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Analysis of discontinuous Galerkin methods using mesh-dependent norms and applications to problems with rough data

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Abstract We prove the inf-sup stability of a discontinuous Galerkin scheme for second order elliptic operators in (unbalanced) mesh-dependent norms for quasi-uniform meshes for all spatial dimensions. This results in a priori error bounds in these norms. As an application we examine some problems with rough source term where the solution can not be characterised as a weak solution and show quasi-optimal error control.

1 Introduction

Discontinuous Galerkin (dG) methods are a popular family of non-conforming finite element-type approximation schemes for partial differential equations (PDEs) involving discontinuous approximation spaces. In the context of elliptic problems their inception can be traced back to the 1970s [5,21]; see also [1] for an accessible overview

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and history of these methods for second order problems. For higher order problems, for example the (nonlinear) biharmonic problem, dG methods are a useful alternative to using C^1-conforming elements whose derivation and implementation can become very complicated [5,13,23].

Inf-sup conditions form one part of the Banach–Nečas–Babuška condition which guarantees the well-posedness of a given variational problem. In this note, we shall describe an analytical framework to examine the stability of dG approximations for L_2 and H^2-like mesh-dependent norms. This is in keeping with the spirit of [3,4], where for continuous finite element methods the authors prove equivalent results for second and fourth order problems respectively. The present approach, however, is quite different and results in inf-sup stability for both L_2- and H^2-like mesh-dependent norms under the assumption that the underlying mesh is quasi-uniform.

The analysis presented utilises a new H^2-*conforming reconstruction operator*, based on Hsieh–Clough–Tocher-type C^1 reconstructions. Such reconstructions, based on nodal averaging, are used for the proof of a posteriori bounds for non-conforming methods for elliptic [7,13,18,24] and hyperbolic problems [12,16]. The new reconstruction operators presented below enjoys certain orthogonality properties; in particular, they are *adjoint orthogonal* to the underlying Hsieh–Clough–Tocher space and maintain the same stability bounds as the H^2-conforming reconstruction from [13].

The argument is quite general and allows the derivation of inf-sup stability results whenever the numerical scheme has a well posed discrete adjoint (dual) problem over an appropriately constructed non-conforming finite element space. This is contrary to the Aubin–Nitsche L_2 duality argument whereby it is the underlying partial differential operator itself that requires the well posedness of the adjoint *continuous* problem.

The use of these recovery operators is not limited to an a posteriori setting, indeed, they have been used to quantify inconsistencies appearing in standard interior penalty methods when the exact solution is not $H^2(\Omega)$ [17]. This allows for quasi optimal a priori bounds for elliptic problems under minimal regularity up to data oscillation. Fundamentally the assumption in this analysis is that the singularity arises from the geometry of the domain rather than through the problem data itself. Our analysis allows us to show quasi-optimal L_2 convergence to problems that have rough problem data. To showcase the result we study the convergence of a method posed for an elliptic problem whose source term is not H^{-1} in both 1 and 2 spatial dimensions. In this case the Aubin–Nitsche, and indeed the standard treatment of Galerkin methods, are not applicable.

The note is set out as follows: In Sect. 2 we introduce the problem and present the analysis cumulating in inf-sup stability for problems with smooth data. In Sect. 3 we examine a particular problem with rough data and prove quasi-optimal convergence in this case. In addition we give some numerical validation of the method.

2 Problem set up and discretisation

To highlight the main steps of the present developments in this area, we consider the Poisson problem with homogeneous Dirichlet boundary conditions. Let $\Omega \subset \mathbb{R}^d$ be an open convex domain and consider the problem: given $f \in L_2(\Omega)$ find $u \in H^2(\Omega) \cap H_0^1(\Omega)$, such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \, v \, dx \quad \forall \, v \in H_0^1(\Omega). \tag{1}$$

We consider $\mathscr{T}$ to be a conforming triangulation of Ω, namely, $\mathscr{T}$ is a finite family of sets such that

(1) $K \in \mathscr{T}$ implies K is an open simplex (segment for $d = 1$, triangle for $d = 2$, tetrahedron for $d = 3$),
(2) for any $K, J \in \mathscr{T}$ we have that $\overline{K} \cap \overline{J}$ is a full lower-dimensional simplex (i.e., it is either $\emptyset$, a vertex, an edge, a face, or the whole of $\overline{K}$ and $\overline{J}$) of both $\overline{K}$ and $\overline{J}$ and
(3) $\bigcup_{K \in \mathscr{T}} \overline{K} = \overline{\Omega}$.

The shape regularity constant of $\mathscr{T}$ is defined as the number

$$\mu(\mathscr{T}) := \inf_{K \in \mathscr{T}} \frac{\rho_K}{h_K}, \tag{2}$$

where ρ_K is the radius of the largest ball contained inside K and h_K is the diameter of K. An indexed family of triangulations $\{\mathscr{T}^n\}_n$ is called *shape regular* if

$$\mu := \inf_n \mu(\mathscr{T}^n) > 0. \tag{3}$$

Further, we define $h : \Omega \to \mathbb{R}$ to be the piecewise constant *meshsize function* of $\mathscr{T}$ given by

$$h(\mathbf{x}) := \max_{\overline{K} \ni \mathbf{x}} h_K.$$

In addition if

$$\frac{\max_{K \in \mathscr{T}} h_K}{\min_{K \in \mathscr{T}} h_K} \leq C_{qu}, \tag{4}$$

we call $\mathscr{T}$ quasiuniform. If an entire indexed family of triangulations satisfy (4), we call it a quasiuniform family. In what follows we shall assume that all triangulations are shape-regular and quasiuniform.

We let $\mathscr{E}$ be the skeleton (set of common interfaces) of the triangulation $\mathscr{T}$ and say $e \in \mathscr{E}$ if e is on the interior of Ω and $e \in \partial\Omega$ if e lies on the boundary $\partial\Omega$ and set h_e to be the diameter of e. We also define the "broken" gradient ∇_h, Laplacian Δ_h and Hessian D_h^2 to be defined element-wise by $\nabla_h w|_K = \nabla w$, $\Delta_h w|_K = \Delta w$, $D_h^2 w|_K = D^2 w$ for all $K \in \mathscr{T}$, respectively, for respectively smooth functions on the interior of K,

We let $\mathbb{P}^k(\mathscr{T})$ denote the space of piecewise polynomials of degree k over the triangulation $\mathscr{T}$, and introduce the *finite element space* $\mathbb{V} := \mathbb{P}^k(\mathscr{T})$ to be the usual space of discontinuous piecewise polynomial functions of degree k. We define average operators for arbitrary scalar functions v and vectors $\mathbf{v}$ over an edge e shared by elements K_1 and K_2 as $\{\!\{ v \}\!\} = \frac{1}{2}(v|_{K_1} + v|_{K_2})$, $\{\!\{ \mathbf{v} \}\!\} = \frac{1}{2}(\mathbf{v}|_{K_1} + \mathbf{v}|_{K_2})$ and jump operators as $[\![v]\!] = v|_{K_1} \boldsymbol{n}_{K_1} + v|_{K_2} \boldsymbol{n}_{K_2}$, $[\![\mathbf{v}]\!] = \mathbf{v}|_{K_1} \cdot \boldsymbol{n}_{K_1} + \mathbf{v}|_{K_2} \cdot \boldsymbol{n}_{K_2}$. Note that

on the boundary of the domain $\partial\Omega$ the jump and average operators are defined as

$$\{\!\{\, v \,\}\!\}\Big|_{\partial\Omega} := v, \ \{\!\{\, \boldsymbol{v} \,\}\!\}\Big|_{\partial\Omega} := \boldsymbol{v}, \ [\![v]\!]\Big|_{\partial\Omega} := v\boldsymbol{n}, \ [\![\boldsymbol{v}]\!]\Big|_{\partial\Omega} := \boldsymbol{v} \cdot \boldsymbol{n},$$

