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Matrices with multiplicative entries are tensor products

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Abstract

We study operators which have (infinite) matrix representation whose entries are multiplicative functions of two variables. We show that such operators are infinite tensor products over the primes. Applications to finding the eigenvalues explicitly of arithmetical matrices are given; also boundedness and norms of Multiplicative Toeplitz and Hankel operators are discussed.

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Introduction

In this paper we shall consider infinite matrices $A = (a_{ij})_{i,j\geq 1}$ whose entries are multiplicative as a function of two variables; i.e. $a_{mn} = f(m,n)$, where $f : \mathbb{N}^2 \to \mathbb{C}$ is not identically zero and satisfies

$$f(m_1n_1, m_2n_2) = f(m_1, m_2)f(n_1, n_2) \quad \text{whenever } (m_1m_2, n_1n_2) = 1.$$

We are interested in knowing when such matrices induce bounded operators (on ℓ^2) and furthermore, what we can say about their (operator) norms and spectra.

The motivation for this investigation is twofold. In a recent paper [10], the singular values (see §1.2 for the definition) of $M_n = M_n(\alpha)$, the $n \times n$ matrix with ij^{th}-entry $(i/j)^{-\alpha}$ if $j|i$ and zero otherwise, were shown to be approximable by the eigenvalues of the operator given by infinite matrix

$$E_\alpha = \Big(\frac{(ij)^\alpha}{[i,j]}\Big)_{i,j\geq 1}.$$

More precisely, with $s_r(M_n)$ denoting the r^{th} largest singular value of M_n and $\lambda_r(E_\alpha)$ the r^{th} largest eigenvalue of E_α, it was shown that for $\alpha < \frac{1}{4}$,

$$s_r(M_n)^2 \sim \lambda_r(E_\alpha)\frac{n^{1-2\alpha}}{1-2\alpha} \quad \text{as } n \to \infty. \tag{1.1}$$

Note that E_α has multiplicative entries. It leads naturally to the question of identifying these eigenvalues and whether (1.1) remains true for $\frac{1}{4} \leq \alpha < \frac{1}{2}$. In particular whether E_α is bounded, indeed compact, for such α — we shall settle the boundedness question here. More generally the above was done for $n \times n$ matrices with entries $f(i/j)$ when $j|i$ and zero otherwise, where f is a square summable multiplicative function on $\mathbb{N}$. See also [14] for related matrices.

Another motivation comes from Multiplicative Toeplitz operators, whose matrix representation has entries of the form $a_{ij} = g(i/j)$ for a given $g : \mathbb{Q}^+ \to \mathbb{C}$. Such operators have been studied in [11], for their connection with Dirichlet series, and in particular the Riemann zeta function. If g is multiplicative as a function on the positive rationals, the matrix has multiplicative entries.

Our main result in this paper is to show that under a natural convergence condition, such matrices A are tensor products of operators over the primes (like an Euler product) with the tensor product corresponding to the prime p having matrix representation $\tilde{A}_p = (f(p^k, p^l))_{k,l\geq 0}$. For finite matrices, this was inspired by a result of Codecá and Nair [4] and generalizes it. The result for infinite matrices can be seen as a limiting case of this.

Thus for example, with $a_{ij} = g(i/j)$ and g multiplicative as above, $\tilde{A}_p$ is the Toeplitz matrix $(g(p^{k-l}))_{k,l\geq 0} = T(a_p)$, where $a_p(t) = \sum_{k=-\infty}^{\infty} g(p^k)t^k$ is the 'symbol'. Then we can deduce that A is bounded if $\prod_p \|a_p\|_\infty$ converges, with $\|A\|$ equal to this product. For many symbols (for instance rational symbols) this can be found explicitly. Similar remarks can be made about multiplicative Hankel operators (i.e. those whose matrix representation has entries $a_{ij} = h(ij)$ with h multiplicative).

For the proof, we need to discuss tensor products and in particular, infinite tensor products. In the case when A is compact, we can further deduce that the spectrum of A has multiplicative structure; namely, every nonzero eigenvalue of A factorizes as an Euler product of eigenvalues of the $\tilde{A}_p$. Whether this continues to hold for more general bounded A is unclear, and is an interesting open question.

After some preliminaries on multiplicative functions on $\mathbb{N}^2$ and tensor products in §1, we discuss and prove the main results in §2. We follow this with examples and applications in §3.

§1. Preliminaries

1.1 Multiplicative functions of several variables; multiplicative matrices

A function of two variables $f : \mathbb{N}^2 \to \mathbb{C}$ is *multiplicative*[1] if f is not identically zero and

$$f(m_1n_1, m_2n_2) = f(m_1, m_2)f(n_1, n_2) \quad \text{whenever } (m_1m_2, n_1n_2) = 1.$$

As such, $f(1, 1) = 1$ and f is determined by the values $f(p^k, p^l)$ for $k, l \in \mathbb{N}_0$ and p prime. Indeed, writing $m = \prod_p p^\alpha$ and $n = \prod_p p^\beta$ for their prime factorizations, we have

$$f(m, n) = f\Big(\prod_p p^\alpha, \prod_p p^\beta\Big) = \prod_p f(p^\alpha, p^\beta).$$

An infinite matrix $A = (a_{ij})_{i,j\geq 1}$ is *multiplicative* if $a_{mn} = f(m, n)$ with f is multiplicative (as a function of two variables).

Examples

(a) A diagonal matrix $D = \text{diag}\ (g(1), g(2), g(3), \ldots)$ is multiplicative if and only if g is multiplicative.

(b) A function $f : \mathbb{Q}^+ \to \mathbb{C}$ defined on the positive rationals is multiplicative if f is not identically zero and $f(p_1^{a_1} \cdots p_k^{a_k}) = f(p_1^{a_1}) \cdots f(p_k^{a_k})$ for distinct primes $p_1, \ldots, p_k$ and $a_1, \ldots, a_k \in \mathbb{Z}$. We note that this holds if and only if the function $\tilde{f} : \mathbb{N}^2 \to \mathbb{C}$ defined by $\tilde{f}(i, j) = f(i/j)$ is multiplicative as a function of two variables.

Thus a multiplicative Toeplitz matrix $A = (f(i/j))_{i,j\geq 1}$ is multiplicative if and only if f is multiplicative as a function on $\mathbb{Q}^+$.

