1} = T(a_+^{-1})T(a_-^{-1})$. □

2 Bounded multiplicative Toeplitz matrices and Operators

2.1 Criterion for boundedness on l^2 Now we consider the linear operators induced by matrices of the form (∗), regarding them as operators on sequence spaces, in particular l^2.

For a function $f : \mathbb{Q}^+ \to \mathbb{C}$ on the positive rationals, we define

$$\sum_{q \in \mathbb{Q}^+} f(q) = \lim_{N\to\infty} \sum_{\substack{m,n \le N \\ (m,n)=1}} f\Big(\frac{m}{n}\Big), \quad \text{whenever this limit exists.}$$

We shall sometimes abbreviate the left-hand sum by $\sum_q f(q)$. We say that $f \in l^1(\mathbb{Q}^+)$ if

$$\sum_{q \in \mathbb{Q}^+} |f(q)|$$

converges. In this case, the function

$$F(t) = \sum_{q \in \mathbb{Q}^+} f(q) q^{it} \qquad t \in \mathbb{R}$$

exists and is uniformly continuous on $\mathbb{R}$. Note that, for $\lambda > 0$, TS3

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t)\lambda^{-it}\, dt = \begin{cases} f(q), & \text{if } \lambda = q \in \mathbb{Q}^+, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Theorem 2.1. *Let $f \in l^1(\mathbb{Q}^+)$ and let φ_f denote the mapping $(a_n) \mapsto (b_n)$ where*

$$b_n = \sum_{m=1}^{\infty} f\Big(\frac{n}{m}\Big) a_m.$$

TS3 Please check: due to reasons of a coherent design of all papers, we had to add the commas here as well as in similar formulas.

Then φ_f is bounded on l^2 with operator norm

$$\|\varphi_f\| = \sup_{t\in\mathbb{R}}\Big|\sum_{q\in\mathbb{Q}^+} f(q)q^{it}\Big| = \|F\|_\infty.$$

Proof. We shall first prove that φ_f is bounded on l^2, showing $\|\varphi_f\| \le \|F\|_\infty$ in the process, and then show that $\|F\|_\infty$ is also a lower bound.

For $q \in \mathbb{Q}^+$ and $N \in \mathbb{N}$ let

$$b_q^{(N)} = \sum_{m=1}^{N} f\Big(\frac{q}{m}\Big)a_m \quad \text{and} \quad b_q = \sum_{m=1}^{\infty} f\Big(\frac{q}{m}\Big)a_m.$$

Note that $b_q^{(N)} \to b_q$ as $N \to \infty$ for every $q \in \mathbb{Q}^+$, whenever a_n is bounded. We have the following formulae:

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} F(t)\Big|\sum_{n=1}^{N} a_n n^{it}\Big|^2 dt = \sum_{n=1}^{N}\overline{a_n}b_n^{(N)} \tag{3.5}$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big|F(t)\sum_{n=1}^{N} a_n n^{it}\Big|^2 dt = \sum_{q\in\mathbb{Q}^+}|b_q^{(N)}|^2. \tag{3.6}$$

(These hold for a_n bounded.) To prove these expand the integrand in a Dirichlet series. For the first formula we have

$$\frac{1}{2T}\int_{-T}^{T} F(t)\Big|\sum_{n=1}^{N} a_n n^{it}\Big|^2 dt = \sum_{m,n\le N} a_m\overline{a_n}\frac{1}{2T}\int_{-T}^{T} F(t)\Big(\frac{n}{m}\Big)^{-it} dt$$

$$\longrightarrow \sum_{m,n\le N} a_m\overline{a_n} f\Big(\frac{n}{m}\Big) = \sum_{n=1}^{N}\overline{a_n}b_n^{(N)}$$

as $T \to \infty$. For the second formula, note first that

$$F(t)\sum_{n=1}^{N} a_n n^{it} = \sum_{q\in\mathbb{Q}^+,n\le N} f(q)a_n(qn)^{it}$$

$$= \sum_{r\in\mathbb{Q}^+}\Big(\sum_{n\le N} f\Big(\frac{r}{n}\Big)a_n\Big)r^{it} = \sum_{r\in\mathbb{Q}^+} b_r^{(N)}r^{it},$$

the series converging absolutely. Thus

$$\frac{1}{2T}\int_{-T}^{T}\Big|F(t)\sum_{n=1}^{N} a_n n^{it}\Big|^2 dt = \sum_{q_1,q_2\in\mathbb{Q}^+} b_{q_1}^{(N)}\overline{b_{q_2}^{(N)}}\frac{1}{2T}\int_{-T}^{T}\Big(\frac{q_1}{q_2}\Big)^{it} dt \longrightarrow \sum_{q\in\mathbb{Q}^+}|b_q^{(N)}|^2$$

as $T \to \infty$.

Since $|F(t)| \le \|F\|_\infty$, we have

$$\sum_{q\in\mathbb{Q}^+} |b_q^{(N)}|^2 \le \|F\|_\infty^2 \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\sum_{n=1}^{N} a_n n^{it}\right|^2 dt = \|F\|_\infty^2 \sum_{n=1}^{N} |a_n|^2.$$

Thus if $a = (a_n) \in l^2$, we have $\sum_{n=1}^{\infty} |b_n^{(N)}|^2 \le \|F\|_\infty^2 \|a\|^2$ for every N. Letting $N \to \infty$ shows that $(b_n) \in l^2$ too (indeed $(b_q) \in l^2(\mathbb{Q}^+)$), and so φ_f is bounded on l^2, with

$$\|\varphi_f\| \le \|F\|_\infty.$$

Now we need a lower bound. By Cauchy–Schwarz,

$$\left|\sum_{n=1}^{\infty} \overline{a_n} b_n\right|^2 \le \sum_{n=1}^{\infty} |a_n|^2 \cdot \sum_{n=1}^{\infty} |b_n|^2.$$

Thus $\|\varphi_f\| \ge |\sum_{n=1}^{\infty} \overline{a_n} b_n|$ for every $a = (a_n) \in l^2$ with $\|a\| = 1$. Choose a_n as follows: let $N \in \mathbb{N}$ (to be determined later) and put

$$a_n = \frac{n^{-it}}{\sqrt{d(N)}} \qquad \text{for } n|N, \text{ and zero otherwise.}$$

Here $d(N)$ is the number of divisors of N. Thus $(a_n) \in l^2$ and $\|a\| = 1$. With this choice,

$$b_n = \frac{1}{\sqrt{d(N)}} \sum_{m|N} f\Big(\frac{n}{m}\Big) m^{-it} \qquad (= b_n^{(N)})$$

and so

$$\begin{aligned}\sum_{n=1}^{\infty} \overline{a_n} b_n &= \frac{1}{d(N)} \sum_{n|N} n^{it} \sum_{m|N} f\Big(\frac{n}{m}\Big) m^{-it} = \frac{1}{d(N)} \sum_{m,n|N} f\Big(\frac{n}{m}\Big)\Big(\frac{n}{m}\Big)^{it} \\ &= \frac{1}{d(N)} \sum_{q\in\mathbb{Q}^+} f(q) q^{it} S_q(N),\end{aligned}$$

where

$$S_q(N) = \sum_{\substack{m,n|N \\ \frac{n}{m} = q}} 1.$$

Put $q = \frac{k}{l}$, where $(k,l) = 1$. Then $\frac{n}{m} = \frac{k}{l}$ if and only if $ln = km$. Since $(k,l) = 1$, this forces $k|n$ and $l|m$. So, for a contribution to the sum, we need $k, l|N$, i.e., $kl|N$. Suppose therefore that $kl|N$. Then

$$\begin{aligned}S_q(N) &= \sum_{\substack{m,n|N \\ ln = km}} 1 = \sum_{rk, rl|N} 1 \qquad m = rl, n = rk \text{ TS4 with } r \in \mathbb{N} \\ &= \sum_{r|\frac{N}{kl}} 1 = d\Big(\frac{N}{kl}\Big).\end{aligned}$$

TS4 Please check if the replacement of $m = rk, n = rl$ with $m = rl, n = rk$ is correct.

Writing $|q| = kl$ whenever $q = \frac{k}{l}$ in its lowest terms, gives

$$\sum_{n=1}^{\infty} \overline{a_n} b_n = \sum_{\substack{q \in \mathbb{Q}^+ \\ |q| | N}} f(q) q^{it} \frac{d(N/|q|)}{d(N)}. \tag{3.7}$$

The idea is now to choose N in such a way that it has all 'small' divisors while $\frac{d(N/|q|)}{d(N)}$ is close to 1 for all such small divisors $|q|$. Take N of the form

$$N = \prod_{p \le P} p^{\alpha_p}, \qquad \text{where } \alpha_p = \left[\tfrac{\log P}{\log p}\right].$$

Thus every natural number up to P is a divisor of N. Every q such that $|q| | N$ is of the form $|q| = \prod_{p \le P} p^{\beta_p}$ $(0 \le \beta_p \le \alpha_p)$, so that

$$\frac{d(N/|q|)}{d(N)} = \prod_{p \le P} \Big(1 - \frac{\beta_p}{\alpha_p + 1}\Big).$$

If we take $|q| \le \sqrt{\log P}$, then $p^{\beta_p} \le \sqrt{\log P}$ for every prime divisor p of $|q|$. Hence, for such p, $\beta_p \le \frac{\log\log P}{2\log p}$ and $\beta_p = 0$ if $p > \sqrt{\log P}$. Thus

$$\begin{aligned} \frac{d(N/|q|)}{d(N)} &= \prod_{p \le \sqrt{\log P}} \Big(1 - \frac{\beta_p}{\alpha_p + 1}\Big) \ge \prod_{p \le \sqrt{\log P}} \Big(1 - \frac{\log\log P}{2\log P}\Big) \\ &= \Big(1 - \frac{\log\log P}{2\log P}\Big)^{\pi(\sqrt{\log P})}, \end{aligned}$$

where $\pi(x)$ is the number of primes up to x. Since $\pi(x) = O(\frac{x}{\log x})$, it follows that for all P sufficiently large, the RHS above is at least

$$1 - \frac{A}{\sqrt{\log P}}$$

for some constant A.

Let $\varepsilon > 0$. Then there exists n_0 such that $\sum_{|q| > n_0} |f(q)| < \varepsilon$. Choose $P \ge e^{n_0^2}$ so that $\sqrt{\log P} \ge n_0$. Then the modulus of the sum in (3.7) can be made as close to $\|F\|_\infty$ as we please by increasing P, for it is at least

$$\begin{aligned} &\Big|\sum_{|q| \le \sqrt{\log P}} f(q) q^{it}\Big| - \frac{A}{\sqrt{\log P}} \sum_{|q| \le \sqrt{\log P}} |f(q)| - \sum_{|q| > \sqrt{\log P}} |f(q)| \\ &\ge \Big|\sum_{q \in \mathbb{Q}^+} f(q) q^{it}\Big| - \frac{A}{\sqrt{\log P}} \sum_{q \in \mathbb{Q}^+} |f(q)| - 2 \sum_{|q| > \sqrt{\log P}} |f(q)| \\ &> |F(t)| - \frac{A'}{\sqrt{\log P}} - 2\varepsilon, \end{aligned}$$

where ε can be made as small as we please by making P large. Since this holds for any t, we can choose t to make $F(t)$ as close as we like to $\|F\|_\infty$. Hence $\|\varphi_f\| \geq \|F\|_\infty$ and so we must have equality. $\square$

In the special case where $f \geq 0$, the supremum of $|F(t)|$ is attained when $t = 0$, in which case $\|F\|_\infty = F(0) = \|f\|_{1,\mathbb{Q}^+}$. Thus:

Corollary 2.2. *Let $f : \mathbb{Q}^+ \to \mathbb{C}$ such that $f \geq 0$. Then φ_f is bounded on l^2 if and only if $f \in l^1(\mathbb{Q}^+)$, in which case $\|\varphi_f\| = \|f\|_{1,\mathbb{Q}^+}$.*

Examples. Take $f(n) = n^{-\alpha}$ for $n \in \mathbb{N}$, and zero otherwise. Then $F(t) = \zeta(\alpha - it)$. We shall denote φ_f by φ_α in this case. Applying Corollary 2.2, we see that φ_α is bounded on l^2 if and only if $\alpha > 1$, and the norm is $\zeta(\alpha)$.

2.2 Viewing φ_f as an operator on function spaces; the Besicovitch space

We can view φ_f as an operator on functions rather than sequences. For this we need to construct the appropriate spaces.

Let A denote the set of trigonometric polynomials; i.e., the elements of A are all finite sums of the form

$$\sum_{k=1}^{n} a_k e^{i\lambda_k t},$$

where $a_k \in \mathbb{C}$ and $\lambda_k \in \mathbb{R}$. The space A has an inner product and a norm given by

$$\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f\overline{g} \qquad \text{and} \qquad \|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f|^2}.$$

Now let B^2 (Besicovitch space) denote the closure of A with respect to this inner product; i.e., $f \in B^2$ if $\|f - f_n\| \to 0$ as $n \to \infty$ for some $f_n \in A$. We turn B^2 into a Hilbert space by identifying f and g whenever $\|f - g\| = 0$. (See [2], Chapter II.)

Now write $\chi_\lambda(t) = \lambda^{it}$ ($\lambda > 0, t \in \mathbb{R}$) and let $\hat{f}(\lambda)$ denote the Fourier coefficient

$$\hat{f}(\lambda) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f\overline{\chi_\lambda} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\lambda^{-it}\, dt \qquad \text{where it exists.}$$

Denote by $\mathcal{F}$ the space of locally integrable $f : \mathbb{R} \to \mathbb{C}$ such that $\hat{f}(\lambda)$ exists for all $\lambda > 0$.