Definition 2.1 (*Mesh dependent norms*) We introduce the mesh dependent L_2-, H^1- and H^2-norms to be

$$\|w_h\|_{0,h}^2 := \|w_h\|_{L_2(\Omega)}^2 + \|h^{3/2} \{\!\{\, \nabla w_h \,\}\!\}\|_{L_2(\mathscr{E}\cup\partial\Omega)}^2 + \|h^{1/2} [\![w_h]\!]\|_{L_2(\mathscr{E}\cup\partial\Omega)}^2$$

$$\|w_h\|_{1,h}^2 := \|\nabla_h w_h\|_{L_2(\Omega)}^2 + \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E}\cup\partial\Omega)}^2$$

$$\|w_h\|_{2,h}^2 := \|\Delta_h w_h\|_{L_2(\Omega)}^2 + \|h^{-1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2 + \|h^{-3/2} [\![w_h]\!]\|_{L_2(\mathscr{E}\cup\partial\Omega)}^2. \quad (5)$$

Note for $w_h \in \mathbb{V}$ in view of scaling each mesh dependent norm is equaivalent to the continuous counterpart, that is $\|w_h\|_{0,h} \sim \|w_h\|_{L_2(\Omega)}$ for example.

Consider the interior penalty (IP) discretisation of (1), to find $u_h \in \mathbb{V}$ such that

$$\mathscr{A}_h(u_h, v_h) = \langle f, v_h \rangle \quad \forall\, v_h \in \mathbb{V}, \quad (6)$$

where

$$\mathscr{A}_h(u_h, v_h) = \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \int_{\mathscr{E}\cup\partial\Omega} [\![u_h]\!] \cdot \{\!\{\, \nabla v_h \,\}\!\} + [\![v_h]\!] \cdot \{\!\{\, \nabla u_h \,\}\!\} \, ds$$

$$+ \int_{\mathscr{E}\cup\partial\Omega} \frac{\sigma_0}{h} [\![u_h]\!] \cdot [\![v_h]\!] \, ds + \int_{\mathscr{E}} \sigma_1 h [\![\nabla u_h]\!] \cdot [\![\nabla v_h]\!] \, ds, \quad (7)$$

where $\sigma_0 > 0$, $\sigma_1 \geq 0$ represent *penalty parameters*. Note that a standard choice is to take $\sigma_1 = 0$. The choice $\sigma_1 \neq 0$ results in a class of *stabilised* dG methods [8].

Proposition 2.2 (Continuity and coercivity of $\mathscr{A}_h(\cdot, \cdot)$ [1,11, c.f.]) *For $\sigma_1 \geq 0$ and σ_0 large enough and any $u_h, v_h \in \mathbb{V}$ the bilinear form $\mathscr{A}_h(\cdot, \cdot)$ satisfies*

$$\mathscr{A}_h(u_h, u_h) \geq C\|u_h\|_{1,h}^2 \quad (8)$$

$$\mathscr{A}_h(u_h, v_h) \leq C\|u_h\|_{1,h}\|v_h\|_{1,h}. \quad (9)$$

Lax-Milgram Theorem guarantees a unique solution to the problem (7). *Also, since $u \in H^2(\Omega)$, the bilinear form is* consistent, *hence, Strang's Lemma yields quasioptimal convergence of the method in the $\|\cdot\|_{1,h}$ norm:*

$$\|u - u_h\|_{1,h} \leq C \inf_{w_h \in \mathbb{V}} \|u - w_h\|_{1,h}. \quad (10)$$

Conforming reconstruction operators The key tool in the proof of the inf-sup condition is the notion of *reconstruction operators*. It is commonplace in the a posteriori analysis of nonconforming schemes to make use of such operators. A simple, quite general methodology for the construction of reconstruction operators is to use an averaging interpolation operator into an H^2-conforming finite element space. For example a C^1 Hsieh–Clough–Tocher (HCT) macro-element conforming space for H^2 conformity [6,13,25, c.f.]. Another option is the use of Argyris-type reconstructions [6].
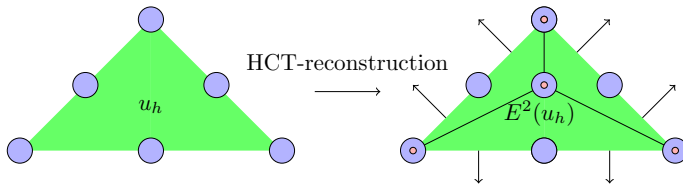
Fig. 1 The $\mathbb{P}^4$ Hsieh–Clough–Tocher-type macro-element, as a $H^2(\Omega)$-conforming reconstruction to the quadratic Lagrange element. Here the large, blue circles represent degrees of freedom associated to a function evaluation, the small, red circles represent those associated to full derivative evaluation and the arrows a normal derivative evaluation

Example 2.3 (Construction of the HCT(4) space for $d = 2$) Since the HCT spaces are an integral part of our analysis, we will illustrate the construction of the HCT(4) space for $d = 2$, over triangles, noting that for $d = 1$, we can simply consider families of cubic splines, while for $d = 3$ corresponding constructions are possible [19, Ch. 18]. Consider a triangle K that is partitioned in 3 subtriangles, $\{K_i\}_{i=1}^3$ by connecting each of the vertices to the barycentre as illustrated in Fig. 1. We then take

$$\text{HCT}(4) := \left\{ \phi \in \mathrm{C}^1(K) : \ u|_{K_i} \in \mathbb{P}^4(K_i) \text{ for } i = 1, 2, 3 \right\}. \tag{11}$$

The dimension of $\mathbb{P}^4(K_i)$ is 15. It can then be verified that, upon enforcing continuity of functions and their derivatives across the subtriangulation edges for the degrees of freedom depicted in Fig. 1, the dimension of the HCT(4) space is 21 and those degrees of freedom and unisolvent.

For the general construction of the macro-element HCT(r) space for simplicial and box-type elements, we refer to [10, 19, 22].