1.2 Bounded operators on ℓ^2; Hilbert-Schmidt operators

A linear operator $A : H_1 \to H_2$ between Hilbert spaces is *bounded* if $\|Ax\| \leq c\|x\|$ for all $x \in H_1$ for some $c \geq 0$, with $\|A\|$, the *operator norm* defined to be the infimum of such c. Let $\mathcal{B}(H_1, H_2)$ denote the Banach space of all such operators; if $H_1 = H_2 = H$, we write $\mathcal{B}(H)$ – this is a Banach algebra. The *spectrum* of $A \in \mathcal{B}(H)$, denoted by $\sigma(A)$, is the set of complex numbers λ for which $\lambda I - A$ is invertible. For A *compact*,

$$\sigma(A) = \{\lambda_n(A) : n \in \mathbb{N}\} \cup \{0\},$$

where $|\lambda_n(A)|$ decreases to 0, the non-zero $\lambda_n(A)$ being eigenvalues. In that case A^*A is self-adjoint and compact, whose spectrum lies in $[0, \infty)$. The (positive) squareroots of the eigenvalues of A^*A

[1] More generally, $f : \mathbb{N}^k \to \mathbb{C}$ is *multiplicative* if f is not identically zero and

$$f(m_1n_1, \ldots, m_kn_k) = f(m_1, \ldots, m_k)f(n_1, \ldots, n_k) \qquad (*)$$

if $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. (See [16] for a survey of results on multi-variable multiplicative functions.)

are called the *singular values*[2] of A. Denote these by $s_1(A), s_2(A), \ldots$, where $s_n(A)$ decreases. For $p > 0$, define the *Schatten* p-class S_p of operators A for which $(s_n(A)) \in \ell^p$.[3] Two important classes are S_1 – the *Trace-class* operators — and S_2 — the *Hilbert-Schmidt* operators.

Some relevant properties (see [7])

(a) *Weyl inequalities*: for all $n \geq 1$, we have (i) $\prod_{m=1}^n |\lambda_m(A)| \leq \prod_{m=1}^n s_m(A)$ and (ii), for any $p > 0$, $\sum_{m=1}^n |\lambda_m(A)|^p \leq \sum_{m=1}^n s_m(A)^p$.

(b) The space of all trace-class operators on a Hilbert space is a Banach algebra with norm

$$\|A\|_1 = \sum_{n=1}^{\infty} s_n(A).$$

For these the usual trace formula holds: $\mathrm{tr}\,(A) = \sum_{n=1}^{\infty} \lambda_n(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle$ for any orthonormal basis (e_n).

(c) If A and B are Hilbert-Schmidt, then AB is trace-class, and $\|AB\|_1 \leq \|A\|_2\|B\|_2$. The Hilbert-Schmidt operators on H form a Hilbert space with inner product and norm given by

$$\langle A, B\rangle = \mathrm{tr}(AB^*) \qquad \|A\|_2 = \sqrt{\mathrm{tr}(AA^*)} = \sqrt{\sum_{n=1}^{\infty} s_n(A)^2}.$$

We have $\|A\| = s_1(A)$, so $\|A\| \leq \|A\|_2 \leq \|A\|_1$.

(d) Suppose A has matrix representation (a_{ij}) w.r.t. some orthonormal basis. Then A is Hilbert-Schmidt if and only if

$$\sum_{i,j\geq 1} |a_{ij}|^2 < \infty,$$

in which case the sum equals $\|A\|_2^2$ (irrespective of the basis).

If $A = (f(m,n))_{m,n\in\mathbb{N}}$ is multiplicative, we see that A is Hilbert-Schmidt if and only if

$$\|A\|_2^2 = \sum_{m,n=1}^{\infty} |f(m,n)|^2 = \prod_p \sum_{k,l\geq 0} |f(p^k,p^l)|^2 \quad \text{converges.} \tag{1.2}$$

Let $\Delta_p = (\sum_{k,l\geq 0} |f(p^k,p^l)|^2 - 1)^{1/2}$. Thus $A \in S_2$ if and only if $\sum_p \Delta_p^2$ converges, in which case $\|A\|_2^2 = \prod_p (1 + \Delta_p^2)$.

(e) Let A, A_n be compact operators and suppose $A_n \to A$ in operator norm. Then $s_r(A_n) \to s_r(A)$ for every $r \geq 1$. This follows directly from the inequality $|s_r(A_n) - s_r(A)| \leq \|A_n - A\|$ (see [7], Chapter VI, Corollary 1.6).

1.3 Tensor products of Hilbert spaces and operators

As tensor products of spaces and bounded linear operators feature heavily, we shall briefly state some relevant facts. First, for finite matrices, the definition is elementary. For matrices $A = (a_{ij})_{i,j\leq n}$ and B, the *Tensor or Kronecker product* $A \otimes B$ (see for example [18]) is the matrix

$$\begin{pmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \ldots & a_{nn}B \end{pmatrix}.$$

[2]More generally, for a bounded linear operator $A : H_1 \to H_2$, one defines the n^{th}-singular value by

$$s_n(A) = \inf\{\|A - F\| : \mathrm{rank}\, F < n\}.$$

Then A is compact if and only if $s_n(A) \to 0$ as $n \to \infty$.

[3]Here, as usual, ℓ^p is the space of complex sequences $(x_n)_{n\in\mathbb{N}}$ for which $\sum |x_n|^p$ converges. More generally, given F a countable set, $\ell^p(F)$ is the space of complex sequences $(x_f)_{f\in F}$ for which $\sum_{f\in F} |x_f|^p$ converges.

Regarding A and B as linear operators on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively (for some m), we see that $A \otimes B$ can be regarded as a linear operator on $\mathbb{R}^{mn}$. Tensor products satisfy many nice properties: thus A, B normal/unitary/invertible implies $A \otimes B$ normal/unitary/invertible respectively. In the last case, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$. More importantly for our purposes, we also have $\|A \otimes B\| = \|A\|\|B\|$ and $\sigma(A \otimes B) = \sigma(A)\sigma(B) = \{\lambda\mu : \lambda \in \sigma(A), \mu \in \sigma(B)\}$; i.e. the eigenvalues of $A \otimes B$ are the products of the eigenvalues of A and B (see [18]).