(a) **Fourier coefficients and series** *For $f \in B^2$, the Fourier coefficients $\hat{f}(\lambda)$ exist and $\hat{f}(\lambda)$ is non-zero on at most a countable set, say $\{\lambda_n\}_{n\in\mathbb{N}}$. The function f has the (formal) Fourier series $\sum_{n\geq 1} \hat{f}(\lambda_n)e^{i\lambda_n t}$.*

(b) **Uniqueness** *$f, g \in B^2$ have the same Fourier series if and only if $\|f - g\| = 0$.*

(c) **Parseval** *For* $f \in B^2$,

$$\|f\| = \lim_{T\to\infty} \sqrt{\frac{1}{2T}\int_{-T}^{T} |f|^2} = \sqrt{\sum_{\lambda} |\hat{f}(\lambda)|^2}, \tag{3.8}$$

and, more generally, for $f, g \in B^2$,

$$\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f\overline{g} = \sum_{\lambda} \hat{f}(\lambda)\overline{\hat{g}(\lambda)}.$$

(d) **Riesz–Fischer Theorem** *Given* $\lambda_n \in \mathbb{R}$ *and* $a_n \in l^2$, *there exists* $f \in B^2$ *such that* $f(t) \sim \sum_{n\geq 1} a_n e^{i\lambda_n t}$.

(e) **Criterion for membership in** B^2: *With* $\mathcal{F}$ *as before, if* $f \in \mathcal{F}$ *and Parseval's identity* (3.8) *holds, then* $f \in B^2$.

Indeed, the set of λ for which $\hat{f}(\lambda) \neq 0$ is necessarily countable and we may write this as $\{\lambda_1, \lambda_2, \ldots\}$ with $\sum_{k=1}^{\infty} |\hat{f}(\lambda_k)|^2 = \|f\|^2$TS5. Let $f_n(t) = \sum_{k\leq n} \hat{f}(\lambda_k)e^{i\lambda_k t}$. Then

$$\|f - f_n\|^2 = \|f\|^2 - \|f_n\|^2 = \sum_{k>n} |\hat{f}(\lambda)|^2 \to 0 \quad \text{as } n \to \infty.$$

The analogues of the Hardy and Wiener spaces: $B^2_{\mathbb{Q}^+}$, $B^2_{\mathbb{N}}$, $W_{\mathbb{Q}^+}$, $W_{\mathbb{N}}$.

(a) Let $B^2_{\mathbb{Q}^+}$ denote the subspace of B^2 of functions with exponents $\lambda = \log q$ for some $q \in \mathbb{Q}^+$. Alternatively, start with the subset of A consisting of finite trigonometric polynomials of the form $\sum a_q \chi_q$, where q ranges over a finite subset of $\mathbb{Q}^+$, and take its closure.

(b) Let $B^2_{\mathbb{N}}$ denote the subspace of B^2 of functions with exponents $\lambda = \log n$ for some $n \in \mathbb{N}$. This is the analogue of the Hardy space.

(c) Let $W_{\mathbb{Q}^+}$ denote the set of all absolutely convergent Fourier series in $B^2_{\mathbb{Q}^+}$; i.e.

$$W_{\mathbb{Q}^+} = \Big\{ \sum_{q\in\mathbb{Q}^+} c(q)\chi_q : \sum_{q\in\mathbb{Q}^+} |c(q)| < \infty \Big\}.$$

This is the analogue of the Wiener algebra. As we saw earlier, ifTS6

$$f = \sum_{q\in\mathbb{Q}^+} c(q)\chi_q \in W_{\mathbb{Q}^+},$$

then $\hat{f}(q) = c(q)$. With pointwise addition and multiplication, $W_{\mathbb{Q}^+}$ becomes an algebra. Further, $W_{\mathbb{Q}^+}$ becomes a Banach algebra with respect to the norm

$$\|f\|_W = \sum_{q\in\mathbb{Q}^+} |\hat{f}(q)|.$$

Analogously, let $W_{\mathbb{N}}$ denote the set of absolutely convergent series $\sum_{n=1}^{\infty} a_n n^{it}$.

TS5 Please check if the replacement of $\|f\|^2_B$ with $\|f\|^2$ is correct.

TS6 The following formula had to be detached from the text because of typesetting reasons. Please check.

$B^2_{\mathbb{Q}+}$ has $(\chi_q)_{q\in\mathbb{Q}^+}$ as an orthonormal basis while $B^2_{\mathbb{N}}$ has $(\chi_n)_{n\in\mathbb{N}}$ as an orthonormal basis. The spaces $B^2_{\mathbb{Q}+}$ and $B^2_{\mathbb{N}}$ are isometrically isomorphic to $l^2(\mathbb{Q}^+)$ and l^2, respectively, via the mappings

$$f \overset{\tau}{\longmapsto} \{\hat{f}(q)\}_{q\in\mathbb{Q}^+} \text{ and } f \overset{\tau}{\longmapsto} \{\hat{f}(n)\}_{n\geq 1}.$$

The operator $M(f)$. Let P be the projection from $B^2_{\mathbb{Q}+}$ to $B^2_{\mathbb{N}}$; that is,

$$P\Big(\sum_{q\in\mathbb{Q}^+} c(q)q^{it}\Big) = \sum_{n\in\mathbb{N}} c(n)n^{it}.$$

For $f \in W_{\mathbb{Q}+}$, we define the operator $M(f)$: $B^2_{\mathbb{N}} \to B^2_{\mathbb{N}}$ by $g \mapsto P(fg)$.

The matrix representation of $M(f)$ w.r.t. $\{\chi_n\}_{n\in\mathbb{N}}$ is the multiplicative Toeplitz matrix $(\hat{f}(i/j))_{i,j\geq 1}$. Indeed, if $f = \sum_q \hat{f}(q)\chi_q$, then

$$\begin{aligned} P(f\chi_j) &= P\Big(\sum_q \hat{f}(q)\chi_q\chi_j\Big) = P\Big(\sum_q \hat{f}(q)\chi_{qj}\Big) \\ &= P\Big(\sum_q \hat{f}(q/j)\chi_q\Big) = \sum_{n=1}^{\infty} \hat{f}(n/j)\chi_n. \end{aligned}$$

Hence

$$\langle M(f)\chi_j, \chi_i\rangle = \langle P(f\chi_j), \chi_i\rangle = \sum_{n=1}^{\infty} \hat{f}(n/j)\langle \chi_n, \chi_i\rangle = \hat{f}\Big(\frac{i}{j}\Big).$$

In terms of the operator φ,

$$M(f) = \tau^{-1}\varphi_{\hat{f}}\tau.$$

Interlude on $\zeta(s)$. Since this work concerns connections to Dirichlet series and the Riemann zeta function in particular, we recall a few relevant facts regarding $\zeta(s)$.

The *Riemann zeta function* is defined for $\Re s > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In this half-plane $\zeta(s)$ is holomorphic and there is an analytic continuation to the whole of $\mathbb{C}$ except for a simple pole at $s = 1$ with residue 1. Furthermore, $\zeta(s)$ satisfies the *functional equation*

$$\zeta(s) = \chi(s)\zeta(1-s)$$

where

$$\chi(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin\Big(\frac{\pi s}{2}\Big) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})}.$$

The connection to prime numbers comes from Euler's remarkable product formula

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \qquad \Re s > 1$$

The order of $\zeta(s)$. Considering $\zeta(\sigma + it)$ as a function of the real variable t for fixed (but arbitrary) σ, it is known that for $|t|$ large

$$\zeta_\sigma(t) \stackrel{\text{def}}{=} \zeta(\sigma + it) = O(|t|^A) \qquad \text{for some } A.$$

The infimum of such A is the *order* of ζ and is called the *Lindelöf* function; i.e.,

$$\mu(\sigma) = \inf\{A : \zeta(\sigma + it) = O(|t|^A)\}.$$

From the general theory of functions it is known that the Lindelöf function is convex and decreasing. Since ζ_σ is bounded for $\sigma > 1$, but $\zeta_\sigma \not\to 0$, it follows that $\mu(\sigma) = 0$ for $\sigma \geq 1$. By the functional equation and continuity of μ we then have

$$\mu(\sigma) = \begin{cases} 0, & \text{if } \sigma \geq 1, \\ \frac{1}{2} - \sigma, & \text{if } \sigma \leq 0. \end{cases} \qquad (\diamond)$$

For $0 < \sigma < 1$, the value of $\mu(\sigma)$ is not known, but it is conjectured that the two line segments in ($\diamond$) above extend to $\sigma = \frac{1}{2}$. This is the *Lindelöf Hypothesis*. It is equivalent to the statement that

$$\zeta(\tfrac{1}{2} + it) = O(t^\varepsilon) \qquad \text{for every } \varepsilon > 0.$$

The Lindelöf Hypothesis is a major open problem TS7 and is a consequence of the Riemann Hypothesis, which states that $\zeta(s) \neq 0$ for $\sigma > \frac{1}{2}$.

Upper and lower bounds for ζ_σ. Let

$$Z_\sigma(T) = \max_{1 \leq |t| \leq T} |\zeta(\sigma + it)|.$$

(The restriction $|t| \geq 1$ is only added to avoid problems for the case $\sigma = 1$.) The following results hold for large T.

(a) $Z_\sigma(T) \to \zeta(\sigma)$ for $\sigma > 1$.

(b) For $\sigma = 1$, unconditionally it is known that $Z_1(T) = O((\log T)^{\frac{2}{3}})$, while on RH

$$Z_1(T) \lesssim 2e^\gamma \log\log T.$$

On the other hand, Granville and Soundararajan [11] proved that

$$Z_1(T) \geq e^\gamma(\log\log T + \log\log\log T - \log\log\log\log T),$$

for some arbitrarily large T. They further conjectured that it equals the above with an $O(1)$ term instead of the quadruple log-term.

TS7 Note from publisher: Paper by Albeverio and Cheng (proof of the Lindelf Hypothesis)?

(c) For $\frac{1}{2} < \sigma < 1$, unconditionally one has $Z_\sigma(T) = O(T^a)$ for various $a > 0$, while on RH

$$\log Z_\sigma(T) \leq A \frac{(\log T)^{2-2\sigma}}{(1-\sigma)\log\log T}$$

for some constant A. Montgomery [25] showed that

$$\log Z_\sigma(T) \geq \frac{\sqrt{\sigma - 1/2}}{20} \frac{(\log T)^{1-\sigma}}{(\log\log T)^\sigma}$$

and, using a heuristic argument, he conjectured that this is (apart from the constant) the correct order of $\log Z_\sigma(T)$. In a recent paper (see [22]), Lamzouri suggests that

$$\log Z_\sigma(T) \sim C(\sigma) \frac{(\log T)^{1-\sigma}}{(\log\log T)^\sigma}$$

with $C(\sigma) = G_1(\sigma)^\sigma \sigma^{-2\sigma}(1-\sigma)^{\sigma-1}$, where

$$G_1(x) = \int_0^\infty \log\left(\sum_{n=0}^\infty \frac{(u/2)^{2n}}{(n!)^2}\right) \frac{du}{u^{1+1/x}}.$$

(d) For $\sigma = \frac{1}{2}$ unconditionally one has $Z_{\frac{1}{2}}(T) = O(T^{\frac{32}{205}}(\log T)^c) = O(T^{0.156..})$ (see [17]), while on RH

$$\log Z_{\frac{1}{2}}(T) \leq A \frac{\log T}{\log\log T}$$

for some constant A. On the other hand, it is known that

$$\log Z_{\frac{1}{2}}(T) \geq c\sqrt{\frac{\log T}{\log\log T}}$$

(see [1], [29]). Using a heuristic argument, Farmer et al. ([6]) conjectured that

$$\log Z_{\frac{1}{2}}(T) \sim \sqrt{\frac{1}{2}\log T \log\log T}.$$

(e) For $\sigma < \frac{1}{2}$, the functional equation for $\zeta(s)$ reduces the problem to the case $\sigma > \frac{1}{2}$. So, for example,

$$Z_\sigma(T) \sim \zeta(1-\sigma)\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma} \qquad \text{for } \sigma < 0.$$

Mean values. For $\sigma \in (\frac{1}{2}, 1)$, the mean-value formula

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(\sigma - it)|^2 \, dt = \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}} = \zeta(2\sigma),$$

is well-known (see [31]). Furthermore,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(\sigma - it)|^4 \, dt = \sum_{n=1}^{\infty} \frac{d(n)^2}{n^{2\sigma}} = \frac{\zeta(2\sigma)^4}{\zeta(4\sigma)}.$$

For higher powers, however, present knowledge is very patchy. It is expected that the above formulas extend to all higher moments, i.e.,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(\sigma - it)|^{2k} \, dt = \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2\sigma}}.$$

This is equivalent to the Lindelöf Hypothesis.

Examples.

(a) The mean values for ζ_σ and ζ_σ^2 imply that $\zeta_\sigma, \zeta_\sigma^2 \in B^2_{\mathbb{N}}$ for $\sigma \in (\frac{1}{2}, 1)$. Note that this also implies $|\zeta_\sigma|^2 \in B^2_{\mathbb{Q}^+}$.

For higher powers, however, only partial results are known. For example, it is known that $\zeta_\sigma^k \in B^2_{\mathbb{N}}$ if $\sigma \in (1 - \frac{1}{k}, 1)$. Slightly better bounds are available, especially for particular values of k, but it is expected that much more holds, namely: $\zeta_\sigma^k \in B^2_{\mathbb{N}}$ for every $k \in \mathbb{N}$ and all $\sigma \in (\frac{1}{2}, 1)$. This is (equivalent to) the Lindelöf Hypothesis.