Definition 2.4 ($H^2(\Omega)$-*reconstructions*) An example of $H^2(\Omega)$ reconstruction operator $E^2(u_h)$ of k-th order Lagrange elements is defined as follows. Let $\mathbf{x}$ be a degree of freedom of the H^2-conforming space HCT($k + 2$) consisting of HCT-type macro-elements of degree $k + 2$, and let $\widehat{K}_{\mathbf{x}}$ be the set of all elements sharing the degree of freedom $\mathbf{x}$. Then, the reconstruction at that specific degree of freedom is given by

$$E^2(u_h)(\mathbf{x}) = \frac{1}{\text{card}(\widehat{K}_{\mathbf{x}})} \sum_{K \in \widehat{K}_{\mathbf{x}}} u_h|_K(\mathbf{x}). \tag{12}$$

For the case $k = 2$, the associated degrees of freedom are illustrated in Fig. 1. Notice that the degrees of freedom of the reconstruction are a superset of those of the original finite element. This is due to the lack of existence of a conforming $H^2(\Omega)$ subspace in $\mathbb{V}$ for low k; for instance, the existence of an $H^2(\Omega)$-conforming space requires $k \geq 5$ in two dimensions (Argyris space). Corresponding reconstructions for higher polynomial degrees have been considered in [6, 13], for instance.

Even though the HCT($k + 2$) space contains functions that are not polynomial, it does include $\mathbb{P}^{k+2}(K)$ and hence the HCT($k + 2$) interpolant preserves $\mathbb{P}^{k+2}(K)$

functions, hence by Bramble-Hilbert the $HCT(k+2)$ space has quasi-optimal approximability.

We refer also to the discussion in [13], regarding reconstructions of box-type k-th order Lagrange elements into $HCT(k+2)$.

Lemma 2.5 (Reconstruction bounds [13, Lem 3.1]) *For $d = 1, 2$, the $HCT(k+2)$ reconstruction operator $E^2 : \mathbb{V} \to \mathrm{H}^2(\Omega)$ satisfies the following bound for all $u_h \in \mathbb{V}$:*

$$\|E^2(u_h) - u_h\|_{1,h}^2 \le C\Big(\|h^{1/2} [\![\nabla u_h]\!]\|_{\mathrm{L}_2(\mathscr{E})}^2 + \|h^{-1/2} [\![u_h]\!]\|_{\mathrm{L}_2(\mathscr{E})}^2\Big), \quad (13)$$

with the constant $C > 0$ independent of u_h and of h.

Remark 2.6 Lemma 2.5 is proven in [13] for $d = 2$. For $d = 1$, we can recover into cubic or quartic splines and the proof is completely analogous. The proof for $d = 3$ using one of the trivariate C^1-elements with nodal and normal derivative degrees of freedom presented in [19] is conjectured to follow along the same lines to the proof of [13, Lem 3.1].

Using this $HCT(k+2)$-reconstruction, we can construct a further $HCT(k+2)$-reconstruction admitting the same bounds, but also satisfying an *adjoint orthogonality* property.

Definition 2.7 ($HCT(k+2)$-*Ritz reconstruction*) We define the Hsieh–Clough–Tocher $\mathrm{H}^2(\Omega)$-conforming Ritz reconstruction operator $E^{\mathscr{R}} : \mathbb{V} \to HCT(k+2)$ such that

$$\int_\Omega \nabla E^{\mathscr{R}}(u_h) \cdot \nabla \widetilde{v} \, dx = \int_\Omega \nabla_h u_h \cdot \nabla \widetilde{v} \, dx - \int_{\mathscr{E} \cup \partial\Omega} [\![u_h]\!] \cdot \nabla \widetilde{v} \, ds \quad \forall \widetilde{v} \in HCT(k+2). \quad (14)$$

Lemma 2.8 (Properties of $E^{\mathscr{R}}$) *The $HCT(k+2)$-Ritz reconstruction is well-defined and satisfies the orthogonality condition:*

$$\int_\Omega \Big(u_h - E^{\mathscr{R}}(u_h)\Big) \Delta \widetilde{v} = 0 \quad \forall \, \widetilde{v} \in HCT(k+2). \quad (15)$$

In addition, for $\alpha = 0, 1, 2$, we have

$$\sum_{K \in \mathscr{T}} \Big| E^{\mathscr{R}}(u_h) - u_h \Big|_{\mathrm{H}^\alpha(K)}^2 \le C\Big(\|h^{3/2-\alpha} [\![\nabla u_h]\!]\|_{\mathrm{L}_2(\mathscr{E})}^2 + \|h^{1/2-\alpha} [\![u_h]\!]\|_{\mathrm{L}_2(\mathscr{E})}^2\Big), \quad (16)$$

for $C > 0$ constants, independent of u_h and of h.

Proof Fixing $u_h \in \mathbb{V}$, $E^{\mathscr{R}}(u_h)$ is well-defined. Indeed, setting $\widetilde{v} = E^{\mathscr{R}}(u_h)$ in (14), along with a standard inverse estimate, we deduce

$$\|\nabla E^{\mathscr{R}}(u_h)\|_{\mathrm{L}_2(\Omega)} \le C\|u_h\|_{1,h},$$

for $C > 0$ independent of u_h. The orthogonality condition follows from integrating both sides of (14) by parts.

To see (16) we note that

$$
\begin{aligned}
\|E^{\mathscr{R}}(u_h) - u_h\|_{1,h}^2 &\le C \mathscr{A}_h\Big(E^{\mathscr{R}}(u_h) - u_h, E^{\mathscr{R}}(u_h) - u_h\Big) \\
&= C \mathscr{A}_h\Big(E^{\mathscr{R}}(u_h) - u_h, E^2(u_h) - u_h\Big) \\
&\le C \|E^{\mathscr{R}}(u_h) - u_h\|_{1,h} \|E^2(u_h) - u_h\|_{1,h}. \qquad (17)
\end{aligned}
$$

Notice that in order to invoke coercivity of $\mathscr{A}_h$ over $W(h)$ we must choose σ_0 larger than if we merely required coercivity over $\mathbb{V}$ since, as already mentioned in Example 2.4, $W(h)$ contains piecewise polynomials two degrees higher than $\mathbb{V}$. Using the properties of $E^2(u_h)$ from Lemma 2.5 shows the claim for $\alpha = 1$. The result for $\alpha = 2$ follows by an inverse inequality.

For $\alpha = 0$ we use a duality argument. Take $z \in H^2(\Omega) \cap H_0^1(\Omega)$ as the solution of the dual problem

$$
-\Delta z = E^{\mathscr{R}}(u_h) - u_h; \qquad (18)
$$

then

$$
\begin{aligned}
\|E^{\mathscr{R}}(u_h) - u_h\|_{L_2(\Omega)}^2 &= \int_{\Omega} -\Delta z \Big(E^{\mathscr{R}}(u_h) - u_h\Big) \, dx \\
&= \int_{\Omega} -(\Delta z - \Delta \widetilde{z}) \Big(E^{\mathscr{R}}(u_h) - u_h\Big) \, ds \quad \forall \widetilde{z} \in \mathrm{HCT}(k+2),
\end{aligned}
$$
$$(19)$$

in view of the orthogonality property (15). Integrating by parts we see

$$
\begin{aligned}
\|E^{\mathscr{R}}(u_h) - u_h\|_{L_2(\Omega)}^2 &= \int_{\Omega} (\nabla z - \nabla \widetilde{z}) \cdot \Big(\nabla E^{\mathscr{R}}(u_h) - \nabla_h u_h\Big) \, dx \\
&\quad + \int_{\mathscr{E}} \{\!\!\{ \nabla z - \nabla \widetilde{z} \}\!\!\} \cdot [\![u_h]\!] \, ds \\
&\le \|h^{-1}(\nabla z - \nabla \widetilde{z})\|_{L_2(\Omega)} \|h(\nabla E^{\mathscr{R}}(u_h) - \nabla_h u_h)\|_{L_2(\Omega)} \\
&\quad + \|h^{-1/2} \{\!\!\{ \nabla z - \nabla \widetilde{z} \}\!\!\}\|_{L_2(\mathscr{E})} \|h^{1/2} [\![u_h]\!]\|_{L_2(\mathscr{E})}. \qquad (20)
\end{aligned}
$$