More generally, tensor products of operators may be defined (see for example [5], [12], or the notes by Berberian [1]). For this, we first need the tensor product of two Hilbert spaces, H_1, H_2. This may be defined to be the Hilbert space

$$H_1 \otimes H_2 = \{T : H_2 \to H_1 \text{ such that } T \text{ is conjugate-linear and Hilbert-Schmidt } \}.$$

This is indeed a Hilbert space with inner product

$$\langle S, T \rangle = \sum_{k=1}^{\infty} \langle S f_k, T f_k \rangle,$$

for an orthonormal basis (f_k) of H_2, the sum being independent of the basis. Now for $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$, their *tensor product* $A \otimes B$ is the bounded linear operator on $H_1 \otimes H_2$ defined by

$$(A \otimes B)(T) = ATB^*.$$

Also for $x \in H_1$ and $y \in H_2$, define $x \otimes y : H_2 \to H_1$ by $(x \otimes y)(z) = \langle y, z \rangle x$. We note that if (e_n) and (f_n) are orthonormal bases of H_1 and H_2 respectively, then $(e_m \otimes f_n)$ is an orthonormal basis for $H_1 \otimes H_2$.

Some relevant properties:

(a) $(A \otimes B)(x \otimes y) = Ax \otimes By$

(b) $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle$.

(c) $\sigma(A \otimes B) = \sigma(A)\sigma(B) = \{\lambda\mu : \lambda \in \sigma(A), \mu \in \sigma(B)\}$ (see [3]).

(d) $\|A \otimes B\| = \|A\|\|B\|$.

(e) A, B normal/unitary/invertible implies $A \otimes B$ normal/unitary/invertible respectively, with $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Infinite tensor products

We shall require infinite tensor products, which were originally considered by von Neumann [17], and further developed by other authors (see for example Guichardet [8], Nakagami [15]). Consider a family of Hilbert spaces $(H_i)_{i \in I}$, where I is some indexing set. For each $i \in I$, let t_i be a unit vector in H_i. First, form the *inductive limit* of the finite tensor product $\otimes_{i \in J} H_i$, for finite subsets J of I with respect to $t = (t_i)_{i \in I}$, which we denote by $\otimes_{i \in I}^t H_i$. Thus if $x_i = t_i$ for all i except for finitely many values, then $\otimes x_i$ is in the infinite product as is every linear combination of such terms. Define an inner product by

$$\langle \otimes x_i, \otimes y_i \rangle = \prod_i \langle x_i, y_i \rangle$$

and extend linearly. Now take its Hilbert space completion. By abusing notation, we shall denote this by $\otimes_{i \in I} H_i$ as our sequence t_i will be fixed throughout. We note that any sequence of unit vectors $u_i \in H_i$ for which $\prod_i \langle t_i, u_i \rangle$ converges absolutely[4] gives rise to the same space ([8], Theorem 2).

Now suppose for each $i \in I$, T_i is a bounded linear operator on H_i. We can form the infinite tensor product of the T_i. We quote Proposition 6 from [8], supplemented by a result of Nakagami

[4] That is, $\sum_i |\langle t_i, u_i \rangle - 1|$ converges.

[15].

Theorem A (after Guichardet [8] and Nakagami [15])
Let $T_i \in \mathcal{B}(H_i)$ for each $i \in I$ and suppose that $\prod_i \|T_i\|$ converges to a non-zero value and the products

$$\prod_i \|T_i t_i\| \quad \textit{and} \quad \prod_i \langle T_i t_i, t_i\rangle$$

both converge absolutely. Then there exists a unique $T \in \mathcal{B}(\otimes_i H_i)$ such that $\otimes_{i\in J} T_i$ converges strongly to T. We write $T = \otimes_i T_i$. Furthermore, $T \neq 0$ and if $\otimes x_i$ exists, then $\otimes T_i x_i$ exists and equals $T(\otimes x_i)$. Finally, $\|T\| = \prod_i \|T_i\|$.

Proof. The assumption that $\prod_i \|T_i\|$ converges implies the existence of T (see Proposition 6 from [8]) and the strong convergence (see Theorem 3.1 (I) from [15]), including the fact that $\|T\| \leq \prod_i \|T_i\|$. The extra assumption that $\prod_i \|T_i\| \neq 0$ further ensures $T \neq 0$ and $\|T\| = \prod_i \|T_i\|$ (from the proof of Theorem 3.1 (II) from [15]).

□

Remarks 1

(a) For the purposes of this paper, our indexing set I will be $\mathbb{P}$ – the set of primes – and each H_p ($p \in \mathbb{P}$) will be a copy of ℓ^2. More precisely, with $\langle p\rangle = \{p^k : k \in \mathbb{N}_0\}$, we shall take $H_p = \ell^2(\langle p\rangle)$. After Proposition 2.3, we shall see that, with $t_p = e := (1, 0, 0, \ldots)$ for all p, we have $\otimes_p \ell^2(\langle p\rangle) \cong \ell^2$.

(b) Although infinite tensor products (of spaces, operators and algebras) have been discussed by many authors, we could not find any ready made result connecting $\sigma(T)$ and $\prod_i \sigma(T_i)$. It may be the case that in general this is false, though for compact T it is true as we show in Theorem 2.4.

§2. Main results

Our main result is to describe the tensor structure of multiplicative A and its spectrum. We distinguish between the finite and infinite cases. The finite case is in many respects simpler, just using the elementary notion of tensor (or Kronecker) product of matrices.

2.1 The finite case. The following result was inspired by a result of Codecá and Nair [4] and generalises it. Let $D(k)$ denote the set of divisors of k, and $d(k) = |D(k)|$, the number of divisors of k.

Theorem 2.1
Let $f : D(k) \times D(k) \to \mathbb{C}$ be multiplicative. For $n|k$, let A_n denote the $d(n) \times d(n)$ matrix with entries $f(c, d)$ where $c, d|n$. Then for $(m, n) = 1$ with $m, n|k$ we have[5]

$$A_m \otimes A_n = A_{mn}.$$

Proof. With $(m, n) = 1$ we have

$$A_m \otimes A_n = (f(c, d)A_n)_{c,d|m}.$$

The r, s entry (with $r, s|n$) inside 'block' cd is $f(c, d)f(r, s) = f(cr, ds)$ by multiplicativity of f (using $(cd, rs) = 1$). Note that cr and ds run through the divisors of mn respectively. Thus the rows and columns of $A_m \otimes A_n$ are the same as those of A_{mn} though possibly in a different order; i.e. they are permutation similar.