(b) Let $g(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ and $h(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ be two Dirichlet series which converge absolutely for $\Re s > \sigma_0$ and $\Re s > \sigma_1$, respectively. Let $\alpha > \sigma_0$ and $\beta > \sigma_1$ and put $f(t) = g(\alpha - it)h(\beta + it)$. Then $f \in W_{\mathbb{Q}^+}$ with

$$\hat{f}(q) = \frac{1}{m^\alpha n^\beta} \sum_{d=1}^{\infty} \frac{a_{md} b_{nd}}{d^{\alpha+\beta}} \qquad \text{for } q = \tfrac{m}{n} \text{ with } (m,n) = 1.$$

We can prove this by multiplying out the series for $g(\alpha - it)$ and $h(\beta + it)$. We have

$$\begin{aligned}
f(t) &= \sum_{m=1}^{\infty} \frac{a_m}{m^{\alpha - it}} \sum_{n=1}^{\infty} \frac{b_n}{n^{\beta + it}} = \sum_{m,n\ge 1} \frac{a_m b_n}{m^\alpha n^\beta} \Big(\frac{m}{n}\Big)^{it} \\
&= \sum_{d=1}^{\infty} \sum_{\substack{m,n\ge 1\\ (m,n)=d}} \frac{a_m b_n}{m^\alpha n^\beta} \Big(\frac{m}{n}\Big)^{it} = \sum_{d=1}^{\infty} \frac{1}{d^{\alpha+\beta}} \sum_{\substack{m,n\ge 1\\ (m,n)=1}} \frac{a_{md} b_{nd}}{m^\alpha n^\beta} \Big(\frac{m}{n}\Big)^{it} \\
&= \sum_{\substack{m,n\ge 1\\ (m,n)=1}} \frac{1}{m^\alpha n^\beta} \Big(\sum_{d=1}^{\infty} \frac{a_{md} b_{nd}}{d^{\alpha+\beta}} \Big) \Big(\frac{m}{n}\Big)^{it}.
\end{aligned}$$

Further Properties of $W_{\mathbb{Q}+}$ and $W_{\mathbb{N}}$. Notation. For a unital Banach algebra $\mathcal{A}$, denote by $G\mathcal{A}$ the group of invertible elements of $\mathcal{A}$, and by $G_0\mathcal{A}$, the connected component of $G\mathcal{A}$ which contains the identity. If $\mathcal{A}$ is commutative, then

$$a \in G_0\mathcal{A} \Longleftrightarrow a = e^b \quad \text{for some } b \in \mathcal{A}.$$

(a) Wiener's Theorem for $W_{\mathbb{Q}+}$ and $W_{\mathbb{N}}$ (see [13], Theorems 1 and 2):

Let $f \in W_{\mathbb{Q}+}$. Then $1/f \in W_{\mathbb{Q}+}$ if and only if $\inf_{\mathbb{R}} |f| > 0$; i.e., $GW_{\mathbb{Q}+} = \{f \in W_{\mathbb{Q}+} : \inf_{\mathbb{R}} |f| > 0\}$.

Let $f(t) = \sum_{n=1}^{\infty} a_n n^{it} \in W_{\mathbb{N}}$ and put $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ for $\Re s \geq 0$. Then $1/f \in W_{\mathbb{N}}$ if and only if there exists $\delta > 0$ such that $|F(s)| \geq \delta$ for all $\Re s \geq 0$.

The example $f(t) = 2^{it}$ shows that the condition $|f(t)| \geq \delta > 0$ for all $t \in \mathbb{R}$ is not sufficient for $1/f \in W_{\mathbb{N}}$.

(b) Let $f \in GW_{\mathbb{Q}+}$. Then the *average winding number*[1] $w(f)$, defined by

$$w(f) = \lim_{T\to\infty} \frac{\arg f(T) - \arg f(-T)}{2T}$$

exists, and $w(f) = \log q$ for some $q \in \mathbb{Q}^+$ (see [19], Theorem 27).

It is easy to see that (i) $w(fg) = w(f) + w(g)$, and (ii) $w(\chi_q) = \log q$.

(c) $G_0 W_{\mathbb{N}} = GW_{\mathbb{N}}$; i.e. for $f \in W_{\mathbb{N}}$, we have $1/f \in W_{\mathbb{N}}$ if and only if $f = e^g$ for some $g \in W_{\mathbb{N}}$

2.3 Factorization and invertibility of multiplicative Toeplitz operators

The analogue of the factorization $T(abc) = T(a)T(b)T(c)$ for $a \in \overline{H^\infty}, b \in L^\infty, c \in H^\infty$ for Toeplitz matrices holds for Multiplicative Toeplitz matrices.

Theorem 2.3. *Let $f \in \overline{W_{\mathbb{N}}}$, $g \in W_{\mathbb{Q}+}$, and $h \in W_{\mathbb{N}}$. Then*

$$M(fgh) = M(f)M(g)M(h).$$

Proof. We show the matrix entries agree. By Proposition 2.1, $fgh \in W_{\mathbb{Q}+}$ and

$$\begin{aligned}\Big(M(f)M(g)M(h)\Big)_{ij} &= \sum_{k,l\geq 1} M(f)_{ik} M(g)_{kl} M(h)_{lj} \\ &= \sum_{\substack{k,l\geq 1 \\ i|k \text{ and } j|l}} \hat{f}\Big(\frac{i}{k}\Big)\hat{g}\Big(\frac{k}{l}\Big)\hat{h}\Big(\frac{l}{j}\Big) \\ &= \sum_{m,n\geq 1} \hat{f}\Big(\frac{1}{m}\Big)\hat{g}\Big(\frac{mi}{nj}\Big)\hat{h}\Big(\frac{n}{1}\Big).\end{aligned}$$

[1] Also known as *mean motion*.

On the other hand, since all $\mathbb{Q}^+$-series converge absolutely we have

$$\begin{aligned} f(t)g(t)h(t) &= \sum_{q_1,q_2,q_3\in\mathbb{Q}^+} \hat{f}(q_1)\hat{g}(q_2)\hat{h}(q_3)(q_1q_2q_3)^{it} \\ &= \sum_{q\in\mathbb{Q}^+}\Big(\sum_{q_1,q_3} \hat{f}(q_1)\hat{g}\Big(\frac{q}{q_1q_3}\Big)\hat{h}(q_3)\Big)q^{it} \\ &= \sum_{q\in\mathbb{Q}^+} \widehat{fgh}(q)q^{it}. \end{aligned}$$

Hence

$$M(fgh)_{ij} = \widehat{fgh}\Big(\frac{i}{j}\Big) = \sum_{q_1,q_3} \hat{f}(q_1)\hat{g}\Big(\frac{i}{jq_1q_3}\Big)\hat{h}(q_3) = \Big(M(f)M(g)M(h)\Big)_{ij},$$

since $1/q_1$ and q_3 must range over $\mathbb{N}$. □

In view of Theorem 2.3, it is of interest to know when a given $f \in W_{\mathbb{Q}^+}$ factorises as $f = gh$ with $g \in \overline{W_{\mathbb{N}}}$ and $h \in W_{\mathbb{N}}$. For then $M(f) = M(g)M(h)$ and the invertibility of $M(f)$ follows from knowing when $M(g)$ and $M(h)$ are invertible. Thus, if h is invertible in $W_{\mathbb{N}}$, then

$$M(h)M(h^{-1}) = M(hh^{-1}) = M(1) = I = M(1) = M(h^{-1}h) = M(h^{-1})M(h),$$

so that $M(h)^{-1} = M(h^{-1})$. Similarly, if g is invertible in $\overline{W_{\mathbb{N}}}$, then $M(g)^{-1} = M(g^{-1})$. It follows then that $M(f)^{-1} = M(h)^{-1}M(g)^{-1}$.

Let $\mathcal{F}W_{\mathbb{Q}^+}$ denote the set of functions in $W_{\mathbb{Q}^+}$ which factorise as

$$f = f_-\chi_q f_+ \tag{3.9}$$

where $f_- \in G\overline{W_{\mathbb{N}}}$, $f_+ \in GW_{\mathbb{N}}$, and $q \in \mathbb{Q}^+$.

Note that with f as above, then $1/f = f_-^{-1}\chi_{(1/q)}f_+^{-1}$, so $1/f \in \mathcal{F}W_{\mathbb{Q}^+}$. In particular, $\mathcal{F}W_{\mathbb{Q}^+} \subset GW_{\mathbb{Q}^+}$. Note that $M(\chi_q)$ is invertible if and only if $q = 1$.

Theorem 2.4. *Let $f \in \mathcal{F}W_{\mathbb{Q}^+}$. Then $M(f)$ is invertible if and only if $w(f) = 0$. If this is the case, then $M(f)^{-1} = M(f_+^{-1})M(f_-^{-1})$, with $f_\pm$ as in (3.9).*

Proof. Write $f = f_-\chi_q f_+$ as in (3.9). Then $M(f) = M(f_-)M(\chi_q)M(f_+)$. Now $M(f_-)$ and $M(f_+)$ are invertible, with inverses $M(f_-^{-1})$ and $M(f_+^{-1})$, respectively. Thus $M(f)$ is invertible if and only if $M(\chi_q)$ is invertible. But this happens if and only if $q = 1$.

Since $w(f) = w(f_-) + w(\chi_q) + w(f_+) = w(\chi_q) = \log q$, we see that $M(f)$ is invertible if and only if $w(f) = 0$.

Now suppose $w(f) = 0$. Then the above gives

$$M(f)^{-1} = \Big(M(f_-)M(f_+)\Big)^{-1} = M(f_+)^{-1}M(f_-)^{-1} = M(f_+^{-1})M(f_-^{-1}). \quad \square$$

Multiplicative coefficients and Euler products.

Definition.

(a) A function $a\colon \mathbb{Q}^+ \to \mathbb{C}$ is *multiplicative* if $a(1) = 1$ and

$$a(p_1^{a_1} \cdots p_k^{a_k}) = a(p_1^{a_1}) \cdots a(p_k^{a_k}),$$

for all distinct primes p_i and all $a_i \in \mathbb{Z}$. We say a is *completely multiplicative* if, in addition to the above,

$$a(p^k) = a(p)^k \quad \text{and} \quad a(p^{-k}) = a(p^{-1})^k,$$

for all primes p and $k \in \mathbb{N}$.

(b) For a subset S of $\mathcal{F}$, let $\mathcal{M}\mathcal{F}$ denote the set of $f \in S$ for which $\hat{f}(\cdot)$ is multiplicative.

(c) Let $f \in \mathcal{F}$ and p prime. Suppose that $\sum_{k\in\mathbb{Z}} |\hat{f}(p^k)|$ converges. Then define

$$f_p = \sum_{k\in\mathbb{Z}} \hat{f}(p^k)\chi_{p^k}.$$

Note that f_p is periodic with period $\frac{2\pi}{\log p}$. Define $f_p^\sharp\colon \mathbb{T} \to \mathbb{C}$ by

$$f_p^\sharp(e^{i\theta}) = f_p(\theta/\log p) = \sum_{k\in\mathbb{Z}} \hat{f}(p^k)e^{ki\theta} \qquad \text{for } 0 \le \theta < 2\pi.$$

Further, denote by $W_{\mathbb{Q}^+,p}$ the set of those $f \in W_{\mathbb{Q}^+}$ whose $\mathbb{Q}^+$-coefficients are supported on $\{p^k : k \in \mathbb{Z}\}$. (Thus $f_p \in W_{\mathbb{Q}^+,p}$ by definition.)

Note that, for fixed p, there is a one-to-one correspondence between $W_{\mathbb{Q}^+,p}$ and $W(\mathbb{T})$ via the mapping $^\sharp$.

In [14], the Euler product formulas

$$f = \sum_{q\in\mathbb{Q}^+} \hat{f}(q)\chi_q = \prod_p f_p \quad \text{and} \quad M(f) = \prod_p M(f_p),$$

were shown to hold whenever $f \in \mathcal{M}W_{\mathbb{Q}^+}$.

The analogue of the Wiener–Hopf factorization holds for $\mathcal{M}W_{\mathbb{Q}^+}$-functions without zeros.

Theorem 2.5. *Let $f \in \mathcal{M}W_{\mathbb{Q}^+}$ such that f has no zeros. Then $f \in \mathcal{F}W_{\mathbb{Q}^+}$.*

Proof. We have $f = \prod_p f_p$, where $f_p \in W_{\mathbb{Q}^+,p}$ and each is non-zero. Hence $f_p^\sharp \in W(\mathbb{T})$ and has no zeros. Let $k_p = \operatorname{wind}(f_p^\sharp, 0)$. Note that $k_p = 0$ for all sufficiently large p, since

$$|f_p^\sharp(t)-1| \le \sum_{m\neq 0} |\hat{f}(p^m)| \le \sum_{|q|\ge p} |\hat{f}(q)| \longrightarrow 0 \quad \text{as } p\to\infty.$$

Let $q=\prod_p p^{k_p}$ (a finite product).
By 1.1(iii)TS8, we have

$$f_p^\sharp = \chi^\sharp_{p^{k_p}} e^{g_p^\sharp},$$

for some $g_p^\sharp \in W(\mathbb{T})$. Hence

$$f_p = \chi_{p^{k_p}} e^{g_p},$$

with $g_p \in W_{\mathbb{Q}^+,p}$. Thus for P so large that $k_p=0$ for $p>P$, we have

$$\prod_{p\le P} f_p = \chi_q \exp\left\{\sum_{p\le P} g_p\right\}.$$

Now $f_p(t)\to 1$ as $p\to\infty$ uniformly in t, so can choose g_p so that $g_p(t)\to 0$ as $p\to\infty$ (uniformly in t). Hence, for all sufficiently large p (and all t), $|f_p-1| = |e^{g_p}-1| \le \frac{1}{2}|g_p|$, so that

$$|g_p| \le 2|f_p-1| \le 2\sum_{m\neq 0} |\hat{f}(p^m)|.$$

Let $g^{(n)}=\sum_{p\le n} g_p$. Then $\{g^{(n)}\}$ is a Cauchy sequence in $W_{\mathbb{Q}^+}$: for $n>m$

$$|g^{(n)}-g^{(m)}| \le \sum_{m<p\le n} |g_p| \le 2\sum_{m<p\le n}\sum_{m\neq 0} |\hat{f}(p^m)| \le 2\sum_{r>m} |\hat{f}(k)| \longrightarrow 0$$

as $m\to\infty$. Thus $g^{(n)}\to g\in W_{\mathbb{Q}^+}$. But each $g^{(n)}$ is of the form h_n+k_n with $h_n\in W_{\mathbb{N}}$ and $k_n\in\overline{W_{\mathbb{N}}}$ (since $g_p\in W_{\mathbb{Q}^+,p}$). Thus $g=h+k$ with $h\in W_{\mathbb{N}}$, $k\in\overline{W_{\mathbb{N}}}$. It follows that $f=\chi_q e^h e^k$, which is of the required form. □

Note that, as such,

$$w(f)=\sum_p w(\chi_p^{k_p}) = \sum_p k_p w(\chi_p) = \sum_p \operatorname{wind}(f_p^\sharp,0)\log p,$$

where the sum is finite.

Corollary 2.6. *Let $f\in\mathcal{M}W_{\mathbb{Q}^+}$ such that f has no zeros and $w(f)=0$. Then $M(f)$ is invertible.*

Example. $M(\zeta_\alpha)$ is invertible for $\alpha>1$ with $M(\zeta_\alpha)^{-1}=M(1/\zeta_\alpha)$.

TS8 What does 1.1 refer to? Theorem 1.1 does not include an enumeration. Is it meant to be Subsection 1.1?

