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Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 29, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

A CHARACTERISATION OF ∞-HARMONIC AND p-HARMONIC MAPS VIA AFFINE VARIATIONS IN L^∞

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Communicated by Peter Bates

ABSTRACT. Let $u:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}^N$ be a smooth map and $n,N\in\mathbb{N}$. The ∞-Laplacian is the PDE system

$$\Delta_\infty u:=\Big(Du\otimes Du+|Du|^2[Du]^\perp\otimes I\Big):D^2u=0,$$

where $[Du]^\perp:=\mathrm{Proj}_{R(Du)^\perp}$. This system constitutes the fundamental equation of vectorial Calculus of Variations in L^∞, associated with the model functional

$$E_\infty(u,\Omega')=\big\||Du|^2\big\|_{L^\infty(\Omega')},\quad \Omega'\Subset\Omega.$$

We show that generalised solutions to the system can be characterised in terms of the functional via a set of designated affine variations. For the scalar case $N=1$, we utilise the theory of viscosity solutions by Crandall-Ishii-Lions. For the vectorial case $N\geq 2$, we utilise the recently proposed by the author theory of $\mathcal{D}$-solutions. Moreover, we extend the result described above to the p-Laplacian, $1<p<\infty$.

1. Introduction

Let $n,N\in\mathbb{N}$. Given a (smooth) map $u:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}^N$ defined on an open set, let $\mathbb{R}^{Nn}$ and $\mathbb{R}^{Nn^2}_s$ denote respectively the space of matrices and the space of symmetric tensors wherein the gradient matrix and the hessian tensor

$$Du(x)=\big(D_iu_\alpha(x)\big)_{i=1,\dots,n}^{\alpha=1,\dots,N},\quad D^2u(x)=\big(D^2_{ij}u_\alpha(x)\big)_{i,j=1,\dots,n}^{\alpha=1,\dots,N}$$

of u are valued. Obviously, $D_i\equiv\partial/\partial x_i$, $x=(x_1,\dots,x_n)^\top$, $u=(u_1,\dots,u_N)^\top$. In this paper we are primarily interested in the so-called ∞-Laplacian which is the quasilinear 2nd order nondivergence system

$$\Delta_\infty u:=\Big(Du\otimes Du+|Du|^2[Du]^\perp\otimes I\Big):D^2u=0. \tag{1.1}$$

Here $[Du]^\perp$ denotes the orthogonal projection on the orthogonal complement of the range of Du and $|Du|$ is the Euclidean norm of Du on $\mathbb{R}^{Nn}$. In index form, (1.1)

2010 *Mathematics Subject Classification.* 35D99, 35D40, 35J47, 35J47, 35J92, 35J70, 35J99.
Key words and phrases. ∞-Laplacian; p-Laplacian; generalised solutions; viscosity solutions; Calculus of Variations in L^∞; Young measures; fully nonlinear systems.

Submitted August 10, 2016. Published January 26, 2017.

reads

$$\sum_{\beta=1}^{N}\sum_{i,j=1}^{n}\Big(D_iu_\alpha D_ju_\beta+|Du|^2[Du]^\perp_{\alpha\beta}\delta_{ij}\Big)D^2_{ij}u_\beta=0,\quad \alpha=1,\dots,N,$$

$$[Du]^\perp:=\mathrm{Proj}_{(R(Du))^\perp}\,.$$

We are also interested in the more classical p-Laplacian for $1<p<\infty$, which is the divergence system

$$\Delta_p u:=\mathrm{div}\left(|Du|^{p-2}Du\right)=0. \tag{1.2}$$

System (1.1) is the fundamental equation which arises in vectorial Calculus of Variations in the space L^∞, that is in connection to variational problems for the model functional

$$E_\infty(u,\Omega'):=\big\||Du|^2\big\|_{L^\infty(\Omega')},\quad \Omega'\Subset\Omega,\ u\in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N). \tag{1.3}$$

The scalar counterpart of (1.1) when $N=1$ simplifies to

$$Du\otimes Du:D^2u=\sum_{i,j=1}^{n}D_iuD_juD^2_{ij}u=0$$

and first arose in the work of Aronsson in the 1960s ([A1, 2] and for a pedagogical introduction see [7, 25]) who pioneered the field of Calculus of Variations in the space L^∞. The full system (1.1) first appeared in recent work of the author [18] who initiated the systematic study of the vectorial case in a series of papers [18]-[24] (see also the recent contributions with Abugirda, Ayanbayev, Croce, Pisante, Manfredi, Moser and Pryer [1, 3, 9, 30, 32, 33, 31]). Let us note also the early vectorial contributions by Barron-Jensen-Wang [4, 5] who, among other deep results, proved existence of absolute minimisers for general supremal functionals in the "rank-one" cases $\min\{n,N\}=1$ and also defined and studied the correct vectorial L^∞-version of quasiconvexity. However, their fundamental contributions were at the level of the functional and the correct (non-obvious) vectorial counterpart of Aronsson's equation was not known at the time.

On the other hand, the p-Laplacian (1.2) is a classical model which arises in conventional Calculus of Variations for integral functionals, in particular as the Euler-Lagrange equation of

$$E_p(u,\Omega'):=\big\||Du|^p\big\|_{L^1(\Omega')},\quad \Omega'\Subset\Omega,\ u\in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N). \tag{1.4}$$

A standard difficulty in both the scalar and the vectorial case of (1.1) is that it is nondivergence and since in general smooth solutions do not exist, the definition of generalised solutions is an issue. In the vectorial case, an additional difficulty is that the system has discontinuous coefficients even if the solution might be smooth (see [19]). This happens because the projection $[Du(x)]^\perp$ "feels" the dimension of the tangent space $R(Du(x))\subseteq\mathbb{R}^N$.

In this article we are concerned with the variational characterisation of appropriately defined generalised solutions to (1.1) and (1.2) in both the scalar and the vectorial case in terms of the supremal functional (1.3). The main results of this paper are contained in the statements of Theorems 4.1, 4.3 and 5.1 (and Corollaries 4.2, 5.2). Roughly speaking, these results claim that for $1<p\le\infty$ we have

$$\Delta_p u=0 \text{ on } \Omega \iff \begin{cases}\text{For all } \Omega'\Subset\Omega \text{ and } A\in\mathcal{A}^p_{\Omega'}(u),\\ E_\infty(u,\Omega')\le E_\infty(u+A,\Omega')\end{cases}$$

where $\mathcal{A}^p_{\Omega'}(u)$ is a designated set of **affine** mappings depending on u and on the subdomain Ω'. This result is quite surprising in that both the ∞-Laplacian (1.1) and the p-Laplacian (1.2) are associated to the respective supremal/integral functionals (1.3), (1.4) (and not both associated to (1.3)) when the classes of variations are *compactly supported*. In the scalar case, the appropriate notion of minimisers for (1.3) characterising ∞-Harmonic functions has been discovered by Aronsson and today we know several more characterisations involving e.g. comparison, Lipschitz extensions and Game Theory (see [7, 25]). In the vectorial case, the correct extension of Aronsson's notion of Absolute Minimals which characterises (1.1) via (1.3) has been introduced in [21].

A central point in both the statements and the proofs of our main results Theorems 4.1, 4.3 and 5.1 is that solutions to (1.1)-(1.2) in general are nonsmooth and they need to be considered in a generalised sense. We discuss below about generalised solutions separately when $N = 1$ and $N \geq 2$.

For the scalar case, we invoke the well established notion of viscosity solutions of Crandall-Ishii-Lions [8] which effectively is based on the maximum principle. Since the p-Laplacian is singular for $1 < p < 2$, we actually use a "feeble" variant of the original viscosity notions taken from [22]. Although (1.2) is in divergence from and the natural definition of weak solution to it is via duality, we find it more fruitful to treat it instead in the viscosity sense. Due to the results in the aforementioned papers, it is known that viscosity and weak solutions of the p-Laplacian coincide.

For the vectorial case, things are much more intricate. Motivated by (1.1), in the very recent works [27, 26] we introduced a new duality-free theory of weak solutions which allows for just measurable maps to be rigorously interpreted and studied as solutions to PDE systems of any order

$$F\big(\cdot, u, Du, D^2u, \dots, D^pu\big) = 0 \quad \text{ on } \Omega, \tag{1.5}$$

which can be allowed to have even discontinuous coefficients. Using this new approach, in the papers [26]-[29] we studied efficiently certain problems which we discuss briefly at the end of the introduction.