□

[5]Here we identify two matrices A and B if they are permutation similar; i.e. if $A = P^{-1}BP$ for some permutation matrix P; equivalently, A and B have the same rows and columns but in a different order.

Writing, as is usual, $p^r \| n$ to mean $p^r | n$ but $p^{r+1} \not| n$, we have:

Corollary 2.2
We have $A_n = \otimes_{p^r \| n} A_{p^r}$ *and*

$$\sigma(A_n) = \prod_{p^r \| n} \sigma(A_{p^r}).$$

Remark 2 Codecá and Nair [4] proved the above for matrices with entries of the form

$$f(m,n) = g((m,n))h([m,n]) \tag{CN}$$

where g, h are multiplicative[6]. As such, f is multiplicative (of two variables) as can be readily verified, but the converse is false: there are multiplicative $f(m,n)$ not of this form. A simple reason is that f need not be symmetrical. But even assuming f is symmetrical, the converse still fails. For if f is of the form (CN) then for p prime and $k \le l$,

$$f(p^k, p^l) f(p^k, 1) = f(p^k, p^k) f(p^l, 1).$$

But in general, $f(p^k, p^l)$ can be any function of k and l (for fixed p).

2.2 The infinite case. Let k be squarefree, say $k = p_1 \dots p_r$, where $p_1, \dots, p_r$ are distinct primes. Denote by $\langle k \rangle$ the set of natural numbers whose prime factors are those of k; i.e. $\langle k \rangle = \{p_1^{a_1} \dots p_r^{a_r} : a_1, \dots, a_r \in \mathbb{N}_0\}$.

For $f : \mathbb{N}^2 \to \mathbb{C}$ multiplicative, let $f_k : \mathbb{N}^2 \to \mathbb{C}$ be defined by

$$f_k(m,n) = \begin{cases} f(m,n) & \text{if } mn \in \langle k \rangle \\ 0 & \text{otherwise} \end{cases}.$$

In particular, for a prime p, $f_p(\cdot, \cdot)$ is supported on $\{(p^k, p^l) \in \mathbb{N}^2 : k, l \ge 0\}$. Let A_k and $\tilde{A}_k$ denote the operators induced by the following infinite matrices:

$$A_k = \Big(f_k(m,n)\Big)_{m,n \ge 1} \quad \text{and} \quad \tilde{A}_k = \Big(f_k(m,n)\Big)_{m,n \in \langle k \rangle}.$$

We can equally regard A_k on ℓ^2 or $\tilde{A}_k$ on $\ell^2(\langle k \rangle)$, and $\|A_k\| = \|\tilde{A}_k\|$.

For p prime, $\tilde{A}_p = (f(p^k, p^l))_{k,l \ge 0}$. Note that for A Hilbert-Schmidt, (1.2) says that

$$\|A\|_2 = \prod_p \|\tilde{A}_p\|_2.$$

Proposition 2.3
Let k and l be coprime and squarefree, and let A_k, A_l be as defined above. If A_k and A_l are bounded on ℓ^2, then A_{kl} is bounded on ℓ^2 and

$$A_{kl} = A_k \otimes A_l,$$

in the sense that the operators on the left and right have the same matrix representations.

Proof. $A_k \otimes A_l$ is the bounded linear operator on the Hilbert space

$$H = \{T : \ell^2 \to \ell^2 \text{ such that } T \text{ is conjugate-linear and Hilbert-Schmidt}\}$$

given by

$$(A_k \otimes A_l)T = A_k T A_l^*.$$

[6] More specifically, they had $g(n) = \frac{l(n)}{n}$ and $h(n) = \frac{1}{n}$ for some multiplicative function $l(n)$, but their argument clearly works for the more general g, h.

Let $(e_n)_{n\in\mathbb{N}}$ denote the usual basis of ℓ^2 (i.e. e_n is the vector with a 1 in the n^{th}-place and zeros elsewhere). Then $(e_m \otimes e_n)_{m,n\geq 1}$ is an orthonormal basis of H. The matrix representation of $A_k \otimes A_l$ is therefore given by

$$\begin{aligned}\langle (A_k \otimes A_l)(e_{m_2} \otimes e_{n_2}), e_{m_1} \otimes e_{n_1}\rangle &= \langle A_k e_{m_2} \otimes A_l e_{n_2}, e_{m_1} \otimes e_{n_1}\rangle && \text{(by 1.3(a))}\\ &= \langle A_k e_{m_2}, e_{m_1}\rangle\langle A_l e_{n_2}, e_{n_1}\rangle && \text{(by 1.3(b))}\\ &= f_k(m_1, m_2) f_l(n_1, n_2).\end{aligned}$$

Now, for the RHS to be non-zero, we need $m_1, m_2 \in \langle k\rangle$ and $n_1, n_2 \in \langle l\rangle$. Since $(k,l) = 1$, as such we require $(m_1m_2, n_1n_2) = 1$. But then

$$f_k(m_1, m_2) f_l(n_1, n_2) = f(m_1, m_2) f(n_1, n_2) = f(m_1n_1, m_2n_2),$$

by multiplicativity of f. Thus

$$\langle (A_k \otimes A_l)(e_{m_2} \otimes e_{n_2}), e_{m_1} \otimes e_{n_1}\rangle = \begin{cases} f(m_1n_1, m_2n_2) & \text{if } m_1m_2 \in \langle k\rangle \text{ and } n_1n_2 \in \langle l\rangle \\ 0 & \text{otherwise}\end{cases}. \tag{A}$$

On the other hand,

$$\langle A_{kl} e_n, e_m\rangle = f_{kl}(m, n) = \begin{cases} f(m,n) & \text{if } mn \in \langle kl\rangle \\ 0 & \text{otherwise}\end{cases}. \tag{B}$$

In (A), if $m_1m_2 \in \langle k\rangle$ and $n_1n_2 \in \langle l\rangle$, then $m_1n_1, m_2n_2 \in \langle kl\rangle$ is forced and moreover every element of $\langle kl\rangle$ is (uniquely) of the form $m'n'$ with $m' \in \langle k\rangle$, $n' \in \langle l\rangle$. Thus there is a one-one correspondence between the matrix entries in (A) and those in (B) by writing $m = m_1n_1$ and $n = m_2n_2$.