3 Unbounded multiplicative Toeplitz operators and matrices

The last section has, in various ways, been a relatively straightforward extension of the theory of bounded Toeplitz operators to the multiplicative setting. The theory of *unbounded* Toeplitz operators is rather less satisfactory and not easy to generalise. Our particular concern is in fact the operator $M(\zeta_\alpha)$ for $\alpha \leq 1$, since we expect a connection with the Riemann zeta function. The hope is that if a satisfactory theory is developed for this case, it can be generalised to other symbols.

In this section, we shall therefore concentrate on the particular case when $f(n) = n^{-\alpha}$, when the symbol is $\zeta(\alpha - it)$. Throughout we write φ_α (equivalently, $M(\zeta_\alpha)$) for φ_f.

From §2 we see that for $\alpha \leq 1$, φ_α is unbounded. It is interesting to see to what extent properties of φ_α are related to properties of ζ_α. The above theory is only valid for absolutely convergent Dirichlet series, when the symbols are bounded. But $\zeta(\alpha - it)$ is unbounded for $\alpha \leq 1$.

How to measure unboundedness? We shall investigate two different measures. The first case can be considered as restricting the range, while in the second case we shall restrict the domain.

3.1 First measure – the function $\Phi_\alpha(N)$

With b_n defined by $a = (a_n) \overset{\varphi_\alpha}{\mapsto} (b_n)$, i.e., $b_n = \sum_{d|n} d^{-\alpha} a_{n/d}$, let

$$\Phi_\alpha(N) = \sup_{\|a\|=1} \left(\sum_{n=1}^{N} |b_n|^2 \right)^{1/2}.$$

Theorem 3.1. *We have the following asymptotic formulae for large N:*

$$\Phi_1(N) = e^\gamma \log\log N + O(1) \qquad (\alpha = 1)$$

$$\log \Phi_\alpha(N) \asymp \frac{(\log N)^{1-\alpha}}{\log\log N} \qquad \left(\frac{1}{2} < \alpha < 1\right)$$

$$\log \Phi_{\frac{1}{2}}(N) \sim \left(\frac{\log N}{\log\log N}\right)^{\frac{1}{2}}. \qquad \left(\alpha = \frac{1}{2}\right)$$

Sketch of proof. (*For the proof see* [15]). We start with upper bounds.

First we note that for any positive arithmetical function $g(n)$,

$$\Phi_\alpha(N)^2 \leq \left(\sum_{n \leq N} \frac{g(n)}{n^\alpha} \right) \cdot \left(\max_{n \leq N} \sum_{d|n} \frac{1}{g(d) d^\alpha} \right). \tag{3.10}$$

This is because

$$|b_n|^2 = \left| \sum_{d|n} \frac{1}{\sqrt{g(d)} d^{\frac{\alpha}{2}}} \cdot \frac{\sqrt{g(d)} a_{n/d}}{d^{\frac{\alpha}{2}}} \right|^2 \leq \left(\sum_{d|n} \frac{1}{g(d) d^\alpha} \right) \cdot \left(\sum_{d|n} \frac{g(d) |a_{n/d}|^2}{d^\alpha} \right),$$

by Cauchy–Schwarz. Writing $G(n) = \sum_{d|n} g(d)^{-1} d^{-\alpha}$, we have

$$\sum_{n\le N} |b_n|^2 \le \sum_{n\le N} G(n) \sum_{d|n} \frac{g(d)|a_{n/d}|^2}{d^\alpha} \le \max_{n\le N} G(n) \sum_{d\le N} \frac{g(d)}{d^\alpha} \sum_{n\le N/d} |a_n|^2.$$

Taking $\|a\|_2 = 1$ yields (3.10). The idea is to choose g appropriately, so that the RHS of (3.10) is as small as possible.

For $\frac{1}{2} < \alpha \le 1$, choose $g(n)$ to be the following multiplicative function: for a prime power p^k let

$$g(p^k) = \begin{cases} 1, & \text{if } p^k \le M, \\ (\frac{M}{p^k})^\beta, & \text{if } p^k > M. \end{cases}$$

Here $M = (2\alpha - 1)\log N$ and β is a constant satisfying $1 - \alpha < \beta < \alpha$. Note that $g(p^k) \le g(p)$ for every $k \in \mathbb{N}$ and p prime.

We estimate the expressions in (3.10) separately. First

$$\sum_{n\le N} \frac{g(n)}{n^\alpha} \le \prod_p \Big(1 + \sum_{k=1}^\infty \frac{g(p^k)}{p^{k\alpha}}\Big) \le \prod_p \Big(1 + \frac{g(p)}{p^\alpha - 1}\Big) \le \exp\Big\{\sum_p \frac{g(p)}{p^\alpha - 1}\Big\} \tag{3.11}$$

and, after some manipulations using the Prime Number Theorem, one finds for the case $\alpha < 1$

$$\log \sum_{n\le N} \frac{g(n)}{n^\alpha} \lesssim \frac{\beta M^{1-\alpha}}{(1-\alpha)(\alpha+\beta-1)\log M}. \tag{3.12}$$

Now consider $G(n)$, which is multiplicative because $g(n)$ is. At the prime powers we have

$$\begin{aligned} G(p^k) &= \sum_{r=0}^k \frac{1}{p^{\alpha r} g(p^r)} = \sum_{\substack{r\ge 0 \\ p^r \le M}} \frac{1}{p^{\alpha r}} + \frac{1}{M^\beta} \sum_{\substack{r\le k \\ p^r > M}} \frac{1}{p^{(\alpha-\beta)r}} \\ &\le 1 + \frac{1}{p^\alpha - 1} + \frac{1}{M^\alpha(1 - p^{\beta-\alpha})}. \end{aligned}$$

(Here we require $\beta < \alpha$.) Note that this is independent of k. It follows that

$$G(n) \le \exp\Big\{\sum_{p|n} \frac{1}{p^\alpha - 1} + \frac{1}{M^\alpha} \sum_{p|n} \frac{1}{1 - p^{\beta-\alpha}}\Big\}.$$

The right-hand side is maximised when n is as large as possible (i.e. N) and N is of the form $N = 2.3\ldots P$. For such a choice, $\log N = \theta(P) \sim P$, so that (using the prime number theorem)

$$\log \max_{n\le N} G(n) \lesssim \sum_{p\le P} \frac{1}{p^\alpha - 1} + \frac{1}{M^\alpha} \sum_{p\le P} 1 \sim \frac{(\log N)^{1-\alpha}}{(1-\alpha)\log\log N} + \frac{\log N}{M^\alpha \log\log N}. \tag{3.13}$$

From (3.12) and (3.13) it follows that M should be of order $\log N$ for optimality. So taking $M = \lambda \log N$ (with $\lambda > 0$), (3.10), (3.12) and (3.13) then imply

$$\log \Phi_\alpha(N) \lesssim \left(\frac{\beta\lambda^{1-\alpha}}{2(1-\alpha)(\alpha+\beta-1)} + \frac{1}{2(1-\alpha)} + \frac{1}{2\lambda^\alpha} \right) \frac{(\log N)^{1-\alpha}}{\log\log N}$$

for every $\beta \in (1-\alpha, \alpha)$ and $\lambda > 0$. Since $\frac{\beta}{\alpha+\beta-1}$ decreases with β, the optimal choice is to take β arbitrarily close to α. Hence we require $\inf_{\lambda>0} h(\lambda)$, where

$$h(\lambda) = \frac{\alpha\lambda^{1-\alpha}}{(1-\alpha)(2\alpha-1)} + \frac{1}{(1-\alpha)} + \frac{1}{\lambda^\alpha}.$$

Since $h'(\lambda) = \frac{\alpha}{\lambda^{\alpha+1}}(\frac{\lambda}{2\alpha-1} - 1)$, we see that the optimal choice is indeed $\lambda = 2\alpha - 1$. Substituting this value of λ gives

$$\log \Phi_\alpha(N) \lesssim \frac{(1 + (2\alpha-1)^{-\alpha})}{2(1-\alpha)} \frac{(\log N)^{1-\alpha}}{\log\log N}.$$

For $\alpha = 1$, we use the same function $g(n)$ as before (though with possibly different values of M and β). From (3.11) it follows that

$$\sum_{n \le N} \frac{g(n)}{n} \le \prod_{p \le M} \left(\frac{1}{1-\frac{1}{p}} \right) \cdot \prod_{p > M} \left(1 + \frac{M^\beta}{p^\beta(p-1)} \right).$$

By Mertens' Theorem, the first product is $e^\gamma \log M + O(1)$, while $M^\beta \sum_{p>M} p^{-1-\beta} = O(1/\log M)$, and so

$$\sum_{n \le N} \frac{g(n)}{n} \le \left(e^\gamma \log M + O(1) \right) \exp\{O(1/\log M)\} = e^\gamma \log M + O(1). \quad (3.14)$$

For the $G(n)$ term we have, as for the $\alpha < 1$ case,

$$G(p^k) \le \frac{1}{1-\frac{1}{p}} + \frac{1}{M(1-p^{\beta-1})}.$$

Thus, with $N = 2.3 \dots P$,

$$\begin{aligned} G(N) &\le \prod_{p \le P} \left(\frac{1}{1-\frac{1}{p}} \right) \left(1 + \frac{1-1/p}{M(1-p^{\beta-1})} \right) \\ &= \left(e^\gamma \log P + O(1) \right) \left(1 + O\left(\frac{P}{M \log P} \right) \right). \end{aligned}$$

Taking $M = \log N$ and noting that $P \sim \log N$, the right-hand side is $e^\gamma \log\log N + O(1)$. Combining with (3.14) shows that

$$\Phi_1(N) \le e^\gamma \log\log N + O(1).$$

The case $\alpha = \frac{1}{2}$. The function g as chosen for $\alpha \in (\frac{1}{2}, 1]$ is not suitable for an upper bound as we would require $\frac{1}{2} < \beta < \frac{1}{2}$! Instead we take g to be the multiplicative function following: for a prime power p^k, let

$$g(p^k) = \min\Big\{1, \Big(\frac{M}{p^k(\log p)^2}\Big)^{\frac{1}{2}}\Big\}.$$

Here $M > 0$ is independent of p and k and will be determined later. Thus $g(p^k) = 1$ if and only if $p^k(\log p)^2 \leq M$. Note that $g(p^k) \leq g(p) \leq 1$ for all $k \geq 1$ and all primes p. Thus (3.11) holds with $\alpha = \frac{1}{2}$ and (using the prime number theorem)

$$\log \sum_{n\leq N} \frac{g(n)}{\sqrt{n}} \lesssim \sum_{p\lesssim \frac{M}{(\log M)^2}} \frac{1}{\sqrt{p}-1} + \sqrt{M} \sum_{p\gtrsim \frac{M}{(\log M)^2}} \frac{1}{p\log p} \sim \frac{\sqrt{M}}{\log M}. \tag{3.15}$$

(The first sum is of order $\sqrt{M}/(\log M)^2$ and the main contribution comes from the second term.)

Regarding $G(n)$, this time we have[2]

$$G(n) = \prod_{p^k\|n} G(p^k) \leq \prod_{p^k\|n} \Big(1 + \sum_{r=1}^{k} \frac{1}{p^{r/2}} + \frac{1}{\sqrt{M}} \sum_{r=1}^{k} \log p\Big),$$

so that

$$\log G(n) \leq \sum_{p|n} \frac{1}{\sqrt{p}-1} + \frac{1}{\sqrt{M}} \sum_{p^k\|n} k\log p \leq \frac{\log n}{\sqrt{M}} + \sum_{p|n} \frac{1}{\sqrt{p}-1}.$$

The right-hand side above is maximal when $n = N = 2.3\ldots P$, hence

$$\log \max_{n\leq N} G(n) \lesssim \frac{\log N}{\sqrt{M}} + \sum_{p\leq P} \frac{1}{\sqrt{p}} \sim \frac{\log N}{\sqrt{M}} + \frac{2\sqrt{\log N}}{\log\log N}.$$

Combining with (3.15), (3.10) then gives

$$\log \Phi_{\frac{1}{2}}(N) \lesssim \frac{\sqrt{M}}{2\log M} + \frac{\log N}{2\sqrt{M}} + \frac{\sqrt{\log N}}{\log\log N}.$$

The optimal choice for M is easily seen to be $M = \log N \log\log N$, and this gives the upper bound in (iii).

Now we proceed to give lower bounds.

For a fixed $n \in \mathbb{N}$, let

$$a_d = \frac{1}{\sqrt{d(n)}} \qquad \text{if } d|n, \text{ and zero otherwise.}$$

[2] Here as usual, $p^k||n$ means $p^k|n$ but $p^{k+1} \nmid n$.

Then $\|a\|_2 = 1$, while for $d|n$

$$b_d = \frac{1}{\sqrt{d(n)}} \sum_{c|d} \frac{1}{c^\alpha} = \frac{\sigma_{-\alpha}(d)}{\sqrt{d(n)}}.$$

Hence for $N \geq n$,

$$\sum_{k \leq N} |b_k|^2 \geq \sum_{d|n} b_d^2 = \frac{1}{d(n)} \sum_{d|n} \sigma_{-\alpha}(d)^2 =: \eta_\alpha(n).$$

With this notation $\Phi_\alpha(N) \geq \max_{n \leq N} \sqrt{\eta_\alpha(n)}$, and the lower bounds follow from the maximal order of $\eta_\alpha(n)$. For $\frac{1}{2} < \alpha < 1$ this can be found easily. Since $\eta_\alpha(p) = 1 + p^{-\alpha} + \frac{1}{2}p^{-2\alpha}$ for p prime, we find for $n = 2.3 \ldots P$ (so that $\log n \sim P$) that

$$\begin{aligned}\eta_\alpha(n) &= \prod_{p \leq P} \left(1 + \frac{1}{p^\alpha} + O\Big(\frac{1}{p^{2\alpha}}\Big)\right) = \exp\left\{(1 + o(1)) \sum_{p \leq P} \frac{1}{p^\alpha}\right\} \\ &= \exp\left\{\frac{(1 + o(1)) P^{1-\alpha}}{(1-\alpha) \log P}\right\} = \exp\left\{\frac{(1 + o(1))(\log n)^{1-\alpha}}{(1-\alpha) \log \log n}\right\}.\end{aligned}$$

Now, if t_k is the k^{th} number of the form $2.3 \cdots P$ (i.e., $t_k = p_1 \cdots p_k$), then $\log t_k \sim k \log k \sim \log t_{k+1}$. Hence for $t_k \leq N < t_{k+1}$, $\log N \sim k \log k$. It follows that

$$\Phi_\alpha(N) \geq \sqrt{\eta_\alpha(t_k)} \geq \exp\left\{\frac{(1+o(1))(\log t_k)^{1-\alpha}}{2(1-\alpha) \log \log t_k}\right\} = \exp\left\{\frac{(1+o(1))(\log N)^{1-\alpha}}{2(1-\alpha) \log \log N}\right\}.$$

For $\alpha = 1$, we have to be a little subtler to obtain $\max_{n \leq N} \sqrt{\eta_1(n)} = e^\gamma \log \log N + O(1)$. We omit the details, which can be found in [15].