The result follows using the approximability of the $\mathrm{HCT}(k+2)$ space [9] that can be inferred through the dimensional analysis in Definition 2.3 and the regularity of the dual problem, specifically

$$
\begin{aligned}
\|h^{-1}(\nabla z - \nabla \widetilde{z})\|_{L_2(\Omega)} + \|h^{-1/2} \{\!\!\{ \nabla z - \nabla \widetilde{z} \}\!\!\}\|_{L_2(\mathscr{E})} &\le C \, |z|_{H^2(\Omega)} \\
&\le C \|E^{\mathscr{R}}(u_h) - u_h\|_{L_2(\Omega)},
\end{aligned}
$$
$$(21)$$

thereby concluding the proof. $\qquad\qquad\square$

Theorem 2.9 (inf–sup stability over $W(h)$) *For polynomial degree $k \geq 2$ there exists a $\gamma_h > 0$, depending on the quasiuniformity constant C_{qu}, such that, when σ_0, σ_1 are chosen large enough, we have for all $w_h \in \mathbb{V}$*

$$\sup_{\widetilde{v} \in W(h)} \frac{\mathscr{A}_h(w_h, \widetilde{v})}{\|\widetilde{v}\|_{0,h}} \geq \gamma_h \|w_h\|_{2,h}, \tag{22}$$

where $W(h) := \mathbb{V} + HCT(k+2)$.

Proof The proof consists of two steps. We first show for a given $w_h \in \mathbb{V}$ that there exists a $\widetilde{v} \in W(h)$ such that

$$\mathscr{A}_h(w_h, \widetilde{v}) \geq C(\min_{x \in \Omega} h)^2 \|w_h\|_{2,h}^2 \tag{23}$$

and that

$$\|\widetilde{v}\|_{L_2(\Omega)} \leq C(\max_{x \in \Omega} h)^2 \|w_h\|_{2,h}, \tag{24}$$

which, along with the quasi-uniformity assumption on the mesh, yields the inf-sup condition (22).

Firstly note that, after an integration by parts, the IP method (7) can be written as

$$\mathscr{A}_h(w_h, v_h) = \int_\Omega -\Delta_h w_h v_h \, dx + \int_\mathscr{E} [\![\nabla w_h]\!] \{\!\!\{ v_h \}\!\!\} \, ds - \int_{\mathscr{E} \cup \partial\Omega} [\![w_h]\!] \{\!\!\{ \nabla v_h \}\!\!\} \, ds$$
$$+ \int_{\mathscr{E} \cup \partial\Omega} \frac{\sigma_0}{h} [\![w_h]\!] \cdot [\![v_h]\!] \, ds + \int_\mathscr{E} \sigma_1 h [\![\nabla w_h]\!] \cdot [\![\nabla v_h]\!] \, ds. \tag{25}$$

Upon setting $\widetilde{v} = w_h - E^{\mathscr{R}}(w_h) - \alpha h^2 \Delta_h w_h$, for some parameter $\alpha \in \mathbb{R}$ to be chosen below, we compute

$$\mathscr{A}_h(w_h, \widetilde{v}) = \alpha \|h \Delta_h w_h\|_{L_2(\Omega)}^2 + \sigma_1 \|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2 + \sigma_0 \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E} \cup \partial\Omega)}^2$$
$$- \int_\Omega \Delta_h w_h \Big(w_h - E^{\mathscr{R}}(w_h) \Big) \, dx$$
$$+ \int_\mathscr{E} [\![\nabla w_h]\!] \{\!\!\{ w_h - E^{\mathscr{R}}(w_h) \}\!\!\} \, ds - \int_{\mathscr{E} \cup \partial\Omega} [\![w_h]\!] \cdot \{\!\!\{ \nabla\Big(w_h - E^{\mathscr{R}}(w_h)\Big) \}\!\!\} \, ds$$
$$- \alpha \int_\mathscr{E} h^2 [\![\nabla w_h]\!] \{\!\!\{ \Delta w_h \}\!\!\} \, ds + \alpha \int_{\mathscr{E} \cup \partial\Omega} h^2 [\![w_h]\!] \cdot \{\!\!\{ \nabla\Delta w_h \}\!\!\} \, ds$$
$$- \alpha\sigma_1 \int_\mathscr{E} h^3 [\![\nabla w_h]\!] [\![\nabla\Delta w_h]\!] \, ds - \sigma_0\alpha \int_{\mathscr{E} \cup \partial\Omega} h [\![w_h]\!] \cdot [\![\Delta w_h]\!] \, ds. \tag{26}$$

The orthogonality property of the $HCT(k+2)$-Ritz reconstruction (15) yields

$$\int_\Omega -\Delta_h w_h \Big(w_h - E^{\mathscr{R}}(w_h) \Big) \, dx = \int_\Omega \Big(\Delta E^{\mathscr{R}}(w_h) - \Delta_h w_h \Big) \Big(w_h - E^{\mathscr{R}}(w_h) \Big) \, dx. \tag{27}$$

Repeated use of the Cauchy–Schwarz inequality, therefore, gives

$$
\begin{aligned}
\mathscr{A}_h(w_h, \tilde{v}) \geq{} & \alpha \|h\Delta_h w_h\|_{L_2(\Omega)}^2 + \sigma_1 \|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2 + \sigma_0 \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E} \cup \partial\Omega)}^2 \\
& - \|h\big(\Delta_h w_h - \Delta E^{\mathscr{R}}(w_h)\big)\|_{L_2(\Omega)} \|h^{-1}\big(w_h - E^{\mathscr{R}}(w_h)\big)\|_{L_2(\Omega)} \\
& - \|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})} \|h^{-1/2} \{\!\{ w_h - E^{\mathscr{R}}(w_h) \}\!\}\|_{L_2(\mathscr{E})} \\
& - \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E} \cup \partial\Omega)} \|h^{1/2} \{\!\{ \nabla w_h - \nabla E^{\mathscr{R}}(w_h) \}\!\}\|_{L_2(\mathscr{E} \cup \partial\Omega)} \\
& - \alpha \|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})} \|h^{3/2} \{\!\{ \Delta w_h \}\!\}\|_{L_2(\mathscr{E})} \\
& - \alpha \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E} \cup \partial\Omega)} \|h^{5/2} \{\!\{ \nabla\Delta w_h \}\!\}\|_{L_2(\mathscr{E} \cup \partial\Omega)} \\
& - \sigma_1 \alpha \|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})} \|h^{5/2} [\![\nabla\Delta w_h]\!]\|_{L_2(\mathscr{E})} \\
& - \sigma_0 \alpha \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E} \cup \partial\Omega)} \|h^{3/2} [\![\Delta w_h]\!]\|_{L_2(\mathscr{E} \cup \partial\Omega)} \\
=:{} & \alpha \|h\Delta_h w_h\|_{L_2(\Omega)}^2 + \sigma_1 \|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2 \\
& + \sigma_0 \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E} \cup \partial\Omega)}^2 - \sum_{i=1}^7 \mathscr{I}_i.
\end{aligned}
\tag{28}
$$