□

It follows directly that $A_k = \bigotimes_{p|k} A_p$ and (from section 1.3) that $\|A_k\| = \prod_{p|k} \|A_p\|$ and $\sigma(A_k) = \prod_{p|k} \sigma(A_p)$.

Theorem 2.4
Let A be a multiplicative matrix such that for every prime p, A_p is bounded (i.e. $\tilde{A}_p$ is bounded on $\ell^2(\langle p\rangle)$) and $\prod_p \|A_p\|$ converges to a non-zero value. Then A is bounded on ℓ^2 and is given by the infinite tensor product[7]

$$A = \bigotimes_p A_p$$

and $\|A\| = \prod_p \|A_p\|$. Furthermore, if each A_p is compact, then $\sigma(A) \supset \prod_p \sigma(A_p)$, while if A is compact, then $\sigma(A) = \prod_p \sigma(A_p)$.

Proof. We check that the conditions of Theorem A are satisfied. With $t_p = e = (1, 0, \ldots)$ and $A = (f(m,n))_{m,n\geq 1}$, we see that $\langle A_p t_p, t_p\rangle = 1$ for all primes p and

$$1 \leq \|A_p t_p\| = \sqrt{\sum_{k=0}^{\infty} |f(p^k, 1)|^2} \leq \|A_p\|.$$

Thus both $\prod_p \langle A_p t_p, t_p\rangle$ and $\prod_p \|A_p t_p\|$ converge absolutely. Applying Theorem A shows there exists a unique non-zero $T \in \mathcal{B}(\otimes_p \ell^2(\langle p\rangle))$ for which

$$T(\otimes_p x_p) = \otimes_p A_p x_p$$

whenever $\otimes_p x_p$ exists.

[7] Always with respect to the sequence $t_p = e\ \forall p \in \mathbb{P}$.

Now, $A_k \to T$ strongly as $k \to \infty$ through the numbers $k = \prod_{p \le P} p$. It follows that

$$\langle (T - A_k)e_n, e_m \rangle \to 0$$

as $k \to \infty$, for all $m, n \in \mathbb{N}$. But $\langle A_k e_n, e_m \rangle = f_k(m,n)$, so

$$\langle Te_n, e_m \rangle = f_k(m,n) + \langle (T - A_k)e_n, e_m \rangle \to f(m,n),$$

as $k \to \infty$. Thus $\langle Te_n, e_m \rangle = \langle Ae_n, e_m \rangle$ for all $m, n \in \mathbb{N}$, and so $T = A$. The formula $\|A\| = \prod_p \|A_p\|$ follows immediately from Theorem A.

Now suppose each A_p is compact. We need to show that if $\lambda_k \in \sigma(A_k)$ and $\lambda_k \to \lambda$, then $\lambda \in \sigma(A)$.

So suppose $\lambda_k \to \lambda$ but that $\lambda \notin \sigma(A)$. Then $(A - \lambda I)^{-1}$ exists and $\|(A - \lambda I)x\| \ge c\|x\|$ for some $c > 0$ and all x. (Indeed we can take $c = 1/\|(A - \lambda I)^{-1}\|$.) Let x_k be such that $\|x_k\| = 1$ and $A_k x_k = \lambda_k x_k$. Then

$$\|(A - \lambda I)x_k\| \le \|(A - A_k)x_k\| + \|(A_k - \lambda I)x_k\| \le \|A - A_k\| + |\lambda_k - \lambda| \to 0$$

as $k \to \infty$. But this contradicts $\|(A - \lambda I)x_k\| \ge c$ from above. Thus $\sigma(A) \supset \prod_p \sigma(A_p)$.

Finally, if also A is compact, then we need to show the converse holds. That is, we need to show every $\lambda \in \sigma(A)$ is a limit of a sequence $\lambda_k \in \sigma(A_k)$. But this follows from a general result about compact operators, namely Lemma 5 from [6] (Chapter XI, section 9.5). Thus in this case $\sigma(A) = \prod_p \sigma(A_p)$.

□

Further results on the spectrum

As we see from the above results, the spectrum of a multiplicative matrix operator has a multiplicative structure, at least in the case that A is compact. Whether $\sigma(A) = \prod_p \sigma(A_p)$ holds more generally remains to be seen. In this case, the non-zero values of $\sigma(A)$ are all eigenvalues, and these are products of the eigenvalues of the simpler operators A_p (equivalently $\tilde{A}_p$ since $\sigma(A_p) = \sigma(\tilde{A}_p)$). Writing

$$\sigma(A) = \{\lambda_n(A) : n \ge 1\} \cup \{0\} \quad \text{and} \quad \sigma(\tilde{A}_p) = \{\lambda_n(A_p) : n \ge 1\} \cup \{0\} \tag{2.1}$$

where $|\lambda_n(A)|$ and $|\lambda_n(A_p)|$ decrease to zero as n increases, we see that every $\lambda \in \sigma(A) \setminus \{0\}$ is of the form

$$\lambda = \prod_p \lambda_{n_p}(A_p) \quad \text{(for some } n_p \in \mathbb{N}\text{)}. \tag{2.2}$$

It is natural to investigate the behaviour of $\lambda_n(A_p)$ as $p \to \infty$ for each n.

Proposition 2.5

Let A be multiplicative, non-zero and compact such that each A_p is in S_2. With $\sigma(A)$ and $\sigma(A_p)$ as given by (2.1)*, we have, as $p \to \infty$*

$$\lambda_n(A_p) \to \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}.$$

Consequently, every non-zero eigenvalue of A is of the form

$$\lambda_1(A) \prod_{p \in F} \frac{\lambda_{n_p}(A_p)}{\lambda_1(A_p)}$$

for some finite set F of primes and some $n_p \in \mathbb{N}$.