For the case $\alpha = \frac{1}{2}$, the above choice doesn't give the correct order and we lose a power of $\log \log N$. Instead, we follow an idea of Soundararajan [29]. Let f be the multiplicative function supported on the squarefree numbers whose values at primes p is

$$f(p) = \begin{cases} (\frac{M}{p})^{1/2} \frac{1}{\log p}, & \text{for } M \leq p \leq R, \\ 0, & \text{otherwise.} \end{cases}$$

Here $M = \log N (\log \log N)$ as before and $\log R = (\log M)^2$.

Now take $a_n = f(n) F(N)^{-1/2}$, where $F(N) = \sum_{n \leq N} f(n)^2$ so that $\sum_{n \leq N} a_n^2 = 1$. Then by Hölder's inequality

$$\begin{aligned}\left(\sum_{n=1}^N b_n^2\right)^{1/2} &\geq \sum_{n=1}^N a_n b_n = \frac{1}{F(N)} \sum_{n=1}^N \frac{f(n)}{\sqrt{n}} \sum_{d|n} \sqrt{d} f(d) \\ &= \frac{1}{F(N)} \sum_{n \leq N} \frac{f(n)}{\sqrt{n}} \sum_{\substack{d \leq N/n \\ (n,d)=1}} f(d)^2. \qquad (3.16)\end{aligned}$$

Now using 'Rankin's trick'[3] we have, for any $\beta > 0$

$$\sum_{n\le N} \frac{f(n)}{n^{1/2}} \sum_{\substack{d\le N/n\\(n,d)=1}} f(d)^2 = \sum_{n\le N} \frac{f(n)}{n^{1/2}}\Bigg(\sum_{\substack{d\ge 1\\(n,d)=1}} f(d)^2 - \sum_{\substack{d> N/n\\(n,d)=1}} f(d)^2\Bigg)$$

$$= \sum_{n\le N} \frac{f(n)}{n^{1/2}}\Bigg(\prod_{p\nmid n}\Big(1+f(p)^2\Big) + O\bigg(\Big(\frac{n}{N}\Big)^{\beta}\prod_{p\nmid n}\Big(1+p^{\beta}f(p)^2\Big)\bigg)\Bigg). \tag{3.17}$$

The O-term in (3.17) is at most a constant times

$$\frac{1}{N^{\beta}}\sum_{n\le N} f(n)n^{\beta-1/2}\prod_{p\nmid n}\Big(1+p^{\beta}f(p)^2\Big) \le \frac{1}{N^{\beta}}\prod_{p}\Big(1+p^{\beta}f(p)^2+p^{\beta-1/2}f(p)\Big),$$

while the main term in (3.17) is (using Rankin's trick again)

$$\prod_{p}\Big(1+f(p)^2+\frac{f(p)}{p^{1/2}}\Big) + O\bigg(\frac{1}{N^{\beta}}\prod_{p}\Big(1+f(p)^2+p^{\beta-1/2}f(p)\Big)\bigg).$$

Hence (3.16) implies

$$\begin{aligned}\Big(\sum_{n=1}^{N} b_n^2\Big)^{1/2} \ \ge\ & \frac{1}{F(N)}\Bigg(\prod_{p}\Big(1+f(p)^2+\frac{f(p)}{p^{1/2}}\Big)\\ & + O\bigg(\frac{1}{N^{\beta}}\prod_{p}\Big(1+p^{\beta}f(p)^2+p^{\beta-1/2}f(p)\Big)\bigg)\Bigg).\end{aligned}$$

The ratio of the O-term to the main term on the right is less than

$$\exp\Big\{-\beta\log N + \sum_{M\le p\le R}(p^{\beta}-1)\Big(f(p)^2+\frac{f(p)}{p^{1/2}}\Big)\Big\},$$

which equals

$$\exp\Big\{-\beta\log N + \sum_{M\le p\le R}(p^{\beta}-1)\Big(\frac{M}{p(\log p)^2}+\frac{M^{1/2}}{p(\log p)}\Big)\Big\}.$$

Take $\beta = (\log M)^{-3}$. The term involving $M^{1/2}$ is at most $(\log N)^{1/2+\varepsilon}$ for every $\varepsilon > 0$, while the remaining terms in the exponent are (by the prime number theorem

[3] If $c_n > 0$, then for any $\beta > 0$, $\sum_{n>x} c_n \le x^{-\beta}\sum_{n=1}^{\infty} n^{\beta}c_n$.

in the form $\pi(x) = \mathrm{li}(x) + O(x(\log x)^{-A})$ for all A)

$$\begin{aligned}
-\beta \log N + M \int_M^R \frac{t^\beta - 1}{t(\log t)^3}\,dt + O\Big(\frac{\log N}{(\log\log N)^A}\Big) \\
&= -\beta \log N + \beta M \int_M^R \frac{dt}{t(\log t)^2} + O\Big(\beta^2 M \int_M^R \frac{dt}{t\log t}\Big) \\
&\sim -\beta \frac{\log N \log\log\log N}{\log\log N},
\end{aligned}$$

after some calculations.

Finally, since $F(N) \le \prod_p (1 + f(p)^2)$, this implies

$$\Phi_{1/2}(N) \ge \frac{1}{2} \prod_{M \le p \le R} \Big(1 + \frac{f(p)}{p^{1/2}(1 + f(p)^2)}\Big),$$

for all N sufficiently large. Hence

$$\log \Phi_{1/2}(N) \gtrsim M^{1/2} \sum_{M \le p \le R} \frac{1}{p(\log p)} \sim \Big(\frac{\log N}{\log\log N}\Big)^{1/2},$$

as required. □

Remark. The result for $\frac{1}{2} < \alpha < 1$ is

$$\frac{(\log N)^{1-\alpha}}{2(1-\alpha)\log\log N} \lesssim \log \Phi_\alpha(N) \lesssim \frac{(1 + (2\alpha - 1)^{-\alpha})}{2(1-\alpha)} \frac{(\log N)^{1-\alpha}}{\log\log N}.$$

It would be nice to obtain an asymptotic formula for $\log \Phi_\alpha(N)$. Indeed, it is possible to improve the lower bound at the cost of more work by using the method for the case $\alpha = \frac{1}{2}$, but we have not been able to obtain the same upper and lower limits.

Connections between $\Phi_\alpha(N)$ and the order of $|\zeta(\alpha + it)|$. The lower bounds obtained for $\Phi_\alpha(N)$ for $\frac{1}{2} < \alpha \le 1$ can be used to obtain information regarding the maximum order of $\zeta(s)$ on the line $\Re s = \alpha$.

Proposition 3.2. *With $b_n = \sum_{d|n} d^{-\alpha} a_{n/d}$ we have, for any α,*

$$\sum_{n \le N} |b_n|^2 = \sum_{m,n \le N} \frac{a_m \overline{a_n} (m,n)^{2\alpha}}{m^\alpha n^\alpha} \sum_{k \le \frac{N}{[m,n]}} \frac{1}{k^{2\alpha}}.$$

Proof. We have

$$|b_n|^2 = b_n \overline{b_n} = \frac{1}{n^{2\alpha}} \sum_{c|n, d|n} c^\alpha d^\alpha a_c \overline{a_d} = \frac{1}{n^{2\alpha}} \sum_{[c,d]|n} c^\alpha d^\alpha a_c \overline{a_d},$$

since $c|n, d|n$ if and only if $[c,d]|n$. Hence

$$\sum_{n\le N}|b_n|^2 = \sum_{c,d\le N} c^\alpha d^\alpha a_c\overline{a_d} \sum_{n\le N,[c,d]|n}\frac{1}{n^{2\alpha}} = \sum_{c,d\le N}\frac{c^\alpha d^\alpha a_c\overline{a_d}}{[c,d]^{2\alpha}}\sum_{k\le\frac{N}{[c,d]}}\frac{1}{k^{2\alpha}},$$

by writing $n=[c,d]k$. Since $(c,d)[c,d]=cd$, the result follows. □

It follows that if $a_n\ge 0$ for all n and $\alpha>\frac12$, then

$$\sum_{n\le N}|b_n|^2 \le \zeta(2\alpha)\sum_{m,n\le N}\frac{a_m\overline{a_n}(m,n)^{2\alpha}}{(mn)^\alpha}. \tag{3.18}$$

Theorem 3.3. *Let $\frac12<\alpha\le 1$ and let $a\in l^2$ with $\|a\|_2=1$. Let $A_N(t)=\sum_{n=1}^N a_n n^{it}$. Let $N\le T^\lambda$, where $0<\lambda<\frac23(\alpha-\frac12)$. Then for some $\eta>0$,*

$$\frac1T\int_1^T|\zeta(\alpha+it)|^2|A_N(t)|^2\,dt = \zeta(2\alpha)\sum_{m,n\le N}\frac{a_m\overline{a_n}(m,n)^{2\alpha}}{(mn)^\alpha}+O(T^{-\eta}). \tag{3.19}$$

Proof. We shall assume $\frac12<\alpha<1$, adjusting the proof for the case $\alpha=1$ afterwards. For $\alpha\ne 1$, we can integrate from 0 to T since the error involved is at most $O(N/T)=O(T^{-\eta})$.

Starting from the approximation $\zeta(\alpha+it)=\sum_{n\le t}n^{-\alpha-it}+O(t^{-\alpha})$, we have

$$|\zeta(\alpha+it)|^2 = \Big|\sum_{n\le t}\frac{1}{n^{\alpha+it}}\Big|^2 + O(t^{1-2\alpha}).$$

Let $k,l\in\mathbb{N}$ such that $(k,l)=1$. Let $M=\max\{k,l\}<T$. The above gives

$$\int_0^T|\zeta(\alpha+it)|^2\Big(\frac{k}{l}\Big)^{it}\,dt = \int_0^T\Big|\sum_{n\le t}\frac{1}{n^{\alpha+it}}\Big|^2\Big(\frac{k}{l}\Big)^{it}\,dt + O(T^{2-2\alpha}).$$

The integral on the right is

$$\int_0^T\sum_{m,n\le t}\frac{1}{(mn)^\alpha}\Big(\frac{km}{ln}\Big)^{it}\,dt = \sum_{m,n\le T}\frac{1}{(mn)^\alpha}\int_{\max\{m,n\}}^T\Big(\frac{km}{ln}\Big)^{it}\,dt.$$

The terms with $km=ln$ (which implies $m=rl, n=rk$ with r integral) contribute

$$\frac{1}{(kl)^\alpha}\sum_{r\le T/M}\frac{T-rM}{r^{2\alpha}} = \frac{\zeta(2\alpha)}{(kl)^\alpha}T + O\Big(\frac{M^{2\alpha-1}T^{2-2\alpha}}{(kl)^\alpha}\Big).$$

The remaining terms contribute at most

$$
\begin{aligned}
2\sum_{\substack{m,n\le T\\ km\neq ln}}\frac{1}{(mn)^{\alpha}|\log\frac{km}{ln}|} &\le 2M^{2\alpha}\sum_{\substack{m,n\le T\\ km\neq ln}}\frac{1}{(kmln)^{\alpha}|\log\frac{km}{ln}|}\\
&\le 2M^{2\alpha}\sum_{\substack{m_1\le kT,n_1\le lT\\ m_1\neq n_1}}\frac{1}{(m_1n_1)^{\alpha}|\log\frac{m_1}{n_1}|}\\
&\le 2M^{2\alpha}\sum_{\substack{m_1,n_1\le MT\\ m_1\neq n_1}}\frac{1}{(m_1n_1)^{\alpha}|\log\frac{m_1}{n_1}|}\\
&= O(M^{2\alpha}(MT)^{2-2\alpha}\log(MT))\\
&= O(M^2T^{2-2\alpha}\log T),
\end{aligned}
$$

using Lemma 7.2 from [31]. Hence

$$\int_0^T|\zeta(\alpha+it)|^2\Big(\frac{k}{l}\Big)^{it}\,dt=\frac{\zeta(2\alpha)}{(kl)^{\alpha}}T+O(M^2T^{2-2\alpha}\log T).$$

It follows that for any positive integers $m,n<T$,

$$\int_0^T|\zeta(\alpha+it)|^2\Big(\frac{m}{n}\Big)^{it}\,dt=\frac{\zeta(2\alpha)(m,n)^{2\alpha}}{(mn)^{\alpha}}T+O(\max\{m,n\}^2T^{2-2\alpha}\log T).$$

Thus, with $A_N(t)=\sum_{n=1}^N a_nn^{it}$,

$$
\begin{aligned}
\int_0^T|\zeta(\alpha+it)|^2|A_N(t)|^2\,dt &= \sum_{m,n\le N}a_m\overline{a_n}\int_0^T|\zeta(\alpha+it)|^2\Big(\frac{m}{n}\Big)^{it}\,dt\\
&=\zeta(2\alpha)T\sum_{m,n\le N}\frac{a_m\overline{a_n}(m,n)^{2\alpha}}{(mn)^{\alpha}}\\
&\quad+O\Big(T^{2-2\alpha}\log T\sum_{m,n\le N}\max\{m,n\}^2|a_ma_n|\Big).
\end{aligned}
$$

The sum in the O-term is at most $N^2(\sum_{n\le N}|a_n|)^2\le N^3$, using Cauchy–Schwarz. Hence

$$\frac{1}{T}\int_0^T|\zeta(\alpha+it)|^2|A_N(t)|^2\,dt=\zeta(2\alpha)\sum_{m,n\le N}\frac{a_m\overline{a_n}(m,n)^{2\alpha}}{(mn)^{\alpha}}+O\Big(\frac{N^3\log T}{T^{2\alpha-1}}\Big).$$

Since $N^3\le T^{3\lambda}$ and $3\lambda<2\alpha-1$, the error term is $O(T^{-\eta})$ for some $\eta>0$.