We proceed to bound each of the terms $\mathscr{I}_i$ individually. Note that in view of scaling and inverse inequalities we have for any $w_h \in \mathbb{V}$:

$$
\|\{\!\{ w_h \}\!\}\|_{L_2(e)} \leq C_1 \|h^{-1/2} w_h\|_{L_2(\bar{K}_1 \cup \bar{K}_2)}
\tag{29}
$$

$$
\|\{\!\{ \nabla w_h \}\!\}\|_{L_2(e)} \leq C_2 \|h^{-3/2} w_h\|_{L_2(\bar{K}_1 \cup \bar{K}_2)}
\tag{30}
$$

for any edge/face $e := \bar{K}_1 \cap \bar{K}_2 \in \mathscr{E}$, and elements $K_1, K_2 \in \mathscr{T}$, with C_1, C_2 depending only on the mesh-regularity and shape-regularity constants.

For $\mathscr{I}_1$, in view of Lemma 2.8, we have

$$
\mathscr{I}_1 \leq C_3 \Big(\|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2 + \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E})}^2 \Big),
\tag{31}
$$

with constant $C_3 > 0$ being the maximum of all constants in (16) for all α.

For $\mathscr{I}_2$, (29) and Lemma 2.8 yield

$$
\begin{aligned}
\mathscr{I}_2 &\leq C_1 C_3^{1/2} \|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})} \Big(\|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2 + \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E})}^2 \Big)^{1/2} \\
&\leq C_1^2 C_3 \|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2 + \frac{C_1^2 C_3}{2} \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E})}^2.
\end{aligned}
\tag{32}
$$

For $\mathscr{I}_3$, (30) and Lemma 2.8 yield

$$
\begin{aligned}
\mathscr{I}_3 &\leq C_2 C_3^{1/2} \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E} \cup \partial\Omega)} \Big(\|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2 + \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E})}^2 \Big)^{1/2} \\
&\leq \frac{C_2^2 C_3}{2} \|h^{1/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E} \cup \partial\Omega)} + C_2^2 C_3 \|h^{-1/2} [\![w_h]\!]\|_{L_2(\mathscr{E})}.
\end{aligned}
\tag{33}
$$

For $\mathscr{I}_4$, we have

$$
\mathscr{I}_4 \leq C_1\alpha\|h^{1/2}[\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}\|h\Delta w_h\|_{L_2(\Omega)} \leq \epsilon_4\alpha\|h\Delta w_h\|_{L_2(\Omega)}^2
$$
$$
+ \frac{C_1^2\alpha}{4\epsilon_4}\|h^{1/2}[\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2, \tag{34}
$$

for any $\epsilon_4 > 0$, while for $\mathscr{I}_5$, we get

$$
\mathscr{I}_5 \leq C_2\alpha\|h^{-1/2}[\![w_h]\!]\|_{L_2(\mathscr{E}\cup\partial\Omega)}\|h\Delta w_h\|_{L_2(\Omega)} \leq \epsilon_5\alpha\|h\Delta w_h\|_{L_2(\Omega)}^2
$$
$$
+ \frac{C_2^2\alpha}{4\epsilon_5}\|h^{-1/2}[\![w_h]\!]\|_{L_2(\mathscr{E}\cup\partial\Omega)}^2. \tag{35}
$$

for any $\epsilon_5 > 0$; similarly for $\mathscr{I}_6$ and for any $\epsilon_6 > 0$, we have

$$
\mathscr{I}_6 \leq C_2\sigma_1\alpha\|h^{1/2}[\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}\|h\Delta w_h\|_{L_2(\Omega)} \leq \epsilon_6\alpha\|h\Delta w_h\|_{L_2(\Omega)}^2
$$
$$
+ \frac{C_2^2\sigma_1^2\alpha}{4\epsilon_6}\|h^{1/2}[\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2. \tag{36}
$$

Finally, the last term $\mathscr{I}_7$ can be bounded as follows:

$$
\mathscr{I}_7 \leq C_1\sigma_0\alpha\|h^{-1/2}[\![w_h]\!]\|_{L_2(\mathscr{E}\cup\partial\Omega)}\|h\Delta_h w_h\|_{L_2(\Omega)} \leq \epsilon_7\alpha\|h\Delta_h w_h\|_{L_2(\Omega)}^2
$$
$$
+ \frac{C_1^2\sigma_0^2\alpha}{4\epsilon_7}\|h^{-1/2}[\![w_h]\!]\|_{L_2(\mathscr{E}\cup\partial\Omega)}^2, \tag{37}
$$

for any $\epsilon_7 > 0$.

Collecting the results (31)–(37) and substituting this into (28) we deduce

$$
\mathscr{A}_h(w_h, \widetilde{v}) \geq \|h\Delta_h w_h\|_{L_2(\Omega)}^2\alpha\left(1 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7\right)
$$
$$
+ \|h^{1/2}[\![\nabla w_h]\!]\|_{L_2(\mathscr{E})}^2\left(\sigma_1 - C_3 - C_1^2 C_3 - \frac{C_2^2 C_3^2}{2} - \frac{C_1^2\alpha}{4\epsilon_4} - \frac{C_2^2\sigma_1^2\alpha}{4\epsilon_6}\right)
$$
$$
+ \|h^{-1/2}[\![w_h]\!]\|_{L_2(\mathscr{E}\cup\Omega)}^2\left(\sigma_0 - C_3 - \frac{C_1^2 C_3}{2} - C_2^2 C_3 - \frac{C_2^2\alpha}{4\epsilon_5} - \frac{C_1^2\sigma_0^2\alpha}{4\epsilon_7}\right). \tag{38}
$$

To arrive to (23), we can choose $\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \frac{1}{5}, \alpha = (\max(\sigma_0^2, \sigma_1^2))^{-1}$ and σ_0 and σ_1 large enough.

For (24), we use Lemma 2.8 to see that

$$
\|\widetilde{v}\|_{L_2(\Omega)} \leq \|w_h - E^{\mathscr{R}}(w_h)\|_{L_2(\Omega)} + \|\alpha h^2\Delta_h w_h\|_{L_2(\Omega)} \leq C\|h^2 w_h\|_{2,h}, \tag{39}
$$

which, together with the quasiuniformity of the meshes, completes the proof. □

Lemma 2.10 (Stability of the Ritz projection) *Let $R : W(h) \to \mathbb{V}$ denote the $\mathscr{A}_h(\cdot, \cdot)$ orthogonal projector into $\mathbb{V}$. Then, for $\widetilde{w} \in W(h)$, there exists a $C > 0$, independent of h but possibly dependent on the quasiuniformity constant, C_{qu}, such that*

$$\|R\widetilde{w}\|_{L_2(\Omega)} \le C \left(\max_{x \in \Omega} h \|\nabla \widetilde{w}\|_{L_2(\Omega)} + \|\widetilde{w}\|_{L_2(\Omega)} \right) \le C \|\widetilde{w}\|_{L_2(\Omega)}. \tag{40}$$