Proof. First note that $\lambda_1(A) = \prod_p \lambda_1(A_p)$ since it is the largest eigenvalue (in modulus). This implies (using $\lambda_1(A) \ne 0$) that $\lambda_1(A_p) \to 1$ as $p \to \infty$. Recall from the end of section 1.2 we had

$$\|A_p\|_2^2 = 1 + \Delta_p^2.$$

Let $E = (e_{ij})_{i,j\geq 0}$ with $e_{00} = 1$ and $e_{ij} = 0$ otherwise. Trivially $s_1(E) = 1$ and $s_n(E) = 0$ for $n > 1$. Then, by 1.2(e),

$$|s_n(A_p) - s_n(E)| \leq \|A_p - E\| \leq \|A_p - E\|_2 = \Delta_p \to 0 \quad \text{as } p \to \infty.$$

Thus $s_1(A_p) = 1 + O(\Delta_p) \to 1$ and $s_n(A_p) \leq \Delta_p \to 0$ for $n > 1$. By 1.2(a), we have

$$|\lambda_1(A_p)\lambda_2(A_p)| \leq s_1(A_p)s_2(A_p).$$

From above it now follows that $\lambda_2(A_p) \ll \Delta_p$ and so $\lambda_n(A_p) \to 0$ for $n \geq 2$.

Thus for $\lambda \neq 0$, we need $n_p = 1$ for all but finitely many p in (2.2). In other words,

$$\lambda = \lambda_1(A) \prod_{p\in F} \frac{\lambda_{n_p}(A_p)}{\lambda_1(A_p)}$$

for some finite set F of primes, as required.

□

Remark 3 If $|\lambda_1(A_p)| > |\lambda_n(A_p)|$ for all $n > 1$ and all p, then $|\lambda_1(A)/\lambda| = \prod_{p\in F} \gamma_p$, where $\gamma_p = |\lambda_1(A_p)/\lambda_{n_p}(A_p)| > 1$; i.e. the (multi-)set

$$\left\{ \left| \frac{\lambda_1(A)}{\lambda} \right| : \lambda \in \sigma(A) \right\}$$

can be regarded as the squarefree numbers of a generalized prime system with generalized primes γ_p. This happens for example when A is *non-negative*; i.e. where $f(m,n) \geq 0$ for all $m,n \in \mathbb{N}$. This is due to the Kreĭn-Rutman theorem [13].

§3. Examples, applications and some open questions

First we give an example to illustrate the results, showing how much rich structure the spectrum can have, even in a simple case. Then we consider more involved examples, including multiplicative Toeplitz and Hankel operators. First we start with a 'trivial' case.

(a) Let $A = (f(m,n))_{m,n\geq 1}$ where $f(m,n) = g(m)h(n)$ with g,h multiplicative[8]. Note that A is bounded if and only if $A \in S_2$ if and only if $g,h \in \ell^2$. As such, we find that the only (non-zero) eigenvalue of A is $\lambda = \sum_{n=1}^{\infty} g(n)h(n)$, with corresponding eigenvector $v_g = (g(n))_{n\geq 1}$. For let $x = (x_n)_{n\geq 1}$. Then

$$Ax = A_g A_h^T x = cv_g,$$

where A_g is the matrix with first column v_g and zero elsewhere (ditto for A_h), and $c = \sum_{n=1}^{\infty} h(n)x_n$. So we have a (non-zero) eigenvector if and only if $x = av_g$ for some a; i.e. $x_n = ag(n)$. As such $c = a\lambda$ and $Ax = \lambda x$; i.e. λ is the only eigenvalue.

Thus for interesting examples, we need f non-firmly multiplicative.

(b) $A = (f(m,n))_{m,n\geq 1}$ with f multiplicative such that $f(p^k,p^l) = 0$ if both $k,l \geq 1$. Note that in this case, for some g,h multiplicative

$$f(m,n) = \begin{cases} g(m)h(n) & \text{if } (m,n) = 1 \\ 0 & \text{if } (m,n) > 1 \end{cases}.$$

Note that A_f is Hilbert-Schmidt if and only if $\sum_{n\in\mathbb{N}}(|g(n)|^2 + |h(n)|^2)$ converges.

Then each A_p has two (non-zero) eigenvalues;

$$\frac{1}{2} \pm \sqrt{\frac{1}{4} + \kappa(p)} \quad \text{where} \quad \kappa(p) = \sum_{n=1}^{\infty} g(p^n)h(p^n).$$

[8]Such functions f are also called *firmly* multiplicative.

To see this, consider (equivalently) the equation $\tilde{A}_p x = \lambda x$ with $\lambda, x = (x_k)_{k\geq 0} \neq 0$; i.e.

$$\tilde{A}_p x = \begin{pmatrix} 1 & h(p) & h(p^2) & \dots \\ g(p) & 0 & 0 & \dots \\ g(p^2) & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}.$$

This gives the equations

$$x_0 + \sum_{n=1}^{\infty} h(p^n) x_n = \lambda x_0, \qquad g(p^n) x_0 = \lambda x_n \quad (n \geq 1).$$

Then $x_0 \neq 0$ (otherwise $x = 0$ follows) and so $\lambda^2 = \lambda + \kappa(p)$. Observe that

$$\sum_p |\kappa(p)| \leq \frac{1}{2} \sum_p \sum_{n=1}^{\infty} |f(p^n, 1)|^2 + |f(1, p^n)|^2 < \infty,$$

so $\kappa(p) \to 0$ as $p \to \infty$. Thus the eigenvalues λ of A are given by

$$\lambda = \prod_p \Big(\frac{1}{2} \pm \sqrt{\frac{1}{4} + \kappa(p)} \Big)$$

where there are only finitely many minus signs. We can rewrite this as

$$\lambda = \lambda_1 \prod_{p \in F} \frac{\frac{1}{2} - \sqrt{\frac{1}{4} + \kappa(p)}}{\frac{1}{2} + \sqrt{\frac{1}{4} + \kappa(p)}}$$

where λ_1 is the largest eigenvalue and F is a finite set of primes. If $\kappa(p) > 0$ for each p, we can further rewrite this as

$$\frac{\lambda_1 (-1)^{|F|}}{\prod_{p \in F} \gamma_p},$$

where $\gamma_p > 1$ for all p. As such, the set of $|\lambda_1/\lambda|$ can be regarded as the squarefree numbers of the generalized prime system with g-primes γ_p.