If $\alpha = 1$ we integrate from 1 to T instead and the O-term above will contain an extra $\log T$ factor, but this is still $O(T^{-\eta})$. $\square$

We note that with more care the N^3 could be turned into an N^2, so that we can take $\lambda < \alpha - \frac{1}{2}$ in the theorem. This is however not too important for us.

Corollary 3.4. *Let $\frac{1}{2} < \alpha \leq 1$. Then for every $\varepsilon > 0$ and N sufficiently large,*

$$\max_{t \leq N} |\zeta(\alpha + it)| \geq \Phi_\alpha(N^{\frac{2}{3}(\alpha-\frac{1}{2})-\varepsilon}) + O(N^{-\eta}) \tag{3.20}$$

for some $\eta > 0$.

Proof. Let $a_n \geq 0$ be such that $\|a\|_2 = 1$, and take $N = T^\lambda$ with $\lambda < \frac{2}{3}(\alpha - \frac{1}{2})$. By (3.18) and (3.19),

$$\begin{aligned}\sum_{n \leq N} |b_n|^2 &\leq \frac{1}{T}\int_0^T |\zeta(\alpha + it)|^2 |A_N(t)|^2\,dt + O(T^{-\eta}) \\ &\leq \max_{t \leq T} |\zeta(\alpha + it)|^2 \frac{1}{T}\int_0^T |A_N(t)|^2\,dt + O(T^{-\eta}) \\ &= \max_{t \leq T} |\zeta(\alpha + it)|^2 \sum_{n \leq N} |a_n|^2 (1 + O(N/T)) + O(T^{-\eta})\end{aligned}$$

using the Montgomery and Vaughan mean value theorem. The implied constants in the O-terms depend only on T and not on the sequence $\{a_n\}$. Taking the supremum over all such a, this gives

$$\Phi_\alpha(N)^2 = \sup_{\|a\|_2 = 1} \sum_{n \leq N} |b_n|^2 \leq \max_{t \leq T} |\zeta(\alpha + it)|^2 + O(T^{-\eta}),$$

for some $\eta > 0$, and (3.20) follows. $\square$

In particular, this gives the (known) lower bounds

$$\max_{t \leq T} |\zeta(\alpha + it)| \geq \exp\left\{\frac{c(\log T)^{1-\alpha}}{\log\log T}\right\}$$

for $\frac{1}{2} < \alpha < 1$ and $\max_{t \leq T} |\zeta(1 + it)| \geq e^\gamma \log\log T + O(1)$ (obtained by Levinson in [23]).

Morever, we can say more about how often $|\zeta(\alpha + it)|$ is as large as this. For $A \in \mathbb{R}$ and $c > 0$, let

$$F_A(T) = \left\{t \in [1, T] : |\zeta(1 + it)| \geq e^\gamma \log\log T - A\right\}. \tag{3.21}$$

$$F_{\alpha,c}(T) = \left\{t \in [0, T] : |\zeta(\alpha + it)| \geq \exp\left\{\frac{c(\log T)^{1-\alpha}}{\log\log T}\right\}\right\}. \tag{3.21$'$}$$

We outline the argument in the case $\frac{1}{2} < \alpha < 1$. We have

$$\sum_{n \le N} |b_n|^2 \le \frac{1}{T}\left(\int_{F_{\alpha,c}(T)} + \int_{[0,T]\setminus F_{\alpha,c}(T)}\right)|\zeta(\alpha + it)|^2 |A_N(t)|^2\, dt + O(T^{-\eta}). \tag{3.22}$$

The second integral on the right is at most

$$\exp\left\{\frac{2c(\log T)^{1-\alpha}}{\log\log T}\right\} \cdot \frac{1}{T}\int_0^T |A_N(t)|^2\, dt = O\left(\exp\left\{\frac{2c(\log T)^{1-\alpha}}{\log\log T}\right\}\right),$$

while, by choosing $a_n = d(N)^{-1/2}$ for $n|N$ and zero otherwise, the LHS of (3.22) is at least $\eta_\alpha(N)$. Every interval $[T^{\lambda/3}, T^\lambda]$ contains an N of the form $2.3.5\ldots P$. As such, $\eta_\alpha(N) \ge \exp\left\{\frac{c'(\log T)^{1-\alpha}}{\log\log T}\right\}$ for some $c' > 0$. Hence, for $2c < c'$,

$$\frac{1}{T}\int_{F_{\alpha,c}(T)} |\zeta(\alpha + it)|^2 |A_N(t)|^2\, dt \ge \exp\left\{\frac{c'(\log T)^{1-\alpha}}{2\log\log T}\right\}.$$

We have $|\zeta(\alpha + it)| = O(T^\nu)$ for some ν and $|A_N(t)|^2 \le d(N) = O(T^\varepsilon)$, so

$$\frac{1}{T}\int_{F_{\alpha,c}(T)} |\zeta(\alpha + it)|^2 |A_N(t)|^2\, dt \le T^{2\nu - 1 + \varepsilon} \mu(F_{\alpha,c}(T)).$$

Thus $\mu(F_{\alpha,c}(T)) \ge T^{1-2\nu-\varepsilon}$ for all c sufficiently small.

In particular, since $\nu < \frac{1-\alpha}{3}$, we have:

Theorem 3.5. *For all A sufficiently large (and positive),*

$$\mu(F_A(T)) \ge T \exp\left\{-a\frac{\log T}{\log\log T}\right\}$$

for some $a > 0$, and for all c sufficiently small, $\mu(F_{\alpha,c}(T)) \ge T^{(1+2\alpha)/3}$ for all sufficiently large T. Furthermore, under the Lindelöf Hypothesis, the exponent can be replaced by $1 - \varepsilon$.

3.2 Second measure – φ_α on $\mathcal{M}^2$ and the function $M_\alpha(T)$ For $\alpha \le 1$, φ_α is unbounded on l^2 and so $\varphi_\alpha(l^2) \not\subset l^2$ (by the closed graph theorem[4] – see [30], p. 183). However, if we *restrict* the domain to $\mathcal{M}^2$, the set of multiplicative elements of l^2, we find that $\varphi(\mathcal{M}^2) \subset l^2$. More generally, if f is multiplicative then, as we shall see, $\varphi_f(\mathcal{M}^2) \subset l^2$ in many cases (and hence $\varphi_f(\mathcal{M}^2) \subset \mathcal{M}^2$).

[4]Being a 'matrix' mapping, φ_α is necessarily a closed operator, and so $\varphi_\alpha(l^2) \subset l^2$ implies φ_α is bounded.

Notation. Let $\mathcal{M}^2$ and $\mathcal{M}_c^2$ denote the subsets of l^2 of multiplicative and completely multiplicative functions, respectively. Further, write $\mathcal{M}^2+$ for the non-negative members of $\mathcal{M}^2$, and similarly for $\mathcal{M}_c^2+$.

Let $\mathcal{M}_0^2$ denote the set of $\mathcal{M}^2$ functions f for which $f * g \in \mathcal{M}^2$ whenever $g \in \mathcal{M}^2$; that is,

$$\mathcal{M}_0^2 = \{f \in \mathcal{M}^2 : g \in \mathcal{M}^2 \implies f * g \in \mathcal{M}^2\}.$$

Thus for $f \in \mathcal{M}_0^2$, $\varphi_f(\mathcal{M}^2) \subset \mathcal{M}^2$.

It was shown in [16] that $\mathcal{M}^2$ is not closed under Dirichlet convolution, so $\mathcal{M}_0^2 \neq \mathcal{M}^2$. A criterion for $f \in \mathcal{M}^2$ to be in $\mathcal{M}_0^2$ was given, namely:

Proposition 3.6. *Let $f \in \mathcal{M}^2$ be such that $\sum_{k=1}^{\infty} |f(p^k)|$ converges for every prime p and that $\sum_{k=1}^{\infty} |f(p^k)| \leq A$ for some constant A independent of p. Then $f \in \mathcal{M}_0^2$.*

On the other hand, if $f \in \mathcal{M}^2$ with $f \geq 0$ and for some prime p_0, $f(p_0^k)$ decreases with k and $\sum_{k=1}^{\infty} f(p_0^k)$ diverges, then $f \notin \mathcal{M}_0^2$.

The proof is based on the following necessary and sufficient condition: *Let $f, g \in \mathcal{M}^2$ be non-negative. Then $f * g \in \mathcal{M}^2$ if and only if*

$$\sum_p \sum_{m,n\geq 1} \sum_{k=0}^{\infty} f(p^m)g(p^n)f(p^{m+k})g(p^{n+k}) \quad \text{converges.}$$

This can be proven in a direct manner.

Thus, in particular, $\mathcal{M}_c^2 \subset \mathcal{M}_0^2$. For $f \in \mathcal{M}_c^2$ if and only if $|f(p)| < 1$ for all primes p and $\sum_p |f(p)|^2 < \infty$. Thus

$$\sum_{k=1}^{\infty} |f(p^k)| = \frac{|f(p)|}{1 - |f(p)|} \leq A,$$

independent of p.

For example, $(n^{-\alpha}) \in \mathcal{M}_0^2$ for $\alpha > \frac{1}{2}$.

The "quasi-norm" $M_f(T)$. Let $f \in \mathcal{M}_0^2$. The discussion above shows that $\varphi_f(\mathcal{M}^2) \subset \mathcal{M}^2$ but, typically, φ_f is not 'bounded' on $\mathcal{M}^2$ (if $f \notin l^1$) in the sense that[5] $\|\varphi_f a\|/\|a\|$ is not bounded by a constant for all $a \in \mathcal{M}^2$. A natural question is: how large can $\|\varphi_f a\|$ become as a function of $\|a\|$? It therefore makes sense to define, for $T \geq 1$,

$$M_f(T) = \sup_{\substack{a \in \mathcal{M}^2 \\ \|a\| = T}} \frac{\|\varphi_f a\|}{\|a\|}.$$

We shall consider only the case $f(n) = n^{-\alpha}$, although the result below can be extended without any significant changes to f completely multiplicative and such that

[5] Here, and throughout this section $\|\cdot\| = \|\cdot\|_2$ is the l^2-norm.

$f|_{\mathbb{P}}$ is *regularly varying* of index $-\alpha$ with $\alpha > 1/2$ in the sense that there exists a regularly varying function $\tilde{f}$ (of index $-\alpha$) with $\tilde{f}(p) = f(p)$ for every prime p. We shall write $M_f(T) = M_\alpha(T)$ in this case.

Theorem 3.7. *As* $T \to \infty$

$$M_1(T) = e^\gamma(\log\log T + \log\log\log T + 2\log 2 - 1 + o(1)), \tag{3.23}$$

while for $\frac{1}{2} < \alpha < 1$,

$$\log M_\alpha(T) = \left(\frac{B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha})^\alpha}{(1-\alpha)2^\alpha} + o(1)\right)\frac{(\log T)^{1-\alpha}}{(\log\log T)^\alpha}. \tag{3.24}$$

Sketch of proof for the case $\frac{1}{2} < \alpha < 1$ *(for the full proof see* [16]*).* We consider first upper bounds. The supremum occurs for $a \geq 0$, which we now assume. Write $a = (a_n)$, $\varphi_\alpha a = b = (b_n)$. Define α_p and β_p for prime p by

$$\alpha_p = \sum_{k=1}^{\infty} a_{p^k}^2 \quad \text{and} \quad \beta_p = \sum_{k=1}^{\infty} b_{p^k}^2.$$

By the multiplicativity of a and b, $T^2 = \|a\|^2 = \prod_p(1 + \alpha_p)$ and $\|b\|^2 = \prod_p(1 + \beta_p)$. Thus

$$\frac{\|\varphi_\alpha a\|}{\|a\|} = \prod_p \sqrt{\frac{1+\beta_p}{1+\alpha_p}}.$$

Now for $k \geq 1$

$$b_{p^k} = \sum_{r=0}^{k} p^{-r\alpha} a_{p^{k-r}} = a_{p^k} + p^{-\alpha} b_{p^{k-1}},$$

whence

$$b_{p^k}^2 = a_{p^k}^2 + 2p^{-\alpha} a_{p^k} b_{p^{k-1}} + p^{-2\alpha} b_{p^{k-1}}^2.$$

Summing from $k = 1$ to ∞ and adding 1 to both sides gives

$$1 + \beta_p = 1 + \alpha_p + 2p^{-\alpha} \sum_{k=1}^{\infty} a_{p^k} b_{p^{k-1}} + p^{-2\alpha}(1 + \beta_p). \tag{3.25}$$

By Cauchy–Schwarz,

$$\sum_{k=1}^{\infty} a_{p^k} b_{p^{k-1}} \leq \left(\sum_{k=1}^{\infty} a_{p^k}^2 \sum_{k=1}^{\infty} b_{p^{k-1}}^2\right)^{1/2} = \sqrt{\alpha_p(1+\beta_p)},$$

so, on rearranging,

$$(1+\beta_p) - \frac{2p^{-\alpha}\sqrt{\alpha_p(1+\beta_p)}}{1-p^{-2\alpha}} \leq \frac{1+\alpha_p}{1-p^{-2\alpha}}.$$

Completing the square we obtain

$$\left(\sqrt{1+\beta_p}-\frac{p^{-\alpha}\sqrt{\alpha_p}}{1-p^{-2\alpha}}\right)^2\le\frac{1+\alpha_p}{(1-p^{-2\alpha})^2}.$$

The term on the left inside the square is non-negative for every p since $1+\beta_p\ge\frac{1+\alpha_p}{1-p^{-2\alpha}}$, which is greater than $\frac{p^{-2\alpha}\alpha_p}{(1-p^{-2\alpha})^2}$ for $\alpha>\frac{1}{2}$. Rearranging gives

$$\sqrt{\frac{1+\beta_p}{1+\alpha_p}}\le\frac{1}{1-p^{-2\alpha}}\left(1+\frac{1}{p^\alpha}\sqrt{\frac{\alpha_p}{1+\alpha_p}}\right).$$

Let $\gamma_p=\sqrt{\frac{\alpha_p}{1+\alpha_p}}$. Taking the product over all primes p gives

$$\frac{\|\varphi_f a\|}{\|a\|}\le\zeta(2\alpha)\prod_p\left(1+\frac{\gamma_p}{p^\alpha}\right)\le\zeta(2\alpha)\exp\left\{\sum_p\frac{\gamma_p}{p^\alpha}\right\}. \tag{3.26}$$

Note that $0\le\gamma_p<1$ and $\prod_p\frac{1}{1-\gamma_p^2}=T^2$. The idea is to show now that the main contribution to the above sum comes from the range $p\asymp\log T\log\log T$.