Our generalised solutions are not based either on integration-by-parts or on the maximum principle. Instead, we build on the probabilistic interpretation of limits of difference quotients by utilizing Young measures valued into compactifications. We caution the reader that we are not using the "standard" Young measures of Calculus of Variations and of PDE theory which are valued into Euclidean spaces (see e.g. [11, 35, 15, 6, 14, V, 34]). In the current setting, Young measures valued into spheres are utilised by applying them to the difference quotients of our candidate solution. The motivation for $W^{1,\infty}_{\mathrm{loc}}$ solutions of 2nd order systems which are relevant to this paper is the following: let $u \in W^{2,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a strong solution to a 2nd order system of the form

$$F\big(Du(x), D^2u(x)\big) = 0, \quad \text{a.e. } x \in \Omega. \tag{1.6}$$

We now rewrite (1.6) in the unconventional form

$$\sup_{\mathbf{X}_x \in \mathrm{supp}(\delta_{D^2u(x)})} \big|F\big(Du(x), \mathbf{X}_x\big)\big| = 0, \quad \text{a.e. } x \in \Omega$$

and we view the hessian D^2u as a probability-valued mapping given by the Dirac mass: δ_{D^2u}. The hope is then that we may relax the requirement to have concentration measures and allow instead general probability-valued maps arising as limits of difference quotients of $W^{1,\infty}_{\rm loc}$ maps. Indeed, if $u:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}^N$ is just $W^{1,\infty}_{\rm loc}$, we may view the usual difference quotients of Du as Young measures into the 1-point compactification

$$\delta_{D^{1,h}Du}:\Omega\subseteq\mathbb{R}^n\to\mathscr{P}\big(\overline{\mathbb{R}}^{Nn^2}_s\big),\quad x\mapsto\delta_{D^{1,h}Du(x)}$$

(see Section 2 for the precise definitions). Since the Young measures are a weakly* compact set, there exist probability-valued limit maps such that along infinitesimal subsequences $(h_\nu)_1^\infty$ we have

$$\delta_{D^{1,h_\nu}Du}\overset{*}{\rightharpoonup}\mathcal{D}^2u,\quad\text{in Young measures, as }\nu\to\infty \tag{1.7}$$

(even if u is merely $W^{1,\infty}_{\rm loc}$). Then, we require

$$\sup_{\mathbf{X}_x\in\operatorname{supp}(\mathcal{D}^2u(x))\setminus\{\infty\}}F\big(Du(x),\mathbf{X}_x\big)=0,\quad\text{a.e. }x\in\Omega, \tag{1.8}$$

for any"diffuse hessian" $\mathcal{D}^2u$. Since (1.7) and (1.8) are independent of the twice differentiability of u, they can be taken as a notion of generalised solution which we call $\mathcal{D}$-solutions. In the event that $u\in W^{2,\infty}_{\rm loc}$, then $\mathcal{D}^2u=\delta_{D^2u}$ and we reduce to strong solutions.

A flaw of our characterisations is that we require our generalised solutions to be C^1 and not just $W^{1,\infty}_{\rm loc}$. This is not a restriction for the p-Laplacian since it is well know that p-Harmonic maps are $C^{1,\alpha}$ ([U]). However, except for the case of $n=2$, $N=1$ (see Savin and Evans-Savin [36, 13]), the C^1 regularity of ∞-Harmonic functions (and a fortiori of maps) is an open problem, at least to date. However, even with the extra C^1 hypothesis, the results are new even in the scalar case. We believe that they are interesting anyway and might allow to glean more information that will unravel the still largely mysterious behaviour of ∞-Harmonic functions (and maps). For the p-Laplacian we restrict our attention only to $N=1$ and we refrain from extending Theorem 4.3 to $N\geq 2$. This however can be done relatively easily along the lines of Theorem 5.1.

Further, we postpone the discussion of the more difficult question of relation of viscosity and $\mathcal{D}$-solutions for future work. It is easily seen though that $\mathcal{D}$-solutions do not have comparison built in the notion as viscosity solutions (in the vectorial case in general not even C^∞-solutions are unique, see [23]) and hence $\mathcal{D}$-solutions are not stronger than viscosity solutions. On the other hand, absolutely minimising $\mathcal{D}$-solutions are viscosity solutions and we conjecture that the opposite is true as well. (Let us note that in [31] is was recently proved that absolutely minimising $\mathcal{D}$-solutions of higher order L^∞ variational problems are unique.)

We conclude this introduction with certain interesting results we have obtained via the new theory of $\mathcal{D}$-solutions. In the paper [26] we proved existence to the Dirichlet problem for (1.1) (uniqueness of smooth solutions has been disproved in [23]). Again in [26], we also proved uniqueness and existence to the Dirichlet problem for the fully nonlinear degenerate elliptic system $F(\cdot,D^2u)=f$. In [27] we proved existence to the Dirichlet problem for the system arising from the functional

$$I_\infty(u,\Omega'):=\big\|H(\cdot,u,u')\big\|_{L^\infty(\Omega')},\quad u:\Omega\subseteq\mathbb{R}\to\mathbb{R}^N,\ \Omega'\Subset\Omega.$$

In [28] we established the equivalence between weak and $\mathcal{D}$-solutions to linear symmetric hyperbolic systems and in [29] we developed a systematic mollification method for $\mathcal{D}$-solutions. We finally note that to the best of our knowledge, the only vectorial contribution by other authors relevant to the content of this paper is the work by Sheffield-Smart [37] which however is restricted to the class of smooth solutions.

2. Basics on generalised solutions to fully nonlinear systems

We begin with some basic material. A much more detailed introduction of the theory of $\mathcal{D}$-solutions can be found in [26]-[29].

Preliminaries. Let $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ be a map defined over an open set. Unless indicated otherwise, Greek indices $\alpha, \beta, \gamma, \dots$ will run in $\{1, \dots, N\}$ and Roman indices $i, j, k, \dots$ will run in $\{1, \dots, n\}$. The norm symbols $|\cdot|$ will always mean the Euclidean ones, whilst Euclidean inner products will be denoted by either "$\cdot$" on $\mathbb{R}^n, \mathbb{R}^N$ or by ":" on $\mathbb{R}^{Nn}, \mathbb{R}_s^{Nn^2}$. For example,

$$|\mathbf{X}|^2 = \mathbf{X} : \mathbf{X} = \sum_{\alpha=1}^{N} \sum_{i,j=1}^{n} \mathbf{X}_{\alpha ij} \mathbf{X}_{\alpha ij}, \quad \mathbf{X} \in \mathbb{R}_s^{Nn^2},$$

etc. Our measure theoretic and function space notation is either standard as e.g. in [11, 12] or self-explanatory. For example, "measurable" means "Lebesgue measurable", the Lebesgue measure will be denoted by $|\cdot|$, the L^p spaces of maps u as above by $L^p(\Omega, \mathbb{R}^N)$, etc. Especially for the space $L^\infty(\Omega, \mathbb{R}^{Nn})$, we will simplify the notation and since the norm on $\mathbb{R}^{Nn}$ is always the Euclidean, we will write

$$\|Du\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_\Omega |Du|.$$

We will systematically use the Alexandroff 1-point compactification of the space $\mathbb{R}_s^{Nn^2}$. Its topology will be the one which makes it homeomorphic to the sphere of dimension $Nn(n+1)/2$ (via the stereographic projection which identifies the north pole with $\{\infty\}$). We will denote it by

$$\overline{\mathbb{R}}_s^{Nn^2} := \mathbb{R}_s^{Nn^2} \cup \{\infty\}.$$

Then, the space $\mathbb{R}_s^{Nn^2}$ will be viewed as a metric vector space, isometrically contained into its 1-point compactification.

Young Measures. Let $\Omega \subseteq \mathbb{R}^n$ be open. The Young measures can be identified with a subset of the unit sphere of a certain L^∞ space of measure-valued maps and this provides very useful properties, such as compactness.