(c) Let $A = (\frac{(ij)^\alpha}{[i,j]})_{i,j\geq 1}$ ($= E_\alpha$ as discussed in the introduction). It is easy to see that A is Hilbert-Schmidt precisely for $\alpha < \frac{1}{4}$ (see [10]), while it is clearly unbounded for $\alpha \geq \frac{1}{2}$. Now we can show that in the remaining range, A is bounded. We do this by finding explicitly the spectrum. Indeed, we shall show that for $\alpha < \frac{1}{2}$, the eigenvalues of A_p are all the λ which satisfy

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1-\frac{1}{p})^n}{p^{(1-2\alpha)\frac{n(n-1)}{2}} \lambda^n} \prod_{r=1}^{n} \Big(\frac{1}{1 - p^{-2\alpha + r(2\alpha - 1)}} \Big) \Big(\frac{1}{1 - p^{r(2\alpha-1)}} \Big) = 1 \tag{3.1}$$

from which we deduce that

$$\|A_p\| = 1 + O\Big(\frac{1}{p^{2-2\alpha}} \Big). \tag{3.2}$$

Thus $\prod_p \|A_p\|$ converges for $\alpha < \frac{1}{2}$ and A is bounded by Theorem 2.4.

Proof of (3.1) *and* (3.2). We have $\tilde{A}_p = (p^{\alpha(k+l) - \max\{k,l\}})_{k,l\geq 0}$, which is of the form $B_{x,y} = (x^{\max\{k,l\}} y^{\min\{k,l\}})_{k,l\geq 0}$, with $x = p^{\alpha-1}$ and $y = p^\alpha$. It is straightforward to show that for general x, y

$$B_{x,y} \text{ is compact} \iff B_{x,y} \in S_2 \iff |x| \text{ and } |xy| < 1$$

in which case

$$\|B_{x,y}\|_2 = \sqrt{\frac{1 + |x|^2}{(1 - |x|^2)(1 - |xy|^2)}}.$$

In our case $xy = p^{2\alpha-1} < 1$ and each $\tilde{A}_p$ is Hilbert-Schmidt, with $\|A_p\|_2 = 1 + O(p^{-2(1-2\alpha)})$. Note further that $\|A_p\| \to 1$ as $p \to \infty$ since $1 \le \|A_p\| \le \|A_p\|_2$.

We find the eigenvalues by solving $B_{x,y}a = \lambda a$ for $\lambda \neq 0$ and $a = (a_0, a_1, \ldots) \in \ell^2$, $a \neq 0$. We do it in general for x, y real such that $0 < x, xy < 1$. Equating coefficients we find that

$$\lambda a_n = x^n \sum_{k=0}^{n} a_k y^k + y^n \sum_{k=n+1}^{\infty} a_k x^k \quad \text{for each } n \ge 0. \tag{3.3}$$

A simple manipulation of (3.3) shows that a_n satisfies

$$a_{n+1} - (x+y)a_n + xya_{n-1} = \frac{x-y}{\lambda}(xy)^n a_n \quad \text{for each } n \ge 1. \tag{3.4}$$

Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$, which has radius of convergence ρ say. We prove that $\rho \ge \frac{1}{x}$. From (3.3) note that $a_n \ll nx^n + (xy)^n$, so $\rho > 1$. Write $y = (\frac{1}{x})^\beta$ for some $\beta < 1$ and assume $\rho \ge (\frac{1}{x})^\gamma$. If $\gamma > \beta$, then $\rho > y$ and so $\lambda a_n = (A(y) + o(1))x^n$ and $\rho \ge \frac{1}{x}$. So suppose $0 \le \gamma \le \beta$; i.e.

$$a_n = O((1+\varepsilon)^n x^{\gamma n}) \quad \text{for all } \varepsilon > 0.$$

Now from (3.3),

$$\lambda a_n \ll_\varepsilon x^n \sum_{k=0}^{n} (1+\varepsilon)^k x^{(\gamma-\beta)k} + y^n \sum_{k=n+1}^{\infty} (1+\varepsilon)^k x^{\gamma k + k} \ll_\varepsilon (1+\varepsilon)^n x^{(\gamma+1-\beta)n}.$$

Hence $\rho \ge (\frac{1}{x})^{\gamma+1-\beta}$; i.e. if $\rho \ge (\frac{1}{x})^\gamma$ and $\gamma \le \beta$, then $\rho \ge (\frac{1}{x})^{\gamma+1-\beta}$. If $\gamma + 1 - \beta \le \beta$, we can apply the same procedure, adding $1 - \beta$ to the exponent. Eventually, we increase the lower bound to find $\gamma > \beta$. As we have seen, $\rho \ge \frac{1}{x}$ follows.

It follows that $y < \rho$ and so

$$\lambda a_n = (A(y) + o(1))x^n.$$

Again using (3.3), we can obtain better approximations. By induction it is easy to prove that for each $R \ge 1$,

$$\lambda a_n = A(y)x^n \sum_{r=0}^{R-1} c_r (xy)^{rn} + O(x^n (xy)^{Rn}),$$

where[9]

$$c_n = \frac{\lambda^{-n}(\frac{x}{y} - 1)^n (xy)^{\frac{n(n+1)}{2}}}{\prod_{r=1}^{n}(1 - \frac{x}{y}(xy)^r)(1 - (xy)^r)}.$$

Let b_n be defined by

$$\lambda b_n = A(y)x^n \sum_{r=0}^{\infty} c_r (xy)^{rn},$$

the series converging absolutely for any n. Then b_n also satisfies (3.4) as can be readily verified. Let $\delta_n = a_n - b_n$ which again satisfies (3.4), and from above we see that $\delta_n \ll A^{-n}$ for all $A > 1$. Thus $D(z) := \sum_{n=0}^{\infty} \delta_n z^n$ is entire and satisfies

$$(1 - xz)(1 - yz)D(z) = \delta_0 + t_0 z + \frac{x-y}{\lambda} z D(xyz).$$

(for some constant t_0). Considering the maximum size of $|D(z)|$ on $|z| = r$ (large) shows this is only possible if $D(z)$ is identically zero; i.e. $\delta_n = 0$. Thus $a_n = b_n$ for all $n \ge 0$; i.e.

$$\lambda a_n = A(y)x^n \sum_{r=0}^{\infty} c_r (xy)^{rn}.$$

[9]We use the convention that $\prod_{r=1}^{0} \cdots = 1$.