Let $\varepsilon>0$ and put $P=\log T\log\log T$. We split up the sum on the right-hand side of (3.26) into $p\le aP$, $aP<p\le AP$, and $p>AP$ (for a small and A large). First,

$$\sum_{p\le aP}p^{-\alpha}\gamma_p\le\sum_{p\le aP}p^{-\alpha}\sim\frac{a^{1-\alpha}P^{1-\alpha}}{(1-\alpha)\log P}<\varepsilon\frac{(\log T)^{1-\alpha}}{(\log\log T)^\alpha}, \tag{3.27}$$

for a sufficiently small. Next, using the fact that $\log T^2=\log\prod_p\frac{1}{1-\gamma_p^2}\ge\sum_p\gamma_p^2$, we have

$$\begin{aligned}\sum_{p>AP}p^{-\alpha}\gamma_p&\le\left(\sum_{p>AP}p^{-2\alpha}\sum_{p>AP}\gamma_p^2\right)^{1/2}\lesssim\left(\frac{2A^{1-2\alpha}P^{1-2\alpha}\log T}{(2\alpha-1)\log P}\right)^{1/2}\\&\sim\frac{(\log T)^{1-\alpha}(\log\log T)^{-\alpha}}{A^{\alpha-1/2}\sqrt{\alpha-1/2}}<\varepsilon\frac{(\log T)^{1-\alpha}}{(\log\log T)^\alpha}\end{aligned} \tag{3.28}$$

for A sufficiently large. This leaves the range $aP<p\le AP$ and the problem therefore reduces to maximising

$$\sum_{aP<p\le AP}\frac{\gamma_p}{p^\alpha}$$

subject to $0\le\gamma_p<1$ and $\prod_p\frac{1}{1-\gamma_p^2}=T^2$. The maximum clearly occurs for γ_p decreasing (if $\gamma_{p'}>\gamma_p$ for primes $p<p'$, then the sum increases in value if we swap γ_p and $\gamma_{p'}$). Thus we may assume that γ_p is decreasing.

By interpolation we may write $\gamma_p = g(\frac{p}{P})$, where $g\colon (0,\infty) \to (0,1)$ is continuously differentiable and decreasing. Of course, g will depend on P. Let $h = \log \frac{1}{1-g^2}$, which is also decreasing. Note that

$$2\log T = \sum_p h\Big(\frac{p}{P}\Big) \geq \sum_{p \leq aP} h\Big(\frac{p}{P}\Big) \geq h(a)\pi(aP) \geq cah(a)\log T,$$

for P sufficiently large, for some constant $c > 0$. Thus $h(a) \leq C_a$ (independently of T).

Now, for $F\colon (0,\infty) \to [0,\infty)$ decreasing, it follows from the Prime Number Theorem in the form $\pi(x) = \mathrm{li}(x) + O(\frac{x}{(\log x)^2})$ that

$$\sum_{ax < p \leq bx} F\Big(\frac{p}{x}\Big) = \frac{x}{\log x}\int_a^b F + O\Big(\frac{xF(a)}{(\log x)^2}\Big), \tag{3.29}$$

where the implied constant is independent of F (and x). Thus

$$2\log T \geq \sum_{aP < p \leq AP} h\Big(\frac{p}{P}\Big) \sim \frac{P}{\log P}\int_a^A h \sim \log T \int_a^A h.$$

Since a and A are arbitrary, $\int_0^\infty h$ must exist and is at most 2. Also, by (3.29),

$$\sum_{aP < p \leq AP} \frac{\gamma_p}{p^\alpha} = \frac{1}{P^\alpha}\sum_{aP < p \leq AP} g\Big(\frac{p}{P}\Big)\Big(\frac{p}{P}\Big)^{-\alpha} \sim \frac{P^{1-\alpha}}{\log P}\int_a^A \frac{g(u)}{u^\alpha}\,du.$$

As a, A are arbitrary, it follows from above and (3.26), (3.27), (3.28) that

$$\log \frac{\|\varphi_f a\|}{\|a\|} \leq \Big(\int_0^\infty \frac{g(u)}{u^\alpha}\,du + o(1)\Big)\tilde{f}(\log T \log\log T)\log T.$$

Thus we need to maximise $\int_0^\infty g(u)u^{-\alpha}du$ subject to $\int_0^\infty h \leq 2$ over all decreasing $g\colon (0,\infty) \to (0,1)$. Since h is decreasing, one finds that $xh(x) \to 0$ as $x \to \infty$ and as $x \to 0^+$.

For the supremum, we can consider only those g (and h) which are continuously differentiable and strictly decreasing, since we can approximate arbitrarily closely by such functions. On writing $g = s \circ h$, where $s(x) = \sqrt{1 - e^{-x}}$, we have

$$\begin{aligned}\int_0^\infty \frac{g(u)}{u^\alpha}\,du &= \Big[\frac{g(u)u^{1-\alpha}}{1-\alpha}\Big]_0^\infty - \frac{1}{1-\alpha}\int_0^\infty g'(u)u^{1-\alpha}\,du \\ &= -\frac{1}{1-\alpha}\int_0^\infty s'(h(u))h'(u)u^{1-\alpha}\,du \\ &= \frac{1}{1-\alpha}\int_0^{h(0^+)} s'(x)l(x)^{1-\alpha}\,dx,\end{aligned}$$

where $l = h^{-1}$, since $\sqrt{u}g(u) \to 0$ as $u \to \infty$. The final integral is, by Hölder's inequality, at most

$$\left(\int_0^{h(0^+)} s'^{1/\alpha}\right)^{\alpha}\left(\int_0^{h(0^+)} l\right)^{1-\alpha}. \tag{3.30}$$

But $\int_0^{h(0^+)} l = -\int_0^\infty uh'(u)du = \int_0^\infty h \le 2$, so

$$\int_0^\infty \frac{g(u)}{u^\alpha}\,du \le \frac{2^{1-\alpha}}{1-\alpha}\left(\int_0^\infty s'^{1/\alpha}\right)^{\alpha}.$$

A direct calculation shows that[6] $\int_0^\infty (s')^{1/\alpha} = 2^{-1/\alpha}B(\frac{1}{\alpha}, 1-\frac{1}{2\alpha})$. This gives the upper bound.

The proof of the upper bound leads to the optimal choice for g and the lower bound. We note that we have equality in (3.30) if $l/(s')^{1/\alpha}$ is constant, i.e., $l(x) = cs'(x)^{1/\alpha}$ for some constant $c > 0$ — chosen so that $\int_0^\infty l = 2$. This means that we take

$$h(x) = (s')^{-1}\left(\left(\frac{x}{c}\right)^{\alpha}\right) = \log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \left(\frac{c}{x}\right)^{2\alpha}}\right),$$

from which we can calculate g. In fact, the required lower bound can be found by taking a_n completely multiplicative, with a_p for p prime defined by

$$a_p = g_0\left(\frac{p}{P}\right),$$

where $P = \log T \log\log T$ and g_0 is the function

$$g_0(x) = \sqrt{1 - \frac{2}{1 + \sqrt{1 + (\frac{c}{x})^{2\alpha}}}},$$

with $c = 2^{1+1/\alpha}/B(\frac{1}{\alpha}, 1-\frac{1}{2\alpha})$. As such, by the same methods as before, we have $\|a\| = T^{1+o(1)}$ and

$$\log\frac{\|\varphi_\alpha a\|}{\|a\|} = \sum_p p^{-\alpha}g_0\left(\frac{p}{P}\right) + O(1) \sim \frac{P^{1-\alpha}}{\log P}\int_0^\infty \frac{g_0(u)}{u^\alpha}\,du.$$

By the choice of g_0, the integral on the right is $\dfrac{B(\frac{1}{\alpha}, 1-\frac{1}{2\alpha})^{\alpha}}{(1-\alpha)2^{\alpha}}$, as required. □

Remark. These asymptotic formulae bear a strong resemblance to the (conjectured) maximal order of $|\zeta(\alpha + iT)|$. It is interesting to note that the bounds found here are just larger than what is known about the lower bounds for $Z_\alpha(T)$ (see the interlude on upper and lower bounds on $\zeta(s)$, especially items (b) and (c)). We note that the constant appearing in (3.24) is not Lamzouri's $C(\alpha)$ since, for α near $\frac{1}{2}$, the former is roughly $\frac{1}{\sqrt{\alpha - \frac{1}{2}}}$, while $C(\alpha) \sim \frac{1}{\sqrt{2\alpha - 1}}$.

[6]The integral is $2^{-1/\alpha}\int_0^\infty e^{-x/\alpha}(1-e^{-x})^{-1/2\alpha}dx = 2^{-1/\alpha}\int_0^1 t^{1/\alpha - 1}(1-t)^{-1/2\alpha}dt$.

Lower bounds for φ_α and some further speculations. We can study lower bounds of φ_α via the function

$$m_\alpha(T) = \inf_{\substack{a \in \mathcal{M}^2 \\ \|a\| = T}} \frac{\|\varphi_\alpha a\|}{\|a\|}.$$

Using very similar techniques, one obtains results analogous to Theorem 3.7:

$$\frac{1}{m_1(T)} = \frac{6e^\gamma}{\pi^2}(\log\log T + \log\log\log T + 2\log 2 - 1 + o(1))$$

and

$$\log \frac{1}{m_\alpha(T)} \sim \frac{B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha})^\alpha (\log T)^{1-\alpha}}{(1-\alpha)2^\alpha (\log\log T)^\alpha} \qquad \text{for } \tfrac{1}{2} < \alpha < 1.$$

We see that $m_\alpha(T)$ corresponds closely to the conjectured minimal order of $|\zeta(\alpha + iT)|$ (see [11] and [25]).

The above formulae suggest that the supremum (respectively infimum) of $\|\varphi_\alpha a\|/\|a\|$ with $a \in \mathcal{M}^2$ and $\|a\| = T$ are close to the supremum (resp. infimum) of $|\zeta_\alpha|$ on $[1, T]$. One could therefore speculate further that there is a close connection between $\|\varphi_\alpha a\|/\|a\|$ (for such a) and $|\zeta(\alpha + iT)|$.

Heuristically, we could argue as follows. Consider

$$\frac{1}{T}\int_0^T |\zeta(\alpha - it)|^2 \left|\sum_{n=1}^\infty a_n n^{it}\right|^2 dt. \tag{3.31}$$

This is less than

$$Z_\alpha(T)^2 \cdot \frac{1}{T}\int_0^T \left|\sum_{n=1}^\infty a_n n^{it}\right|^2 dt \sim Z_\alpha(T)^2 \|a\|^2,$$

by the Montgomery–Vaughan mean value theorem (under appropriate conditions). On the other hand, (3.31) is expected to be approximately

$$\frac{1}{T}\int_0^T \left|\sum_{n=1}^\infty b_n n^{it}\right|^2 dt \approx \sum_{n=1}^\infty |b_n|^2 = \|\varphi_\alpha a\|^2.$$

Putting these together gives

$$\frac{\|\varphi_\alpha a\|}{\|a\|} \lesssim Z_\alpha(T).$$

The left-hand side, as a function of $\|a\|$, can be made as large as $F(\|a\|)$, where $F(x) = \exp\{c_\alpha \frac{(\log x)^{1-\alpha}}{(\log\log x)^\alpha}\}$. If the above continues to hold for $\|a\|$ as large as T, then $M_\alpha(T) \le Z_\alpha(T)$ would follow. Even if it holds for $\|a\|$ as large as a smaller power of T, one would recover Montgomery's Ω-result.

Alternatively, considering (3.31) over $[T, 2T]$,

$$\frac{1}{T}\int_T^{2T} |\zeta(\alpha-it)|^2 \bigg|\sum_{n=1}^{\infty} a_n n^{it}\bigg|^2 dt = |\zeta(\alpha+it_0)|^2 \cdot \frac{1}{T}\int_T^{2T} \bigg|\sum_{n=1}^{\infty} a_n n^{it}\bigg|^2 dt \quad (3.32)$$

for some $t_0 \in [T, 2T]$ and, using the Montgomery–Vaughan mean value theorem (assuming it applies), this is approximately

$$|\zeta(\alpha+it_0)|^2 \sum_{n=1}^{\infty} |a_n|^2 = |\zeta(\alpha+it_0)|^2 \|a\|^2.$$

On the other hand, formally multiplying out the integrand, by writhing $\zeta(\alpha - it) = \sum_{n=1}^{\infty} \frac{n^{it}}{n^{\alpha}}$ and formally multiplying out the integrand, the left-hand side of (3.32) becomes (heuristically)

$$\frac{1}{T}\int_T^{2T} \bigg|\sum_{n=1}^{\infty} b_n n^{it}\bigg|^2 dt \approx \sum_{n=1}^{\infty} |b_n|^2 = \|\varphi_\alpha a\|^2.$$

Equating these gives

$$\frac{\|\varphi_\alpha a\|}{\|a\|} \approx |\zeta(\alpha+it_0)|.$$

Clearly there are a number of problems with this. For a start, we need $\varphi_\alpha a \in l^2$. More importantly, the error term in the Montgomery–Vaughan theorem contains $\sum_{n=1}^{\infty} n|a_n|^2$, which may diverge. Also, a_n and hence $\|a\|$ may depend on T, and finally, the series for $\zeta(\alpha - it)$ doesn't converge for $\alpha \le 1$.

If $a_n = 0$ for $n > N$, the above argument can be made to work, even for N a (small) power of T (see for example [15]), but difficulties arise for larger powers of T.