Definition 2.1. The set of Young Measures from Ω to $\overline{\mathbb{R}}_s^{Nn^2}$ is the subset of the unit sphere of the space $L^\infty_{w^*}\big(\Omega, \mathcal{M}\big(\overline{\mathbb{R}}_s^{Nn^2}\big)\big)$ which contains probability-valued maps:

$$\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}_s^{Nn^2}\big) := \Big\{ \vartheta \in L^\infty_{w^*}\big(\Omega, \mathcal{M}\big(\overline{\mathbb{R}}_s^{Nn^2}\big)\big) : \vartheta(x) \in \mathscr{P}\big(\overline{\mathbb{R}}_s^{Nn^2}\big), \text{ for a.e. } x \in \Omega \Big\}.$$

The space $L^\infty_{w^*}\big(\Omega, \mathcal{M}\big(\overline{\mathbb{R}}_s^{Nn^2}\big)\big)$ is a dual Banach space and consists of measure-valued maps $\Omega \ni x \longmapsto \vartheta(x) \in \mathcal{M}\big(\overline{\mathbb{R}}_s^{Nn^2}\big)$ which are weakly* measurable, in the sense that for any Borel set $\mathcal{U} \subseteq \overline{\mathbb{R}}_s^{Nn^2}$, the function $\Omega \ni x \mapsto [\vartheta(x)](\mathcal{U}) \in \mathbb{R}$ is measurable. The norm of the space is given by

$$\|\vartheta\|_{L^\infty_{w^*}(\Omega, \mathcal{M}(\overline{\mathbb{R}}_s^{Nn^2}))} := \operatorname{ess\,sup}_{x \in \Omega} \|\vartheta(x)\|\big(\overline{\mathbb{R}}_s^{Nn^2}\big)$$

where "$\|\cdot\|$" denotes the total variation. For background material on these spaces we refer e.g. to [15, 10, V] and to [26]-[29]. The $L^\infty_{w^*}$ space above is the dual space of the space $L^1\big(\Omega, C^0\big(\overline{\mathbb{R}}_s^{Nn^2}\big)\big)$ of Bochner integrable maps. The points of this L^1 space are the Carathéodory functions $\Phi : \Omega \times \overline{\mathbb{R}}_s^{Nn^2} \to \mathbb{R}$ which satisfy

$$\|\Phi\|_{L^1(\Omega, C^0(\overline{\mathbb{R}}_s^{Nn^2}))} := \int_\Omega \big\|\Phi(x,\cdot)\big\|_{C^0(\mathbb{R}_s^{Nn^2})} dx < \infty.$$

It is well known that the unit ball of $L^\infty_{w^*}$ is sequentially weakly* compact. Hence, for any bounded sequence $(\vartheta^m)_1^\infty \subseteq L^\infty_{w^*}$, there is a limit map ϑ and a subsequence of m's along which $\vartheta^m \overset{*}{\rightharpoonup} \vartheta$ as $m \to \infty$.

Remark 2.2. We note the following facts about Young measures (proofs can be found e.g. in [14]):

- (i) [Functions as Y.M.] The set of measurable maps $U : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}_s^{Nn^2}$ can be identified with a subset of the Young measures via the embedding $U \mapsto \delta_U$, $\delta_U(x) := \delta_{U(x)}$.
- (ii) [Weak* compactness of Y.M.] The set of Young measures is convex and sequentially compact in the weak* topology induced from $L^\infty_{w^*}$.

The next lemma is a minor variant of a classical result (see [14, 15, 26]) but it plays a fundamental role in our setting because it guarantees the compatibility of classical/strong solutions with $\mathcal{D}$-solutions.

Lemma 2.3. *Let $U^\nu, U^\infty : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}_s^{Nn^2}$ be measurable maps, $\nu \in \mathbb{N}$. Then, up the passage to a subsequence, the following equivalence holds*

$$\delta_{U^\nu} \overset{*}{\rightharpoonup} \delta_{U^\infty} \ \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}_s^{Nn^2}\big) \iff U^\nu \to U^\infty \ \text{a.e. on } \Omega.$$

Notion of $\mathcal{D}$-Solutions to fully nonlinear 2nd order systems. Herein we consider the special case of once differentiable solutions to second order systems which is relevant to the ∞-Laplacian. For the general case of measurable solutions to pth order system we refer to [26, 29].

Let $D^{1,h}$ denote the usual difference quotient operator on $\mathbb{R}^n$, i.e. given a map $v : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ and $h \neq 0$, we understand v as being extended by zero on $\mathbb{R}^n \setminus \Omega$ and we set

$$D_i^{1,h} v(x) := \frac{v(x + he^i) - v(x)}{h}, \quad x \in \Omega,$$
$$D^{1,h} v(x) := \Big(D_1^{1,h} v(x), \dots, D_n^{1,h} v(x)\Big), \quad x \in \Omega.$$

Definition 2.4. *Let $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ be a locally Lipschitz continuous map. We define the* diffuse hessians $\mathcal{D}^2 u$ of u *as the subsequential weak* limits of the difference quotients of the gradient in the space of Young measures along infinitesimal sequences $(h_\nu)_1^\infty$:*

$$\delta_{D^{1,h_{\nu_k}} Du} \overset{*}{\rightharpoonup} \mathcal{D}^2 u \quad \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}_s^{Nn^2}\big), \ \text{as } k \to \infty.$$

Next is our notion of generalised solution for the vectorial case. We will use the notation "supp_*" to denote the *reduced support* of a probability measure ϑ on $\overline{\mathbb{R}}_s^{Nn^2}$ "off infinity", namely,

$$\mathrm{supp}_*(\vartheta) := \mathrm{supp}(\vartheta) \setminus \{\infty\}, \quad \vartheta \in \mathscr{P}\big(\overline{\mathbb{R}}_s^{Nn^2}\big).$$

Definition 2.5 (Lipschitz $\mathcal{D}$-solutions to 2nd order systems)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $F : \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s \to \mathbb{R}^N$ a mapping which is Borel measurable with respect to the first argument and continuous with respect to the second argument. Consider the PDE system

$$F\big(Du, D^2u\big) = 0 \quad \text{on } \Omega. \tag{2.1}$$

We say that the locally Lipschitz continuous map $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ is a $\mathcal{D}$-*solution of* (2.1) when for any diffuse hessian $\mathcal{D}^2u$ of u, we have

$$\sup_{\mathbf{X}_x \in \mathrm{supp}_*(\mathcal{D}^2u(x))} \big|F\big(Du(x), \mathbf{X}_x\big)\big| = 0, \quad \text{a.e. } x \in \Omega. \tag{2.2}$$

In particular, for the ∞-Laplace system (1.1), a $W^{1,\infty}_{\rm loc}$ map $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ is ∞-Harmonic in the $\mathcal{D}$-sense, when for a.e. $x \in \Omega$ and all $\mathbf{X}_x \in \mathrm{supp}_*(\mathcal{D}^2u(x))$, we have

$$\big(Du(x) \otimes Du(x) + |Du(x)|^2[Du(x)]^\perp \otimes I\big) : \mathbf{X}_x = 0.$$

Note that at certain points it may happen that $\mathcal{D}^2u(x) = \delta_{\{\infty\}}$ which implies that the reduced support of $\mathcal{D}^2u(x)$ is empty. The criterion then is understood to be trivially satisfied. Further, the $\mathcal{D}$-notions are compatible with the strong/classical notions of solution: this is a direct consequence of Lemma 2.3 and the definition of diffuse hessians.

Remark 2.6 (An alternative formulation of $\mathcal{D}$-solutions)**.** We give an alternative "integral" form of Definition 2.5 above which we put foremost in [26]-[28] because of its technical convenience for the existence/uniqueness proofs therein. We will not use this version herein, however. Note first that (2.2) can be rephrased as the following differential inclusion for the support:

$$\mathrm{supp}(\mathcal{D}^2u(x)) \subseteq \Big\{\mathbf{X} \in \mathbb{R}^{Nn^2}_s : |F\big(Du(x), \mathbf{X}\big)| = 0\Big\} \cup \{\infty\}, \qquad \text{a.e. } x \in \Omega.$$

Then, for any compactly supported $\Phi \in C^0_c\big(\mathbb{R}^{Nn^2}_s\big)$ off infinity and for a.e. $x \in \Omega$, the continuous function

$$\overline{\mathbb{R}}^{Nn^2}_s \ni \mathbf{X} \mapsto \Phi(\mathbf{X})F\big(Du(x), \mathbf{X}\big) \in \mathbb{R}^N$$

is well-defined on the compactification and also vanishes on the support of any diffuse hessian measure. As a consequence, we have the statement

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\mathbf{X})F\big(Du(x), \mathbf{X}\big)d[\mathcal{D}^2u(x)](\mathbf{X}) = 0, \quad \text{a.e. } x \in \Omega, \tag{2.3}$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn^2}_s\big)$ and any diffuse hessian $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$. It can be easily seen that the converse is true as well (see [26]) and hence (2.3) is a restatement of (2.2).

For more details on the material of this section (e.g. analytic properties, equivalent formulations of Definition 2.5, etc) we refer to [26]-[29].

Notion of feeble viscosity solutions to fully nonlinear 2nd order equations. The definitions of this paragraph are taken from [22] (see also [16, 17] where the "feeble" counterparts of the "usual" viscosity notion first appeared) but here we apply them only to the case of the p-Laplacian for $1 < p < \infty$. The standard viscosity notions as in [8, 7, 25] do not apply here because we treat also the singular case of the p-Laplacian when $p < 2$ which is not even defined when the gradient vanishes.