Multiplying through by y^n and summing over $n \geq 0$ gives

$$\lambda = \sum_{r=0}^{\infty} \frac{c_r}{1-(xy)^{r+1}}. \tag{3.5}$$

Let $Q_0(x,y) = 1$ and for $n \in \mathbb{N}$,

$$Q_n(x,y) = \prod_{r=1}^{n} (1 - x^r y^{r-2})(1 - x^r y^r).$$

Define the entire function

$$H(z) = H_{x,y}(z) = \sum_{n=0}^{\infty} \frac{(xy)^{\frac{n(n+1)}{2}}}{Q_n(x,y)} z^n.$$

Inserting the formula for c_r into (3.5), we see that the non-zero eigenvalues of $B_{x,y}$ are all the λ which satisfy

$$H\Big(\frac{\frac{x}{y} - 1}{\lambda xy}\Big) = 0.$$

In our case, $x = p^{\alpha-1}$ and $y = p^{\alpha}$. With these values, the above equation becomes (3.1). Now take $\lambda = \lambda_1$, the largest eigenvalue. Since A_p is non-negative we have $\lambda_1 = \|A_p\|$, and so $\lambda_1 \to 1$ as $p \to \infty$. Write $\beta = 1 - 2\alpha (> 0)$ and put $\mu_p = \frac{1}{\lambda_1}(1 - \frac{1}{p})$. Thus $\mu_p \to 1$. Then (3.1) says

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \mu_p^n p^{-\frac{\beta n(n-1)}{2}}}{Q_n(p^{\alpha-1}, p^{\alpha})} = 1. \tag{3.6}$$

Now, for $n \geq 1$,

$$Q_n(p^{\alpha-1}, p^{\alpha}) = \Big(1 - \frac{1}{p}\Big) \prod_{r=1}^{n} \Big(1 - \frac{1}{p^{\beta r}}\Big) \cdot \Big(1 + O\Big(\frac{1}{p^{1+\beta}}\Big)\Big).$$

Insert this into the LHS of (3.6) to give

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \mu_p^n p^{-\frac{\beta n(n-1)}{2}}}{\prod_{r=1}^{n} (1 - p^{-\beta r})} = \Big(1 - \frac{1}{p}\Big)\Big(1 + O\Big(\frac{1}{p^{1+\beta}}\Big)\Big).$$

Using the identity $\prod_{n=1}^{\infty}(1 + ax^n) = \sum_{n=0}^{\infty} \frac{a^n x^{\frac{n(n+1)}{2}}}{\prod_{r=1}^{n}(1-x^r)}$ (see [9], Theorem 348) leads to

$$\prod_{n=0}^{\infty} (1 - \mu_p p^{-\beta n}) = \frac{1}{p} + O\Big(\frac{1}{p^{1+\beta}}\Big).$$

Separating the $n = 0$ term from the rest gives $1 - \mu_p = \frac{1}{p} + O(\frac{1}{p^{1+\beta}})$, which leads to $\lambda_1 = 1 + O(\frac{1}{p^{1+\beta}})$; i.e. $\|A_p\| = 1 + O(\frac{1}{p^{2-2\alpha}})$. Thus A is bounded for $\alpha < \frac{1}{2}$.

Problem. Can we deduce that A is compact or even in some Schatten class?

(d) In [11], Multiplicative Toeplitz operators – i.e. those induced by matrices with ij^{th} entry of the form $c(i/j)$ – were studied, particularly in the case where $c(\cdot)$ is multiplicative as a function on the rationals; i.e. $C(m,n) := c(m/n)$ is multiplicative on $\mathbb{N}^2$.

Let f denote the 'symbol' given by the formal series

$$f(t) = \sum_{q \in \mathbb{Q}^+} c(q) q^{it}.$$

For p prime, let $f_p(t) = \sum_{k\in\mathbb{Z}} c(p^k)p^{kit}$, which we assume converges absolutely. Further let $f_p^\sharp : \mathbb{T} \to \mathbb{C}$ denote the function $f_p^\sharp(e^{i\theta}) = f_p(\frac{\theta}{\log p})$. Denoting the operator $A = (c(i/j))_{i,j\geq 1}$ by M_f, we have $\tilde{A}_p = (c(p^{k-l}))_{k,l\geq 0} = T(f_p^\sharp)$, the usual Toeplitz operator with symbol $f_p^\sharp$. Note that $\|T(f_p^\sharp)\| = \|f_p^\sharp\|_\infty$. Theorem 2.4 says that if $\prod_p \|f_p^\sharp\|_\infty$ converges, then M_f is bounded with

$$M_f = \bigotimes_p T(f_p^\sharp) \quad \text{and} \quad \|M_f\| = \prod_p \|f_p^\sharp\|_\infty.$$

Problem: When can we say that $\sigma(M_f) = \prod_p \sigma(T(f_p^\sharp))$?

As the spectrum of Toeplitz operators can often easily be identified, a positive answer means we can find $\sigma(M_f)$. Note that in this case the $T(f_p^\sharp)$ are not even compact (except in the trivial case when $f_p^\sharp = 0$).

(e) In the same way, Multiplicative Hankel operators may be defined as operators induced by matrices of the form $(c(ij))_{i,j\geq 1}$ for some arithmetical function $c(\cdot)$. A simple exercise shows that $c(ij)$ is multiplicative (as a function of two variables) if and only if $c(\cdot)$ is multiplicative.

As such, let A denote the matrix induced by $(c(ij))_{i,j\geq 1}$. Then $\tilde{A}_p$ is the Hankel operator $H(f_p^\sharp) = (c(p^{k+l}))_{k,l\geq 0}$. Thus, if $\prod_p \|H(f_p^\sharp)\|$ converges, we have

$$A = \bigotimes_p H(f_p^\sharp) \quad \text{and} \quad \|A\| = \prod_p \|H(f_p^\sharp)\|.$$

This tensor property was noted in [2]. If furthermore A is compact, then also

$$\sigma(A) = \prod_p \sigma(H(f_p^\sharp)).$$

This certainly holds if A is Hilbert-Schmidt (i.e. when $\sum_{n=1}^\infty d(n)|c(n)|^2$ converges).

Problem: More generally, when does this hold?

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