There seem to be some reasons to believe that the error from the Montgomery–Vaughan theorem should be much smaller when considering products. These occur when a_n is multiplicative. For example (with $Q = \prod_{p\le P} p$ so that $\log Q = \theta(P) \sim P$ by the Prime Number Theorem),

$$\frac{1}{2T}\int_{-T}^{T} \bigg|\prod_{p\le P}(1+p^{it})\bigg|^2 dt = \frac{1}{2T}\int_{-T}^{T}\bigg|\sum_{d|Q} d^{it}\bigg|^2 dt = \sum_{d|Q} 1 + O\bigg(\frac{1}{T}\sum_{d|Q} d\bigg).$$

The 'main term' is $d(Q) = 2^{\pi(P)}$, while the error is at least $\frac{Q}{T}$. However the left-hand side is trivially at most $4^{\pi(P)}$, so the error dominates the other terms if $P > (1+\delta)\log T$. If, say, P is of order $\log T \log\log T$ (which is the range of interest), then $\pi(P) \asymp \log T$, so $2^{\pi(P)}$ is like a power of T, but Q is roughly like $T^{\log\log T}$ – far too large.

Thus it may be that for a_n completely multiplicative, it holds that

$$\frac{1}{T}\int_T^{2T} |\zeta(\alpha-it)|^2 \bigg|\prod_{p\le P}\frac{1}{1-a_p p^{it}}\bigg|^2 dt \sim |\zeta(\alpha+it_0)| \prod_{p\le P}\frac{1}{1-|a_p|^2}$$

for P up to $c \log T \log\log T$. This suggests the following might be true:

(a) *given $a \in \mathcal{M}^2$ with $\|a\| = T$, there exists $t \in [T, 2T]$ such that*

$$\frac{\|\varphi_\alpha a\|}{\|a\|} \approx |\zeta(\alpha + it)|.$$

(b) *given $T \geq 1$, there exists $a \in \mathcal{M}^2$ with $\|a\| = T$ such that*

$$\frac{\|\varphi_\alpha a\|}{\|a\|} \approx |\zeta(\alpha + iT)|.$$

Here, $\approx$ means something like log-asymptotic, $\sim$, or possibly even $=$. Thus (a) implies $M_\alpha(T) \lesssim Z_\alpha(T)$, while (b) implies the opposite. Together they would imply we can *encode* real numbers into $\mathcal{M}^2$-functions with equal l^2-norm, such that φ_α has a similar action as ζ_α.

Closure of $\mathcal{M}B^2_{\mathbb{N}}$? We finish these speculations with a final plausible conjecture regarding the closure under multiplication of functions in $B^2_{\mathbb{N}}$ with multiplicative coefficients.

Let $\mathcal{M}_0 B^2_{\mathbb{N}}$ denote the subset $\mathcal{M}B^2_{\mathbb{N}}$ of functions f for which $\hat{f} \in \mathcal{M}_0$. Recall that $\mathcal{M}^2_0$ is the subset of $\mathcal{M}^2$ for which $g \in \mathcal{M}^2 \Rightarrow f * g \in \mathcal{M}^2$. This suggests the following conjecture:

Conjecture. Let $f \in \mathcal{M}B^2_{\mathbb{N}}$ and $g \in \mathcal{M}_0 B^2_{\mathbb{N}}$. Then $fg \in \mathcal{M}B^2_{\mathbb{N}}$.

In particular, $\mathcal{M}B^2_{\mathbb{N}}\mathcal{M}_c B^2_{\mathbb{N}} = \mathcal{M}B^2_{\mathbb{N}}$. Since $\zeta_\alpha \in \mathcal{M}_c B^2_{\mathbb{N}}$ for $\alpha > \frac{1}{2}$, this would imply $\zeta_\alpha^k \in \mathcal{M}B^2_{\mathbb{N}}$ for every $k \in \mathbb{N}$ and $\alpha > \frac{1}{2}$, which implies the Lindelöf hypothesis.

4 Connections to matrices of the form $(f(ij/(i,j)^2))_{i,j \leq N}$

The asymptotic formulae for $\Phi_f(N)$ in Theorem 3.1 can be used to obtain information on the largest eigenvalue of certain arithmetical matrices. Various authors have discussed asymptotic estimates of eigenvalues and determinants of arithmetical matrices (see for example [4], [5], [24] to name just a few).

Let $A_N(f)$ denote the $N \times N$ matrix with ij^{th}-entry $f(i/j)$ if $j|i$ and zero otherwise. (i.e. $A_N(f) = M_N(\hat{f})$). As noted in the introduction, these matrices behave much like Dirichlet series with coefficients $f(n)$; namely,

$$A_N(f)A_N(g) = A_N(f * g).$$

In particular, $A_N(f)$ is invertible if f has a Dirichlet inverse, i.e., $f(1) \neq 0$, in which case $A_N(f)^{-1} = A_N(f^{-1})$.

Suppose for simplicity that f is a real arithmetical function. (For complex values we can easily adjust.) Observe that $\Phi_f(N)^2$ is the largest eigenvalue of the matrix

$$A_N(f)^T A_N(f).$$

Indeed, we have (with $b_n = \sum_{d|n} f(d)a_{n/d}$)

$$b_n^2 = \sum_{i,j|n} f\Big(\frac{n}{i}\Big) f\Big(\frac{n}{j}\Big) a_i a_j,$$

so that, on noting $i, j|n$ if and only if $[i, j]|n$ (where $[i, j]$ denotes the lcm of i and j)

$$\sum_{n=1}^{N} b_n^2 = \sum_{n=1}^{N} \sum_{[i,j]|n} f\Big(\frac{n}{i}\Big) f\Big(\frac{n}{j}\Big) a_i a_j = \sum_{i,j \le N} b_{ij}^{(N)} a_i a_j,$$

where (using $(i, j)[i, j] = ij$)

$$b_{ij}^{(N)} = \sum_{k \le \frac{N}{[i,j]}} f\Big(\frac{ki}{(i, j)}\Big) f\Big(\frac{kj}{(i, j)}\Big).$$

But $b_{ij}^{(N)}$ is also the ij^{th}-entry of $A_N(f)^T A_N(f)$, as an easy calculation shows. Thus

$$\Phi_f(N)^2 = \sup_{a_1^2+\cdots+a_N^2=1} \sum_{i,j \le N} b_{ij}^{(N)} a_i a_j \tag{3.33}$$

is the largest eigenvalue of $A_N(f)^T A_N(f)$, i.e., $\Phi_f(N)$ the largest *singular value*[7] of $A_N(f)$. Thus an equivalent formulation of Corollary 2.2 for f supported on $\mathbb{N}$ is: *For $f \in l^1$ non-negative, the largest singular value of $A_N(f)$ tends to $\|f\|_1$.*

Now if f is completely multiplicative, then

$$b_{ij}^{(N)} = f\Big(\frac{ij}{(i, j)^2}\Big) \sum_{k \le \frac{N}{[i,j]}} f(k)^2,$$

which for large N is roughly $\|f\|_2^2 f(\frac{ij}{(i,j)^2})$ for $f \in l^2$. This suggests that the matrix $\big(f(\frac{ij}{(i,j)^2})\big)_{i,j \le N}$ has its largest eigenvalue close to $\Phi_f(N)^2/\|f\|_2^2$. This is indeed the case.

[7] The singular values of a matrix A are the square roots of the eigenvalues of $A^T A$ (or $A^* A$ if A has complex entries).

Corollary 4.1. *Let $f \in l^2$ be non-negative and completely multiplicative. Let Λ_N denote the largest eigenvalue of $\left(f(\frac{ij}{(i,j)^2})\right)_{i,j\le N}$. Then*

$$\frac{\Phi_f(N)^2}{\|f\|_2^2} \le \Lambda_N \le \frac{\Phi_f(N^3)^2}{\sum_{k=1}^N f(k)^2}.$$

In particular, for $f \in l^1$,

$$\lim_{N\to\infty} \Lambda_N = \frac{\|f\|_1^2}{\|f\|_2^2}.$$

Proof. We have

$$\Lambda_N = \sup_{a_1^2+\cdots+a_N^2=1} \sum_{i,j\le N} f\Big(\frac{ij}{(i,j)^2}\Big) a_i \overline{a_j} \tag{3.34}$$

When $f \ge 0$, the supremums in (3.33) and (3.34) are reached for $a_n \ge 0$. Thus,

$$\Phi_f(N)^2 \le \|f\|_2^2 \Lambda_N$$

follows immediately.

On the other hand, for $i, j \le N$, $[i, j] \le N^2$ so

$$\Phi_f(N^3)^2 \ge \sum_{i,j\le N} f\Big(\frac{ij}{(i,j)^2}\Big) a_i a_j \sum_{k\le N} f(k)^2.$$

Taking the supremum over all such a_n gives, $\Phi_f(N^3)^2 \ge \Lambda_N \sum_{k\le N} f(k)^2$, as required.

Finally, if $f \in l^1$, then $\Phi_f(N) \to \|f\|_1$ and so $\Lambda_N \to \frac{\|f\|_1^2}{\|f\|_2^2}$ follows. □

The approximate formulae for $\Phi_\alpha(N)$ in Theorem 3.1 lead to:

Corollary 4.2. *Let $f(n) = n^{-\alpha}$ and let $\Lambda_N(\alpha)$ denote the largest eigenvalue of $\left(f(\frac{ij}{(i,j)^2})\right)_{i,j\le N}$. Then*

$$\Lambda_N(1) = \frac{6}{\pi^2}(e^\gamma \log\log N + O(1))^2,$$

$$\log \Lambda_N(\alpha) \asymp \frac{(\log N)^{1-\alpha}}{\log\log N} \qquad \text{for } \tfrac{1}{2} < \alpha < 1,$$

$$\log \Lambda_N\Big(\frac{1}{2}\Big) \asymp \sqrt{\frac{\log N}{\log\log N}}.$$

Bibliography

[1] R. Balasubramanian and K. Ramachandra, On the frequency of Titchmarsh's phenomenom for $\zeta(s)$. III, *Proc. Indian Acad. Sci.* 86 A (1977), 341–351.

[2] A. S. Besicovitch, *Almost Periodic Functions*. Dover Publications, 1954.

[3] A. Böttcher and B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices*. Springer-Verlag, 1999.

[4] K. Bourque and S. Ligh, Matrices associated with classes of arithmetical functions. *J. Number Theory* 45 (1993), 367–376.

[5] D. A. Cardon, Matrices related to Dirichlet series. *J. Number Theory* 130 (2010), 27–39.

[6] D. W. Farmer, S. M. Gonek and C. P. Hughes, The maximum size of L-functions. *J. Reine Angew. Math.* 609 (2007), 215–236.

[7] I. Gohberg, On an application of the theory of normed rings to singular integral equations. *Usp. Mat. Nauk* 7 (1952), 149–156.

[8] I. Gohberg, S. Goldberg and M. A. Kaashoek, *Classes of Linear Operators Vol. I*. Birkhäuser, 1990.

[9] S. M. Gonek, Finite Euler products and the Riemann Hypothesis. Preprint, 2007; arXiv:0704.3448v1.

[10] S. M. Gonek and J. P. Keating, Mean values of finite Euler products. *J. London Math. Soc.* 82 (2010), 763–786.

[11] A. Granville and K. Soundararajan, Extreme values of $|\zeta(1+it)|$. In *Ramanujan Math. Soc. Lect. Notes Ser 2*, Ramanujan Math. Soc., Mysore, 2006, 65–80.

[12] P. Hartman, On completely continuous Hankel operators. *Proc. Am. Math. Soc.* 9 (1958), 862–866.

[13] E. Hewitt and J. H. Williamson, Note on absolutely convergent Dirichlet series. *Proc. Am. Math. Soc.* 8 (1957), 863–868.

[14] T. W. Hilberdink, Determinants of multiplicative Toeplitz matrices. *Acta Arith.* 125 (2006), 265–284.

[15] T. W. Hilberdink, An arithmetical mapping and applications to Ω-results for the Riemann zeta function. *Acta Arith.* 139 (2009), 341–367.

[16] T. W. Hilberdink, 'Quasi'-norm of an arithmetical convolution operator and the order of the Riemann zeta function. *Funct. Approx.* 49 (2013), 201–220.

[17] M. Huxley, Exponential sums and the Riemann zeta function V. *PLMS* 90 (2005), 1–41.

[18] T. J. Jarvis, A dominant negative eigenvalue of a matrix of Redheffer. *Linear Algebra Appl.* 142 (1990), 141–152.

[19] B. Jessen and H. Tornehave, Mean motions and zeros of almost periodic functions. *Acta Math.* 77 (1945), 137–279.

[20] J. Keating and N. Snaith, *Random matrix theory and some zeta-function moments*. Lecture at the Erwin Schrödinger Institute (1998).

[21] M. Krein, Integral equations on a half-line with kernel depending upon the difference of the arguments. *Am. Math. Soc. Transl.* (2) 22 (1962), 163–288.

[22] Y. Lamzouri, On the distribution of extreme values of zeta and L-functions in the strip $\frac{1}{2} < \sigma < 1$. *Int. Math. Res. Nat.*, to appear; arXiv:1005.4640v2 [math.NT].

[23] N. Levinson, Ω-theorems for the Riemann zeta function. *Acta Arith.* 20 (1972), 319–332.

[24] P. Lindqvist and K. Seip, Note on some greatest common divisor matrices. *Acta Arith.* 84 (1998), 149–154.

[25] H. L. Montgomery, Extreme values of the Riemann zeta-function. *Comment. Math. Helv.* 52 (1977), 511–518.

[26] Z. Nehari, On bounded bilinear forms. *Ann. Math.* 65 (1957), 153–162.

[27] R. M. Redheffer, Eine explizit lösbare Optimierungsaufgabe. *Int. Schriftenr. Numer. Math.* 36 (1977).

[28] L. Rodman, I. M. Spitkowsky and H. J. Woerdeman, Fredholmness and Invertibility of Toeplitz operators with matrix almost periodic symbols. *Proc. Am. Math. Soc.* 130 (2001), 1365–1370.

[29] K. Soundararajan, Extreme values of zeta and L-functions. *Math. Ann.* 342 (2008), 467–486.

[30] A. E. Taylor, *Introduction to Functional Analysis*. Wiley and Sons, 1958.

[31] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*. Second edition, Oxford University Press, 1986.

[32] O. Toeplitz, Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veränderlichen. *Math. Ann.* 70 (1911), 351–376.

[33] O. Toeplitz, Zur Theorie der Dirichletschen Reihen. *Am. J. Math.* 60 (1938), 880–888.