Let $F : (\mathbb{R}^n\backslash\{0\}) \times \mathbb{R}^{n^2}_s \to \mathbb{R}$ be a continuous function which satisfies the monotonicity hypothesis $F(P, \mathbf{X}) \le F(P, \mathbf{Y})$ when $\mathbf{X} \le \mathbf{Y}$ in $\mathbb{R}^{n^2}_s$. We consider the PDE

$$F\big(Du, D^2u\big) = 0 \quad \text{on } \Omega.$$

Let $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Given a triplet $(x, P, \mathbf{X}) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n^2}_s$, we define the quadratic polynomial $T_{P,\mathbf{X},x}u$ by setting

$$T_{P,\mathbf{X},x}u(z) := u(x) + P \cdot z + \frac{1}{2}\mathbf{X} : z \otimes z, \quad z \in \mathbb{R}^n.$$

We then set

$$J^{2,\pm}_0 u(x) := \Big\{(P, \mathbf{X}) \in (\mathbb{R}^n\backslash\{0\})\times\mathbb{R}^{n^2}_s : u(z+x) \underset{\ge}{\le} T_{P,\mathbf{X},x}u(z)+o(|z|^2), \text{ as } z \to 0\Big\}$$

and call $J^{2,\pm}_0 u(x)$ the *feeble 2nd order sub/superjet of u at x*. We say that u is a feeble viscosity solution of $F\big(Du, D^2u\big) \ge 0$ (resp. of $F\big(Du, D^2u\big) \le 0$) on Ω when for any $x \in \Omega$

$$\inf_{(P,\mathbf{X})\in J^{2,+}_0 u(x)} F(P, \mathbf{X}) \ge 0 \quad \Big(\text{resp.} \sup_{(P,\mathbf{X})\in J^{2,-}_0 u(x)} F(P, \mathbf{X}) \le 0\Big).$$

Feeble viscosity solutions of $F\big(Du, D^2u\big) = 0$ are defined as the combination of the above one-sided sub/super solution statements.

If $u \in C^1(\Omega)$, then any pair $(P, \mathbf{X})$ in $J^{2,\pm}_0 u(x)$ satisfies $P = Du(x)$. In this case we will use the notation

$$D^{2,\pm}u(x) := \Big\{\mathbf{X} \in \mathbb{R}^{n^2}_s : (Du(x), \mathbf{X}) \in J^{2,\pm}_0 u(x)\Big\}$$

and we will call $D^{2,\pm}u(x)$ the set of feeble 2nd order sub/super derivatives of u at $x \in \Omega$.

3. Two elementary lemmas

In this brief section we isolate a couple of very simple technical results which contain an essential common part of the proofs of the main results in both the scalar and the vectorial case.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in C^1(\Omega, \mathbb{R}^N)$. Given $\Omega' \Subset \Omega$, we set*

$$\Omega'(u) := \big\{x \in \overline{\Omega'} : |Du(x)| = \|Du\|_{L^\infty(\Omega')}\big\}$$

Let further $A : \mathbb{R}^n \to \mathbb{R}^N$ be an affine map.

(a) Suppose that for some $\Omega' \Subset \Omega$ and any $\lambda > 0$, u satisfies

$$\|Du\|_{L^\infty(\Omega')} \le \big\|Du + \lambda DA\big\|_{L^\infty(\Omega')}.$$

Then, we have

$$\max_{z\in\overline{\Omega'}} \big\{Du(z) : DA\big\} \ge 0.$$

(b) Given $x \in \Omega$ and $0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)$, the set

$$\Omega_\varepsilon(x) := \big\{y \in \Omega\big||Du(y)| < |Du(x)|\big\} \cap \mathbb{B}_\varepsilon(x)$$

is open and compactly contained in Ω and also $x \in \big(\Omega_\varepsilon(x)\big)(u)$, that is

$$|Du(x)| = \|Du\|_{L^\infty(\Omega_\varepsilon(x))}.$$

Proof. (a) By assumption we have

$$\|Du\|^2_{L^\infty(\Omega')} \le \|Du + \lambda DA\|^2_{L^\infty(\Omega')}$$

and hence

$$\begin{aligned}\operatorname{ess\,sup}_{\Omega'} |Du|^2 &\le \operatorname{ess\,sup}_{\Omega'} \big\{|Du|^2 + 2\lambda Du : DA + \lambda^2|DA|^2\big\}\\ &\le \operatorname{ess\,sup}_{\Omega'} |Du|^2 + 2\lambda \operatorname{ess\,sup}_{\Omega'} \big\{Du : DA\big\} + \lambda^2|DA|^2.\end{aligned}$$

Consequently,

$$\operatorname{ess\,sup}_{\Omega'} \big\{Du : DA\big\} + \frac{\lambda}{2}|DA|^2 \ge 0$$

and by letting $\lambda \to 0^+$, we obtain the desired inequality. (b) is immediate from the definitions. ☐

Lemma 3.1 is in general true for locally Lipschitz maps, once we replace $|Du|$ by the *local* L^∞ *norm*

$$\|Du\|_\infty(x) := \lim_{\varepsilon\to 0} \|Du\|_{L^\infty(\mathbb{B}_\varepsilon(x))}$$

which has enough upper semi-continuity properties.

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in C^1(\Omega, \mathbb{R}^N)$. Given $\Omega' \Subset \Omega$, let $\Omega'(u)$ be as in Lemma 3.1. Let further $A : \mathbb{R}^n \to \mathbb{R}^N$ be an affine map. We set*

$$h(t) := \big\|Du + tDA\big\|^2_{L^\infty(\Omega')} - \|Du\|^2_{L^\infty(\Omega')}, \quad t \ge 0.$$

Then, h is convex, $h(0) = 0$ and also the lower right Dini derivative of h at zero satisfies

$$\underline{D}h(0^+) := \liminf_{t\to 0^+} \frac{h(t) - h(0)}{t} \ge \max_{y\in\Omega'(u)} \big\{2Du(y) : DA\big\}.$$

Proof. Effectively, this is an application of Danskin's theorem [D], but we may also prove it directly. By setting

$$H(t, y) := \big|Du(y) + tDA\big|^2$$

we have

$$h(t) = \max_{y\in\overline{\Omega'}} H(t, y) - \max_{y\in\overline{\Omega'}} H(0, y)\,.$$

Also for any $t \ge 0$ the maximum $\max_{y\in\overline{\Omega'}} H(t, y)$ is realised at (at least one) point $y^t \in \overline{\Omega'}$. Hence

$$\begin{aligned}\frac{1}{t}\big(h(t) - h(0)\big) &= \frac{1}{t}\Big[\max_{y\in\overline{\Omega'}} H(t, y) - \max_{y\in\overline{\Omega'}} H(0, y)\Big]\\ &= \frac{1}{t}\Big[H(t, y^t) - H(0, y^0)\Big]\\ &= \frac{1}{t}\Big[\big(H(t, y^t) - H(t, y^0)\big) + \big(H(t, y^0) - H(0, y^0)\big)\Big]\\ &\ge \frac{1}{t}\big(H(t, y^0) - H(0, y^0)\big),\end{aligned}$$

where $y^0 \in \overline{\Omega'}$ is any point such that

$$|Du(y^0)| = H(0, y^0) = \max_{\overline{\Omega'}} H(0, \cdot) = \|Du\|_{L^\infty(\Omega')}.$$

Hence, by the definition of the set $\Omega'(u)$ in Lemma 3.1, we have

$$\begin{aligned}\underline{D}h(0^+) &= \liminf_{t\to 0^+}\frac{1}{t}\big(h(t)-h(0)\big)\\ &\geq \max_{y\in\Omega'(u)}\Big\{\liminf_{t\to 0^+}\frac{1}{t}\big(H(t,y)-H(0,y)\big)\Big\}\\ &= \max_{y\in\Omega'(u)}\Big\{\liminf_{t\to 0^+}\frac{1}{t}\big(\big|Du(y)+tDA\big|^2-|Du(y)|^2\big)\Big\}\\ &= \max_{y\in\Omega'(u)}\big\{2Du(y):DA\big\}.\end{aligned}$$

The lemma follows. □

Let us also record for later use the elementary inequality

$$h(t)-h(0)\geq \underline{D}h(0^+)t,\quad t\geq 0,$$

which is an immediate consequence of the definitions of convexity and of the lower right Dini derivative.

4. Scalar case $N=1$

The following is the first main result of this section, for C^1 ∞-harmonic functions.