Proof Let $\widetilde{g} \in W(h)$ be the solution to the discrete dual problem such that

$$\mathscr{A}_h(\widetilde{v}, \widetilde{g}) = \langle R\widetilde{w}, \widetilde{v} \rangle \quad \forall \, \widetilde{v} \in W(h). \tag{41}$$

Note that this is well posed owing to coercivity as long as the penalty parameters are tuned to account for the fact that $W(h)$ contains piecewise polynomials over the subpartition two degrees higher than $\mathbb{V}$ itself. Then, we have

$$\begin{aligned}
\|R\widetilde{w}\|_{L_2(\Omega)}^2 &= \langle R\widetilde{w} - \widetilde{w}, R\widetilde{w} \rangle + \langle \widetilde{w}, R\widetilde{w} \rangle \\
&= \mathscr{A}_h(R\widetilde{w} - \widetilde{w}, \widetilde{g}) + \langle \widetilde{w}, R\widetilde{w} \rangle .
\end{aligned} \tag{42}$$

Let $\Pi : H^1(\Omega) \to \mathbb{V} \cap H_0^1(\Omega)$ a suitable projection with optimal approximation properties. Then

$$\begin{aligned}
\|R\widetilde{w}\|_{L_2(\Omega)}^2 &= \mathscr{A}_h(R\widetilde{w} - \widetilde{w}, \widetilde{g} - \Pi\widetilde{g}) + \langle \widetilde{w}, R\widetilde{w} \rangle \\
&\le \|h(R\widetilde{w} - \widetilde{w})\|_{1,h} \|h^{-1}(\widetilde{g} - \Pi\widetilde{g})\|_{1,h} + \|\widetilde{w}\|_{L_2(\Omega)} \|R\widetilde{w}\|_{L_2(\Omega)}, \quad (43)
\end{aligned}$$

through the continuity of $\mathscr{A}_h(\cdot, \cdot)$. From the optimal approximation properties of the projection/interpolant Π, we have

$$\|h^{-1}(\widetilde{g} - \Pi\widetilde{g})\|_{1,h}^2 \le C_{qu}\tilde{C}\|\widetilde{g}\|_{2,h}^2, \tag{44}$$

and, using the discrete regularity of $\widetilde{g}$ induced by the inf-sup condition in Theorem 2.9

$$\gamma_h \|\widetilde{g}\|_{2,h} \le \sup_{\widetilde{v} \in W(h)} \frac{\mathscr{A}_h(\widetilde{g}, \widetilde{v})}{\|\widetilde{v}\|_{0,h}} = \sup_{\widetilde{v} \in W(h)} \frac{\langle R\widetilde{w}, \widetilde{v} \rangle}{\|\widetilde{v}\|_{0,h}} \le C \|R\widetilde{w}\|_{L_2(\Omega)}. \tag{45}$$

Hence we see that

$$\begin{aligned}
\|R\widetilde{w}\|_{L_2(\Omega)}^2 &\le C \left(\|h(R\widetilde{w} - \widetilde{w})\|_{1,h} \|R\widetilde{w}\|_{L_2(\Omega)} + \|\widetilde{w}\|_{L_2(\Omega)} \|R\widetilde{w}\|_{L_2(\Omega)} \right) \\
&\le C \left(\|h\widetilde{w}\|_{1,h} \|R\widetilde{w}\|_{L_2(\Omega)} + \|\widetilde{w}\|_{L_2(\Omega)} \|R\widetilde{w}\|_{L_2(\Omega)} \right), \quad (46)
\end{aligned}$$

in view of the quasi-best approximation in $\|\cdot\|_{1,h}$ from (10). The conclusion follows from standard inverse inequalities. $\qquad \square$

Theorem 2.11 (inf–sup stability over $\mathbb{V}$) *For polynomial degree $k \ge 2$ there exists a $\gamma_h > 0$, independent of h, but dependent on C_{qu} such that, when $\sigma_0, \sigma_1 \ge 1$ are chosen as in Theorem 2.9, we have for all $w_h \in \mathbb{V}$*

$$\sup_{v_h \in \mathbb{V}} \frac{\mathscr{A}_h(w_h, v_h)}{\|v_h\|_{2,h}} \geq \gamma_h \|w_h\|_{0,h}. \tag{47}$$

Proof To show (47) we fix w_h and let $\Phi \in \mathbb{V}$ be the solution of the dual problem

$$\mathscr{A}_h(\Psi, \Phi) = \int_\Omega w_h \Psi \, dx \quad \forall \, \Psi \in \mathbb{V}. \tag{48}$$

Following the same arguments as in the proof of Theorem 2.9, it is clear that there exists a $C > 0$ such that

$$Ch^2 \|\Phi\|_{2,h}^2 \leq \mathscr{A}_h(\Phi, \widetilde{v}), \tag{49}$$

where $\widetilde{v} := \Phi - E^{\mathscr{R}}(\Phi) - \alpha h^2 \Delta_h \Phi$. Now it is clear that

$$\|\widetilde{v}\|_{L_2(\Omega)} \|\Phi\|_{2,h} \leq Ch^2 \|\Phi\|_{2,h}^2 \leq C \mathscr{A}_h(\Phi, \widetilde{v}), \tag{50}$$

and hence in view of Lemma 2.10 we have, with R denoting the $\mathscr{A}_h$ orthogonal projector into $\mathbb{V}$, that

$$\|R\widetilde{v}\|_{L_2(\Omega)} \leq C \max_{x \in \Omega} h \left(\|\nabla \widetilde{v}\|_{L_2(\Omega)} + \|\widetilde{v}\|_{L_2(\Omega)} \right) \leq C \|\widetilde{v}\|_{L_2(\Omega)}, \tag{51}$$

through inverse inequalities. Hence

$$\|R\widetilde{v}\|_{L_2(\Omega)} \|\Phi\|_{2,h} \leq C \|\widetilde{v}\|_{L_2(\Omega)} \|\Phi\|_{2,h}. \tag{52}$$

Now arguing as in the Proof of Theorem 2.9, and noting the constant will now depend on C_{qu}, we may show that

$$\|\widetilde{v}\|_{L_2(\Omega)} \leq C \|h^2 \Phi\|_{2,h}. \tag{53}$$

Combining the previous two inequalities yields

$$\|R\widetilde{v}\|_{L_2(\Omega)} \|\Phi\|_{2,h} \leq C \|h\Phi\|_{2,h}^2 \leq C \mathscr{A}_h(\Phi, \widetilde{v}) = C \mathscr{A}_h(\Phi, R\widetilde{v}), \tag{54}$$

concluding the proof. □

Corollary 2.12 (Convergence) *Let u solve (1) and $u_h \in \mathbb{V}$ be the interior penalty approximation from (7), then, under the assumptions of Theorem 2.11,*

$$\|u - u_h\|_{0,h} \leq \left(1 + \frac{C_B}{\gamma_h} \right) \inf_{w_h \in \mathbb{V}} \|u - w_h\|_{0,h} + \frac{1}{\gamma_h} \sup_{v_h \in \mathbb{V}} \frac{\mathscr{A}_h(u_h - u, v_h)}{\|v_h\|_{2,h}}, \tag{55}$$

where γ_h is the discrete inf-sup constant and C_B is the continuity constant. If $u \in H^{k+1}(\Omega)$ for $k \geq 2$ the following a priori bound holds:

$$\|u - u_h\|_{0,h} + \|h(u - u_h)\|_{1,h} + \|h^2(u - u_h)\|_{2,h} \leq Ch^{k+1} |u|_{k+1}. \tag{56}$$

Proof Using the inf-sup condition from Theorem 2.11 we see for all $w_h \in \mathbb{V}$

$$
\begin{aligned}
\gamma_h \|w_h - u_h\|_{0,h} &\leq \sup_{v_h \in \mathbb{V}} \frac{\mathscr{A}_h(w_h - u_h, v_h)}{\|v_h\|_{2,h}} \\
&\leq \sup_{v_h \in \mathbb{V}} \frac{\mathscr{A}_h(w_h - u, v_h)}{\|v_h\|_{2,h}} + \sup_{v_h \in \mathbb{V}} \frac{\mathscr{A}_h(u - u_h, v_h)}{\|v_h\|_{2,h}}.
\end{aligned} \tag{57}
$$

Now using the natural continuity bound

$$
\mathscr{A}_h(u - w_h, v_h) \leq C_B \|u - w_h\|_{0,h} \|v_h\|_{2,h} \tag{58}
$$

we see

$$
\|w_h - u_h\|_{0,h} \leq \frac{C_B}{\gamma_h} \|u - w_h\|_{0,h} + \frac{1}{\gamma_h} \sup_{v_h \in \mathbb{V}} \frac{\mathscr{A}_h(u - u_h, v_h)}{\|v_h\|_{2,h}}. \tag{59}
$$

Hence, in view of the triangle inequality

$$
\|u - u_h\|_{0,h} \leq \left(1 + \frac{C_B}{\gamma_h}\right) \|u - w_h\|_{0,h} + \frac{1}{\gamma_h} \sup_{v_h \in \mathbb{V}} \frac{\mathscr{A}_h(u - u_h, v_h)}{\|v_h\|_{2,h}}. \tag{60}
$$

The bound (55) follows since w_h was arbitrary and (56) follows from the best approximation of $\mathbb{V}$. $\qquad\square$

3 Applications to problems with rough data

In this section we examine some problems of the form

$$
\begin{aligned}
-\Delta u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{aligned} \tag{61}
$$

where f may be as rough as $\mathrm{H}^{-2}(\Omega) \setminus \mathrm{H}^{-1}(\Omega)$ and so, $u \in \mathrm{L}_2(\Omega) \setminus \mathrm{H}^1(\Omega)$. This means that the problem (75) cannot be characterised through a weak formulation, rather an *ultra weak formulation*, whereby we seek $u \in \mathrm{L}_2(\Omega)$ such that

$$
\int_\Omega -u \Delta v \, \mathrm{d}x = \langle f \mid v \rangle_{\mathrm{H}^{-2}(\Omega) \times \mathrm{H}^2_0(\Omega)} \quad \forall v \in \mathrm{H}^2_0(\Omega), \tag{62}
$$

and the right hand side of (62) is understood as a *duality pairing*. In this setting standard tools pertaining to the analysis of Galerkin methods may not apply, for example the Aubin–Nitsche duality argument.

However, the stabilised IP method is still well defined and the inf-sup condition still holds. Indeed, we define the modified IP method: seek $u_h \in \mathbb{V}$ such that

$$
\mathscr{A}_h(u_h, v_h) = \left\langle f \mid E_0^{\mathscr{R}}(v_h) \right\rangle_{\mathrm{H}^{-2}(\Omega) \times \mathrm{H}^2_0(\Omega)} \quad \forall v_h \in \mathbb{V}, \tag{63}
$$

where $E_0^2 : \mathbb{V} \to H_0^2(\Omega)$ is the modification of E^2 recovering onto $H_0^2(\Omega)$ which is constructed by setting to zero all the degrees of freedom on $\partial\Omega$. We note that Lemma 2.5 still holds verbatim when E^2 is replaced by E_0^2. The nonstandard definition of the right-hand side allows us to make sense of extremely rough source terms [15] by interpreting the right-hand side a a duality pairing. Correspondingly, we also denote by $E_0^{\mathscr{R}}$ the recovery given by Definition 2.7 when E^2 is replaced by E_0^2. Also, we note that Lemma 2.8 holds for $E_0^{\mathscr{R}}$ also.

Since the inf-sup condition given in Theorem 2.11 is a condition only on the operator itself, the best approximation result given in Corollary 2.12 holds true. The only uncertainty with the bound is the behaviour of the inconsistency term. The control of this term is the main motivation in the nonstandard definition of the right hand side of (63).

Theorem 3.1 (quasi-optimal error control for problems with rough data) *Let* $u \in$ $L_2(\Omega)$ *solve* (62) *and* $u_h \in \mathbb{V}$ *be the approximation defined through* (63)*, then*

$$\|u - u_h\|_{L_2(\Omega)} \leq C \inf_{w_h \in \mathbb{V}} \left(\|u - w_h\|_{L_2(\Omega)} + \|h^{3/2} [\![\nabla w_h]\!]\|_{L_2(\mathscr{E})} + \|h^{1/2} [\![w_h]\!]\|_{L_2(\mathscr{E})} \right).$$
(64)

Proof The proof takes some inspiration from that of [17], where inconsistency terms arise from the fact that the solution of an elliptic problem may only lie in $H^1(\Omega)$, for which the operator $\mathscr{A}_h(u, v_h)$ may not be well defined. Here, the situation is more involved, since the solution $u \in L_2(\Omega) \backslash H^1(\Omega)$.

Using the inf-sup condition from Theorem 2.11 we have

$$\gamma_h \|w_h - u_h\|_{0,h} \leq \sup_{v_h \in \mathbb{V}} \frac{\mathscr{A}_h(w_h - u_h, v_h)}{\|v_h\|_{2,h}}.$$
(65)

Now, by adding and subtracting appropriate terms and using (62) and (63), we see

$$\mathscr{A}_h(w_h - u_h, v_h) = \mathscr{A}_h\left(w_h, E_0^{\mathscr{R}}(v_h)\right) + \int_\Omega u \Delta E_0^{\mathscr{R}}(v_h) \, \mathrm{d}x + \mathscr{A}_h\left(w_h, v_h - E_0^{\mathscr{R}}(v_h)\right)$$

$$= \mathscr{A}_h\left(w_h, E_0^{\mathscr{R}}(v_h)\right) + \int_\Omega u \Delta E_0^{\mathscr{R}}(v_h) \, \mathrm{d}x$$

$$+ \mathscr{A}_h\left(w_h - E_0^2(v_h), v_h - E_0^{\mathscr{R}}(v_h)\right)$$
(66)

by the orthogonality properties of $E_0^{\mathscr{R}}(w_h)$ given in (14). Now we may use that

$$\int_\Omega u \Delta E_0^{\mathscr{R}}(v_h) \, \mathrm{d}x + \mathscr{A}_h\left(w_h, E_0^{\mathscr{R}}(v_h)\right) \leq C \|u - w_h\|_{L_2(\Omega)} \|\Delta E_0^{\mathscr{R}}(v_h)\|_{L_2(\Omega)}$$

$$\leq C \|u - w_h\|_{L_2(\Omega)} \|v_h\|_{2,h},$$
(67)

through the stability of $E^{\mathscr{R}}(v_h)$.