Theorem 4.1. *Let $\Omega\subseteq\mathbb{R}^n$ be open and $u\in C^1(\Omega)$. Given $\Omega'\Subset\Omega$, let $\Omega'(u)$ be as in Lemma 3.1 and consider the sets of affine functions*

$$\begin{aligned}\mathcal{A}^{\pm,\infty}_{\Omega'}(u):=&\Big\{A:\mathbb{R}^n\to\mathbb{R}: D^2A\equiv 0 \text{ and there exist } \xi\in\mathbb{R}^{\pm},\\ &\quad x\in\Omega'(u) \text{ and } \mathbf{X}_x\in D^{2,\pm}u(x) \text{ s. t. } DA\equiv\xi\mathbf{X}_xDu(x)\Big\}\\ &\cup\mathbb{R}.\end{aligned}$$

Then, we have the equivalences

$$\left.\begin{array}{l} Du\otimes Du:D^2u\geq 0 \text{ on } \Omega,\\ \text{in the Viscosity sense}\end{array}\right\}\iff\left\{\begin{array}{l}\text{For all } \Omega'\Subset\Omega \text{ and } A\in\mathcal{A}^{+,\infty}_{\Omega'}(u),\\ \|Du\|_{L^\infty(\Omega')}\leq\|Du+DA\|_{L^\infty(\Omega')},\end{array}\right.$$

and

$$\left.\begin{array}{l} Du\otimes Du:D^2u\leq 0 \text{ on } \Omega,\\ \text{in the Viscosity sense}\end{array}\right\}\iff\left\{\begin{array}{l}\text{For all } \Omega'\Subset\Omega \text{ and } A\in\mathcal{A}^{-,\infty}_{\Omega'}(u),\\ \|Du\|_{L^\infty(\Omega')}\leq\|Du+DA\|_{L^\infty(\Omega')}.\end{array}\right.$$

We note that by the C^1 regularity results for ∞-harmonic functions of Savin and Evans-Savin [36, 13], if $n=2$ the hypothesis that u is a $C^1(\Omega)$ viscosity solution is superfluous.

Obviously, for certain subdomains it may happen that $\mathcal{A}^{\pm,\infty}_{\Omega'}(u)$ contain only the trivial (i.e. constant) functions if $J^{2,\pm}u(x)=\emptyset$ for all points $x\in\Omega'(u)$. Hence, the minimality property above with respect to affine functions is an effective restatement of the definition of viscosity sub/super solutions.

In the event that the solution is smooth, Theorem 4.1 above simplifies to the following statement for classical solutions of the ∞-Laplacian, i.e. for C^2 ∞-Harmonic functions.

Corollary 4.2. *Suppose that $\Omega \subseteq \mathbb{R}^n$ is open and $u \in C^2(\Omega)$. Then, we have the equivalence*

$$Du \otimes Du : D^2u = 0 \text{ on } \Omega \iff \begin{cases} \text{For all } \Omega' \Subset \Omega \text{ and } A \in \big(\mathcal{A}^{+;\infty}_{\Omega'} \cup \mathcal{A}^{-;\infty}_{\Omega'}\big)(u), \\ \|Du\|_{L^\infty(\Omega')} \le \|Du + DA\|_{L^\infty(\Omega')} \end{cases}$$
$$\iff \begin{cases} \text{For all } \Omega' \Subset \Omega \text{ and } A \in \mathcal{A}^{\infty}_{\Omega'}(u), \\ \|Du\|_{L^\infty(\Omega')} \le \|Du + DA\|_{L^\infty(\Omega')}. \end{cases}$$

Here $\mathcal{A}^{\infty}_{\Omega'}(u)$ is the set of affine functions

$$\mathcal{A}^{\infty}_{\Omega'}(u) = \Big\{A : \mathbb{R}^n \to \mathbb{R} : D^2A \equiv 0 \text{ and there exist } \xi \in \mathbb{R},\, x \in \Omega'(u)$$
$$\text{such that } A \text{ is parallel to the tangent of } \xi|Du|^2 \text{ at } x\Big\}.$$

Proof of Theorem 4.1. Suppose that for any $\Omega' \Subset \Omega$ and any affine function in $\mathcal{A}^{+;\infty}_{\Omega'}(u)$, we have

$$\|Du\|_{L^\infty(\Omega')} \le \|Du + DA\|_{L^\infty(\Omega')}.$$

Fix any $x \in \Omega$ such that $(Du(x), \mathbf{X}_x) \in J^{2,+}u(x)$, whence $\mathbf{X}_x \in D^{2,+}u(x)$. Consider the affine function

$$A(z) := \xi \mathbf{X}_x : Du(x) \otimes (z - x), \quad z \in \mathbb{R}^n,$$

where $\xi \ge 0$. Fix also $\varepsilon > 0$ and let $\Omega_\varepsilon(x)$ be as in Lemma 3.1. Then, for any $\lambda > 0$, the affine function λA is contained in $\mathcal{A}^{+,\infty}_{\Omega_\varepsilon(x)}(u)$. Hence,

$$\|Du\|_{L^\infty(\Omega_\varepsilon(x))} \le \|Du + \lambda DA\|_{L^\infty(\Omega_\varepsilon(x))}.$$

By applying Lemma 3.1 to u and A, we have

$$\begin{aligned} 0 &\le \underline{\max_{z\in\Omega_\varepsilon(x)}} \big\{Du(z) \cdot DA\big\} \\ &= \underline{\max_{z\in\Omega_\varepsilon(x)}} \big\{Du(z) \cdot \big(\xi \mathbf{X}_x : Du(x)\big)\big\} \\ &= \underline{\max_{z\in\Omega_\varepsilon(x)}} \big\{\xi\big(\mathbf{X}_x : Du(x) \otimes Du(z)\big)\big\} \\ &\to \xi\big(\mathbf{X}_x : Du(x) \otimes Du(x)\big), \end{aligned}$$

as $\varepsilon \to 0$. Hence, $Du \otimes Du : D^2u \ge 0$ on Ω in the viscosity sense.

Conversely, fix any $\Omega' \Subset \Omega$ and $x \in \Omega'(u)$. If it happens $J^{2,+}u(x) \ne \emptyset$, then any $A \in \mathcal{A}^{+;\infty}_{\Omega'}(u)$ can be written as

$$A(z) = a + \xi \mathbf{X}_x : Du(x) \otimes z, \quad z \in \mathbb{R}^n,$$

for some $a \in \mathbb{R}$, $\xi \ge 0$ and $\mathbf{X}_x \in D^{2,+}u(x)$. Let h be the function of Lemma 3.2 for such an A. By applying Lemma 3.2 to this setting, we have

$$\begin{aligned} \underline{D}h(0^+) &\ge \max_{y\in\Omega'(u)} \big\{2Du(y) \cdot DA\big\} \\ &\ge 2Du(x) \cdot DA \\ &= 2Du(x) \cdot \big(\xi \mathbf{X}_x : Du(x)\big)\big\} \\ &= 2\xi\big(\mathbf{X}_x : Du(x) \otimes Du(x)\big) \ge 0, \end{aligned}$$

since by assumption $Du \otimes Du : D^2u \ge 0$ on Ω in the viscosity sense. Since $h(0) = 0$ and h is convex, it follows that

$$h(t) \ge h(0) + \underline{D}h(0^+)t \ge 0, \quad t \ge 0,$$

and hence, by the definition of h we obtain

$$\|Du\|_{L^\infty(\Omega')} \leq \|Du + DA\|_{L^\infty(\Omega')}$$

for any $\Omega' \Subset \Omega$ and any $A \in \mathcal{A}^{+,\infty}_{\Omega'}(u)$. The case of supersolutions follows similarly and hence the theorem has been established. □

Proof of Corollary 4.2. The first equivalence of the statement is immediate. Since by assumption $u \in C^2(\Omega)$, we have

$$J^{2,+}u(x) \cap J^{2,-}u(x) = \big\{\big(Du(x), D^2u(x)\big)\big\}$$

and hence $D^{2,+}u(x) \cap D^{2,-}u(x) = \{D^2u(x)\}$. The second equivalence of the statement follows by making the choice $\mathbf{X}_x \in D^{2,\pm}u(x)$ in the proof of Theorem 4.1 above and repeating all the steps. Then, by noting that

$$\mathbf{X}_x Du(x) = D\big(\frac{1}{2}|Du|^2\big)(x)$$

it follows that for any $\Omega' \Subset \Omega$ the set $\mathcal{A}^{\infty}_{\Omega'}(u)$ contains only affine functions of the form

$$A(z) = a + \xi D\big(|Du|^2\big)(x)\cdot(z-x), \quad z \in \mathbb{R}^n,$$

for $a, \xi \in \mathbb{R}$ and $x \in \Omega'(u)$. The corollary ensues. □

Theorem 4.1 extends relatively easily to the case of the p-Laplacian for $1 < p < \infty$ which, quite surprisingly, can also be characterised by the L^∞ functional via affine variations. In view of the well known $C^{1,\alpha}$ regularity results for p-Harmonic mappings [U], the hypothesis that solutions are C^1 is actually superfluous.

Theorem 4.3 (p-harmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in C^1(\Omega)$. Given $\Omega' \Subset \Omega$, let $\Omega'(u)$ be as above and consider the sets of affine functions*

$$\begin{aligned}\mathcal{A}^{\pm,p}_{\Omega'}(u) := \Big\{&A : \mathbb{R}^n \to \mathbb{R} : D^2A \equiv 0 \text{ and there exist } \xi \in \mathbb{R}^{\pm}, x \in \Omega'(u)\\ &\text{and } \mathbf{X}_x \in D^{2,\pm}u(x) \text{ s. t. } DA \equiv \xi\big((p-2)\mathbf{X}_x + (I:\mathbf{X}_x)I\big)Du(x)\Big\}\\ &\cup \mathbb{R},\end{aligned}$$

where $p \in (1,\infty)$. Then, the following statements are equivalent:

(a) $\operatorname{div}\big(|Du|^{p-2}Du\big) \geq 0$ *weakly on Ω;*
(b) $\big((p-2)Du \otimes Du + |Du|^2 I\big) : D^2u \geq 0$ *on Ω, in the feeble Viscosity sense.*
(c) *For all $\Omega' \Subset \Omega$ and all $A \in \mathcal{A}^{+,p}_{\Omega'}(u)$, we have*

$$\|Du\|_{L^\infty(\Omega')} \leq \|Du + DA\|_{L^\infty(\Omega')}.$$

The case "≤ 0" of supersolutions is symmetrical and corresponds to $\mathcal{A}^{-,p}_{\Omega'}(u)$ as in Theorem 4.1.

In the case of the usual Laplacian for $p = 2$, the affine functions in $\mathcal{A}^{+,2}_{\Omega'}(u)$ of Theorem 4.3 satisfy $DA = \xi(\mathbf{X}_x : I)Du(x)$, where $\xi \geq 0$, $\mathbf{X}_x \in D^{2,\pm}u(x)$, $\Omega' \Subset \Omega$ and $x \in \Omega'(u)$.

Proof of Theorem 4.3. The idea is similar to that of the proof of Theorem 4.1, so we basically need to indicate the points where it differs. We begin by noting by the results of the papers [22, 17, 16], it follows that a function is weakly p-subharmonic on Ω (that is we have $\operatorname{div}\big(|Du|^{p-2}Du\big) \geq 0$ holding weakly on Ω) if and only if it is p-subharmonic on Ω in the feeble viscosity sense for the p-Laplacian expanded:

$$|Du|^{p-4}\big((p-2)Du \otimes Du + |Du|^2 I\big) : D^2u \geq 0, \quad \text{on } \Omega.$$

Since by definition of the feeble Jets we do not check anything in the viscosity criterion when the gradient vanishes, the p-Laplacian is equivalent in the feeble viscosity sense to

$$\big((p-2)Du \otimes Du + |Du|^2 I\big) : D^2u \geq 0, \quad \text{on } \Omega.$$

As a consequence, $(a) \Leftrightarrow (b)$. We suppose now that for any $\Omega' \Subset \Omega$ and any affine function $A \in \mathcal{A}^{+,\infty}_{\Omega'}(u)$, we have

$$\|Du\|_{L^\infty(\Omega')} \leq \|Du + DA\|_{L^\infty(\Omega')}.$$

Fix any $x \in \Omega$ such that $(Du(x), \mathbf{X}_x) \in J^{2,+}_0 u(x)$, whence $\mathbf{X}_x \in D^{2,+}u(x)$. Consider the affine function

$$A(z) := \big((p-2)\mathbf{X}_x + (I : \mathbf{X}_x)I\big) : Du(x) \otimes (z - x), \quad z \in \mathbb{R}^n.$$

Fix also $\varepsilon > 0$ and let $\Omega_\varepsilon(x)$ be as in Lemma 3.1 and note that for any $\lambda > 0$, $\lambda A \in \mathcal{A}^{+,p}_{\Omega_\varepsilon(x)}(u)$. Hence, by arguing as in Theorem 4.1 we have that

$$\begin{aligned} 0 &\leq Du(x) \cdot DA \\ &= Du(x) \cdot \Big((p-2)\mathbf{X}_x Du(x) + (I : \mathbf{X}_x)Du(x)\Big) \\ &= \Big((p-2)Du(x) \otimes Du(x) + |Du(x)|^2 I\Big) : \mathbf{X}_x. \end{aligned}$$

Hence, u is a feeble viscosity solution on Ω.

Conversely, fix any $\Omega' \Subset \Omega$ and $x \in \Omega'(u)$. If $J^{2,+}_0 u(x) \neq \emptyset$, then any $A \in \mathcal{A}^{+,p}_{\Omega'}(u)$ can be written as

$$A(z) = a + \xi\big((p-2)\mathbf{X}_x + (I : \mathbf{X}_x)I\big) : Du(x) \otimes z, \quad z \in \mathbb{R}^n,$$

for some $a \in \mathbb{R}$, $\xi \geq 0$ and some $(Du(x), \mathbf{X}_x) \in J^{2,+}_0 u(x)$. Let h be the function of Lemma 3.2 for such an A. By applying Lemma 3.2, we have

$$\begin{aligned} \underline{D}h(0^+) &\geq 2Du(x) \cdot DA \\ &= 2\xi\big((p-2)Du(x) \otimes Du(x) : \mathbf{X}_x + |Du(x)|^2 I : \mathbf{X}_x\big) \geq 0, \end{aligned}$$

since by assumption u is a subsolution on Ω in the feeble viscosity sense. By using that $h(0) = 0$ and that h is convex, we deduce as in Theorem 4.1 that $h(t) \geq 0$ for $t \geq 0$ and hence

$$\|Du\|_{L^\infty(\Omega')} \leq \|Du + DA\|_{L^\infty(\Omega')}$$

for any $A \in \mathcal{A}^{+,p}_{\Omega'}(u)$ and any $\Omega' \Subset \Omega$. Thus, $(b) \Leftrightarrow (c)$. The case of supersolutions follows analogously and hence the theorem ensues. □

5. Vectorial case $N \geq 2$

In this section we extend the results of the previous section to the full case of the ∞-Laplace system. We begin by noting that (1.1) actually consists of two independent systems, the second of which is identically trivial in the scalar case. Namely, if $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ is smooth, then

$$\Delta_\infty u = 0 \iff \begin{cases} Du \otimes Du : D^2u = 0, \\ |Du|^2 [Du]^\perp \Delta u = 0. \end{cases}$$

This is an immediate consequence of the mutual perpendicularity of the vector fields $Du \otimes Du : D^2u$ and $|Du|^2[Du]^\perp \Delta u$; indeed, it suffices to recall that $[Du]^\perp$ is the projection on the orthogonal complement of $R(Du)$ and to note the identity

$$2Du \otimes Du : D^2u = DuD\big(|Du|^2\big).$$

Our last main result is the following resutl for C^1 ∞-Harmonic mappings.

Theorem 5.1. *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in C^1(\Omega, \mathbb{R}^N)$. Given a set $\Omega' \Subset \Omega$, let $\Omega'(u)$ be as in Lemma 3.1. Consider first the set of affine maps*

$$\begin{aligned}
&\mathcal{A}^{\top,\infty}_{\Omega'}(u) \\
&:= \Big\{A : \mathbb{R}^n \to \mathbb{R}^N : D^2A \equiv 0 \text{ and there exist } \xi \in \mathbb{R}^N, x \in \Omega'(u) \\
&\quad \mathcal{D}^2u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big), \mathbf{X}_x \in \operatorname{supp}_*\big(\mathcal{D}^2u(x)\big) \text{ s. t. } DA \equiv \xi \otimes \big(\mathbf{X}_x : Du(x)\big)\Big\} \\
&\cup \mathbb{R}^N.
\end{aligned}$$

Then, we have the equivalence

$$\left.\begin{array}{l} Du \otimes Du : D^2u = 0 \\ \text{on } \Omega, \text{ in the } \mathcal{D}\text{-sense} \end{array}\right\} \iff \left\{\begin{array}{l} \text{For all } \Omega' \Subset \Omega \text{ and } A \in \mathcal{A}^{\top,\infty}_{\Omega'}(u), \\ \|Du\|_{L^\infty(\Omega')} \le \|Du + DA\|_{L^\infty(\Omega')}. \end{array}\right.$$

Further, consider the set of affine maps

$$\begin{aligned}
\mathcal{A}^{\perp,\infty}_{\Omega'}(u) := \Big\{&A : \mathbb{R}^n \to \mathbb{R}^N : D^2A \equiv 0 \text{ there exist } x \in \Omega'(u), \mathcal{D}^2u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big), \\
&\mathbf{X}_x \in \operatorname{supp}_*\big(\mathcal{D}^2u(x)\big) \text{ s. t. } A(x) \in R\big(Du(x)\big)^\perp, DA \in \mathscr{L}^{\mathbf{X}_x}\big(A(x)\big)\Big\} \\
&\cup \mathbb{R}^N
\end{aligned}$$

where for any $a \in \mathbb{R}^N$, $\mathscr{L}^{\mathbf{X}_x}(a)$ is an affine matrix space defined as

$$\mathscr{L}^{\mathbf{X}_x}(a) := \begin{cases} \big\{X \in \mathbb{R}^{Nn} : Du(x) : X = -(a \otimes I) : \mathbf{X}_x\big\}, & \text{if } Du(x) \neq 0 \\ \{0\}, & \text{if } Du(x) = 0. \end{cases}$$

Then, we have the equivalence

$$\left.\begin{array}{l} |Du|^2[Du]^\perp \Delta u = 0 \\ \text{on } \Omega, \text{ in the } \mathcal{D}\text{-sense} \end{array}\right\} \iff \left\{\begin{array}{l} \text{For all } \Omega' \Subset \Omega \text{ and } A \in \mathcal{A}^{\perp,\infty}_{\Omega'}(u), \\ \|Du\|_{L^\infty(\Omega')} \le \|Du + DA\|_{L^\infty(\Omega')}. \end{array}\right.$$

In view of Theorem 5.1, a mapping is ∞-Harmonic in the $\mathcal{D}$-sense if and only if it minimises with respect to the union of the sets of affine variations of the tangential and the normal component:

$$\left.\begin{array}{l} \Delta_\infty u = 0 \text{ on } \Omega, \\ \text{in the } \mathcal{D}\text{-sense} \end{array}\right\} \iff \left\{\begin{array}{l} \text{For all } \Omega' \Subset \Omega \text{ and } A \in \big(\mathcal{A}^{\top,\infty}_{\Omega'} \cup \mathcal{A}^{\perp,\infty}_{\Omega'}\big)(u), \\ \|Du\|_{L^\infty(\Omega')} \le \|Du + DA\|_{L^\infty(\Omega')}. \end{array}\right.$$

In the event that $u \in C^2(\Omega, \mathbb{R}^N)$, Theorem 5.1 simplifies to the following statement for classical solutions of the ∞-Laplace system, i.e. for C^2 ∞-Harmonic mappings.

Corollary 5.2. *Suppose that $\Omega \subseteq \mathbb{R}^n$ is open and $u \in C^2(\Omega, \mathbb{R}^N)$. Then, we have the equivalence*

$$\Delta_\infty u = 0 \text{ on } \Omega \iff \left\{\begin{array}{l} \text{For all } \Omega' \Subset \Omega \text{ and } A \in \big(\mathcal{A}^{\top,\infty}_{\Omega'} \cup \mathcal{A}^{\perp,\infty}_{\Omega'}\big)(u), \\ \|Du\|_{L^\infty(\Omega')} \le \|Du + DA\|_{L^\infty(\Omega')}, \end{array}\right.$$

where $\mathcal{A}^{\top,\infty}_{\Omega'}(u)$, $\mathcal{A}^{\perp,\infty}_{\Omega'}(u)$ *are the sets of affine maps*

$$\begin{aligned}\mathcal{A}^{\top,\infty}_{\Omega'}(u) = \Big\{ A : \mathbb{R}^n \to \mathbb{R}^N : D^2A \equiv 0 \textit{ and there exist } \xi \in \mathbb{R}^N, \textit{and } x \in \Omega'(u) \\ \textit{s. t. } A \textit{ is parallel to the tangent of } \xi|Du|^2 \textit{ at } x \Big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}^{\perp,\infty}_{\Omega'}(u) = \Big\{ A : \mathbb{R}^n \to \mathbb{R}^N : D^2A \equiv 0 \textit{ and there exists } x \in \Omega'(u) \textit{such that} \\ A \textit{ is normal to } Du \textit{ at } x \textit{ and } A^\top Du \textit{ is divergenceless at } x \ \Big\}.\end{aligned}$$

Proof of Theorem 5.1. We begin by a general observation about the notion of $\mathcal{D}$-solutions $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ in $C^1(\Omega, \mathbb{R}^N)$ to a homogeneous 2nd order quasilinear system of the form

$$\mathbf{A}(Du) : D^2u = 0, \quad \text{on } \Omega,$$

when $\mathbf{A}$ is Borel measurable. By definition, every diffuse hessian $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$ of a candidate solution u is defined a.e. on Ω as a weakly* measurable probability valued map $\Omega \to \mathbb{R}^{Nn^2}_s \cup \{\infty\}$. Hence, we may modify each $\mathcal{D}^2u$ on a Lebesgue nullset and choose from each equivalence class the representative which is redefined as $\delta_{\{0\}}$ at points where $\mathcal{D}^2u(x)$ does not exist. Moreover, let u be a fix map in $C^1(\Omega, \mathbb{R}^N)$. Since $Du(x)$ exists for all $x \in \Omega$, by perhaps a further re-definition of every $\mathcal{D}^2u$ on a Lebesgue nullset, it follows that u is $\mathcal{D}$-solution to the system if and only if for (any fixed such representative of) any diffuse hessian, we have

$$\mathbf{A}\big(Du(x)\big) : \mathbf{X}_x = 0, \quad \text{for all } x \in \Omega \text{ and } \mathbf{X}_x \in \operatorname{supp}_*\big(\mathcal{D}^2u(x)\big).$$

(We remind that at points $x \in \Omega$ for which $\mathcal{D}^2u(x) = \delta_{\{\infty\}}$ and hence $\operatorname{supp}_*\big(\mathcal{D}^2u(x)\big) = \emptyset$, the above condition is understood as being trivially satisfied.) We will apply this observation to the two independent systems

$$Du \otimes Du : D^2u = 0, \ |Du|^2\big([Du]^\perp \otimes I\big) : D^2u = 0$$

comprising the ∞-Laplace system.

Suppose now that for some $\Omega' \Subset \Omega$ and some affine mapping $A \in \mathcal{A}^{\top,\infty}_{\Omega'}(u)$, we have

$$\|Du\|_{L^\infty(\Omega')} \le \|Du + DA\|_{L^\infty(\Omega')}.$$

Fix any $x \in \Omega$ and any diffuse hessian $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$ such that $\operatorname{supp}_*\big(\mathcal{D}^2u(x)\big) \neq \emptyset$ and pick any $\mathbf{X}_x \in \operatorname{supp}_*\big(\mathcal{D}^2u(x)\big)$. Fix also $\xi \in \mathbb{R}^N$ and consider the affine map which is defined by

$$A(z) := \xi \otimes \big(\mathbf{X}_x : Du(x)\big) \cdot (z - x), \quad z \in \mathbb{R}^n.$$

In index form this means

$$A_\alpha(z) = \xi_\alpha \sum_{\beta=1}^{N} \sum_{i,j=1}^{n} \Big((\mathbf{X}_x)_{\beta ji} D_j u_\beta(x)\Big)(z - x)_i, \quad \alpha = 1, \dots, N.$$

For $\varepsilon > 0$ small, let $\Omega_\varepsilon(x)$ be as in Lemma 3.1. Then, $\lambda A \in \mathcal{A}^{\top,\infty}_{\Omega_\varepsilon(x)}(u)$ for any $\lambda > 0$. Thus,

$$\|Du\|_{L^\infty(\Omega_\varepsilon(x))} \le \|Du + \lambda DA\|_{L^\infty(\Omega_\varepsilon(x))}$$

and by applying Lemma 3.1 to u and A, we have

$$\begin{aligned}
0 &\le \max_{z\in\Omega_\varepsilon(x)} \Big\{ Du(z) : \big(\xi \otimes \mathbf{X}_x : Du(x)\big)\Big\} \\
&= \max_{z\in\Omega_\varepsilon(x)} \Big\{ \sum_{\alpha=1}^{N}\sum_{i=1}^{n} D_i u_\alpha(z)\xi_\alpha \sum_{\beta=1}^{N}\sum_{j=1}^{n} (\mathbf{X}_x)_{\beta ji} D_j u_\beta(x)\Big\} \\
&\le \max_{z\in\Omega_\varepsilon(x)} \Big\{ \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} \xi_\alpha D_i u_\alpha(z) D_j u_\beta(x)(\mathbf{X}_x)_{\beta ji}\Big\} \\
&\to \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} \xi_\alpha \Big(D_i u_\alpha(x) D_j u_\beta(x)(\mathbf{X}_x)_{\beta ji}\Big)
\end{aligned}$$

as $\varepsilon \to 0$, and hence

$$\xi \cdot \big(Du(x)\otimes Du(x) : \mathbf{X}_x\big) \ge 0,$$

for any $\xi \in \mathbb{R}^N$. By the arbitrariness of ξ we deduce that $Du(x)\otimes Du(x) : \mathbf{X}_x = 0$. As a consequence, $Du \otimes Du : D^2u = 0$ in the $\mathcal{D}$-sense on Ω.

Now we argue similarly for the normal component of the system. Suppose that for any $\Omega' \Subset \Omega$ and any $A \in \mathcal{A}^{\perp;\infty}_{\Omega'}(u)$, we have

$$\|Du\|_{L^\infty(\Omega')} \le \|Du + DA\|_{L^\infty(\Omega')}.$$

We fix as before $x \in \Omega$ and $\mathbf{X}_x \in \operatorname{supp}_*\big(\mathcal{D}^2u(x)\big)$. If $Du(x) = 0$, then the system $|Du|^2[Du]^\perp\Delta u = 0$ is trivially satisfied at x. If $Du(x) \neq 0$, then we choose any direction normal to $Du(x)$; that is,

$$n_x \in R\big(Du(x)\big)^\perp \subseteq \mathbb{R}^N,$$

which means that $n_x^\top Du(x) = 0$. We note that if $Du(x) : \mathbb{R}^n \to \mathbb{R}^N$ is surjective, then we can find only the trivial $n_x = 0$, but the system $|Du|^2[Du]^\perp\Delta u = 0$ is satisfied at x anyhow because $[Du(x)]^\perp = 0$. We also fix any matrix N_x in the affine space $\mathscr{L}^{\mathbf{X}_x}(n_x)$. By the definition of $\mathscr{L}^{\mathbf{X}_x}(n_x)$, this means that

$$N_x : Du(x) = -(n_x \otimes I) : \mathbf{X}_x.$$

We consider the affine map which is defined by

$$A(z) := n_x + N_x(z - x), \quad z \in \mathbb{R}^n.$$

We now claim that $\lambda A \in \mathcal{A}^{\perp;\infty}_{\Omega'}(u)$ for any $\lambda \in \mathbb{R}$. Indeed, this is a consequence of our choices and of the following homogeneity property of the space $\mathscr{L}^{\mathbf{X}_x}(a)$:

$$\mathscr{L}^{\mathbf{X}_x}(\lambda a) = \lambda \mathscr{L}^{\mathbf{X}_x}(a), \quad \lambda \in \mathbb{R}.$$

Hence, we have

$$\|Du\|_{L^\infty(\Omega')} \le \|Du + \lambda DA\|_{L^\infty(\Omega')}.$$

By applying Lemma 3.1 to u and A, we have

$$0 \le \max_{z\in\Omega_\varepsilon(x)} \big\{Du(z) : N_x\big\} \to Du(x) : N_x = -(n_x \otimes I) : \mathbf{X}_x,$$

as $\varepsilon \to 0$. Hence, we have $(n_x \otimes I) : \mathbf{X}_x \le 0$ and by the arbitrariness of the direction $n_x \perp R\big(Du(x)\big)$, we obtain that $(n_x \otimes I) : \mathbf{X}_x = 0$. Thus, $\big([Du(x)]^\perp \otimes I\big) : \mathbf{X}_x = 0$ and as a consequence $|Du|^2[Du]^\perp\Delta u = 0$ in the $\mathcal{D}$-sense on Ω.

Conversely, we fix $\Omega' \Subset \Omega$ and $x \in \Omega'(u)$ and any $A \in \mathcal{A}^{\top;\infty}_{\Omega'}(u)$ corresponding to a diffuse hessian $\mathcal{D}^2 u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$ and some $\mathbf{X}_x \in \mathrm{supp}_*(\mathcal{D}^2 u(x))$ and $\xi \in \mathbb{R}^N$. We take as h to be the function of Lemma 3.2. By applying Lemma 3.2 to this setting, we have

$$\begin{aligned}\underline{D}h(0^+) &\geq \max_{y\in\Omega'(u)} \big\{2Du(y) : DA\big\}\\ &\geq 2Du(x) : DA\\ &\geq 2\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} D_i u_\alpha(x)\xi_\alpha(\mathbf{X}_x)_{\beta ji} D_j u_\beta(x)\end{aligned}$$

and hence

$$\underline{D}h(0^+) \geq 2\xi\cdot\big(Du(x)\otimes Du(x) : \mathbf{X}_x\big) = 0,$$

since by assumption $Du\otimes Du : D^2u = 0$ on Ω in the $\mathcal{D}$-sense. In view of the fact that $h(0) = 0$ and h is convex, it follows that

$$h(t) \geq h(0) + \underline{D}h(0^+)t \geq 0, \quad t\geq 0,$$

and hence

$$\|Du\|_{L^\infty(\Omega')} \leq \|Du + DA\|_{L^\infty(\Omega')}, \quad A\in \mathcal{A}^{\top;\infty}_{\Omega'}(u),\ \Omega'\Subset\Omega.$$

The case of $A \in \mathcal{A}^{\perp;\infty}_{\Omega'}$ is completely analogous: any such nonconstant A satisfies $A(x)\perp R(Du(x))$ and $DA \in \mathscr{L}^{\mathbf{X}_x}\big(A(x)\big)$ for some $\mathbf{X}_x \in \mathrm{supp}_*(\mathcal{D}^2u(x))$ and some $x\in\Omega'(u)$. By applying Lemma 3.2 again, we have

$$\underline{D}h(0^+) \geq \max_{y\in\Omega'(u)} \big\{2Du(y) : DA\big\} \geq 2Du(x) : DA.$$

If $Du(x)\neq 0$, then by the definition of $\mathscr{L}^{\mathbf{X}_x}\big(A(x)\big)$ we have

$$\begin{aligned}\underline{D}h(0^+) &\geq 2DA : Du(x)\\ &= -2(n_x\otimes I) : \mathbf{X}_x\\ &= -2n_x^\top\Big(\big([Du(x)]^\perp\otimes I\big) : \mathbf{X}_x\Big) = 0\end{aligned}$$

because by assumption $|Du|^2[Du]^\perp\Delta u = 0$ on Ω in the $\mathcal{D}$-sense. If $Du(x) = 0$, then again $\underline{D}h(0^+)\geq 0$. In either cases, we obtain

$$h(t) \geq h(0) + \underline{D}h(0^+)t \geq 0, \quad t\geq 0,$$

and hence

$$\|Du\|_{L^\infty(\Omega')} \leq \|Du + DA\|_{L^\infty(\Omega')}, \quad A\in \mathcal{A}^{\perp;\infty}_{\Omega'}(u),\ \Omega'\Subset\Omega.$$

The proof is complete. □

Proof of Corollary 5.2. If $u\in C^2(\Omega,\mathbb{R}^N)$, then it is an immediate consequence of Lemma 2.3 that any diffuse hessian of u satisfies

$$\mathcal{D}^2u(x) = \delta_{D^2u(x)}, \quad x\in\Omega,$$

and by the remarks in the beginning of the proof of Theorem 5.1, this happens for all $x\in\Omega$. Hence, the only possible $\mathbf{X}_x$ in the reduced support of $\mathcal{D}^2u(x)$ is $\mathbf{X}_x = D^2u(x)$. For $\mathcal{A}^{\top;\infty}_{\Omega'}$, we have that any possible A satisfies $DA\equiv D\big(\xi|Du|^2\big)(x)$. For $\mathcal{A}^{\perp;\infty}_{\Omega'}$, we have that any possible A satisfies

$$A(x)^\top Du(x) = 0, \quad DA\in\mathscr{L}^{D^2u(x)}\big(A(x)\big),$$

which gives

$$DA : Du(x) = -\big(A(x) \otimes I\big) : D^2u(x) = -A(x) \cdot \Delta u(x).$$

Thus,

$$\mathrm{div}\,\big(A^\top Du\big)(x) = DA : Du(x) + A(x) \cdot \Delta u(x) = 0.$$

The proof is complete. □

Acknowledgments. The author has been financially supported by the EPSRC grant EP/N017412/1. The author would like to thank Craig Evans, Robert Jensen, Jan Kristensen and Juan Manfredi for inspiring scientific discussion relevant to the content of this particle, as well as for their encouragement. He is also indebted to the anonymous referee for the careful reading of the manuscript and for preparing their report so swiftly.

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