Finally, using the approximation properties of $E_0^{\mathscr{R}}(\cdot)$ and $E_0^2(\cdot)$ we see

$$
\begin{aligned}
&\mathscr{A}_h\left(w_h - E_0^2(w_h),\, v_h - E_0^{\mathscr{R}}(v_h)\right) \\
&\leq C\|w_h - E_0^2(w_h)\|_{1,h}\|v_h - E_0^{\mathscr{R}}(v_h)\|_{1,h} \\
&\leq C\left(\|h^{3/2}\,[\![\nabla w_h]\!]\|_{L_2(\mathscr{E})} + \|h^{1/2}\,[\![w_h]\!]\|_{L_2(\mathscr{E})}\right)\|v_h\|_{2,h}.
\end{aligned}
\tag{68}
$$

Substituting (67) and (68) into (66) we have

$$
\begin{aligned}
&\mathscr{A}_h(w_h - u_h,\, v_h) \\
&\leq C\left(\|u - w_h\|_{L_2(\Omega)} + \|h^{3/2}\,[\![\nabla w_h]\!]\|_{L_2(\mathscr{E})} + \|h^{1/2}\,[\![w_h]\!]\|_{L_2(\mathscr{E})}\right)\|v_h\|_{2,h},
\end{aligned}
\tag{69}
$$

and hence

$$
\begin{aligned}
\|u - u_h\|_{L_2(\Omega)} &\leq \|u - w_h\|_{L_2(\Omega)} + \|u_h - w_h\|_{L_2(\Omega)} \\
&\leq \|u - w_h\|_{L_2(\Omega)} + \|u_h - w_h\|_{0,h} \\
&\leq C\left(\|u - w_h\|_{L_2(\Omega)} + \|h^{3/2}\,[\![\nabla w_h]\!]\|_{L_2(\mathscr{E})} + \|h^{1/2}\,[\![w_h]\!]\|_{L_2(\mathscr{E})}\right),
\end{aligned}
\tag{70}
$$

as required. □

3.1 Numerical experiments

The implementation of all the numerical experiments was performed in Matlab®
on a laptop computer with a 2.3GHz Intel i7 processor and 16 GB of RAM. All
computations took less than 2 min on this machine.

3.1.1 Test 1: an one-dimensional example

We begin by assessing the method (63) for $d = 1$. We set $\Omega = (0, 1)$ and consider the
problem of finding u such that

$$
\begin{aligned}
-u'' &= \delta_{\bar{x}}' \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{71}
$$

where $\delta_{\bar{x}}'$ denotes the distributional derivative of the Dirac distribution at a point
$\bar{x} \in \Omega$. This one-dimensional problem was a motivating example in the classical
work of Babuška and Osborn [3]; this computation is included here as a tribute to that
inspiring work.

For this problem we can even characterise a distributional solution, indeed we have
that

$$
u(x) = \begin{cases} -x & \text{when } x < \bar{x} \\ 1 - x & \text{when } x > \bar{x}, \end{cases}
\tag{72}
$$

solves (75). If we assume that $\bar{x}$ does not lie on the skeleton of the triangulation we can define our approximation as seeking $u_h \in \mathbb{V}$ such that

$$\mathscr{A}_h(u_h, v_h) = E_0^{\mathscr{R}}(v_h)'(\bar{x}) \quad \forall\, v_h \in \mathbb{V}. \tag{73}$$

Using Theorem 3.1 we are able to show this approximation satisfies the a priori bound

$$\|u - u_h\|_{L_2(\Omega)} \le Ch^{1/2-\epsilon} \quad \forall\, \epsilon > 0, \tag{74}$$

since $\delta_{\bar{x}} \in H^{-s}(\Omega)$ for all $s > 1/2$.

We fix $k = 2$ and solve (73) over a sequence of uniform meshes in 1d with $h = 1/2, 1/4, \ldots, 1/1024$. We take $\bar{x} = 1/2 + \sqrt{2}/100000$ as to not align it with the nodes of the mesh. In Fig. 2 we show the numerical approximation over the finest mesh along with the experimental order of convergence.

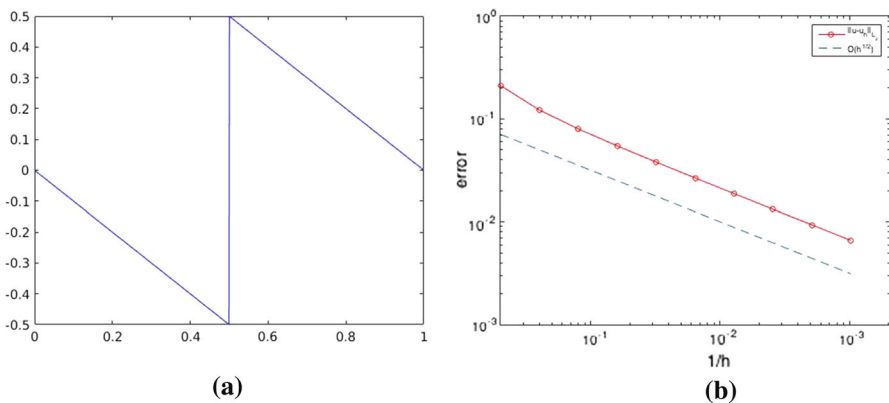### 3.1.2 Test 2: $d = 2$ with a Dirac source term at a point

We now take $\Omega = B(0, 1)$, the open ball of radius 1 centred at the origin, and consider the problem of finding u such that

$$\begin{aligned} -\Delta u &= \delta_0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{75}$$
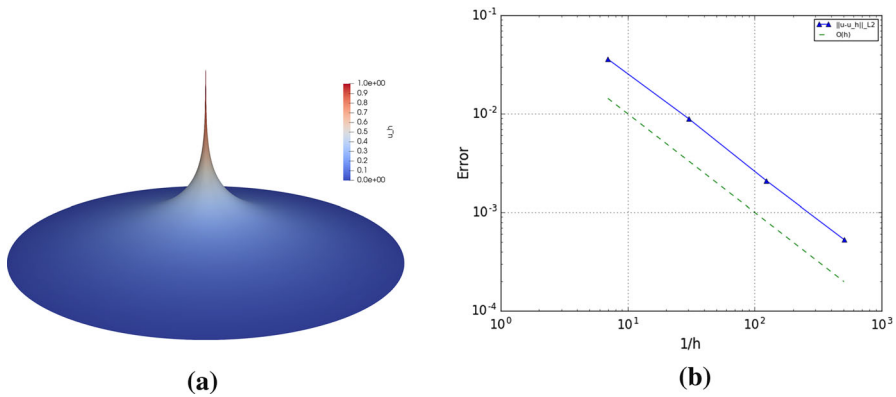
The exact solution u is the fundamental solution of Laplace's problem

$$u(\mathbf{x}) = -\frac{1}{2\pi} \log(|\mathbf{x}|), \tag{76}$$

for which we have $u \in L_2(\Omega) \setminus H^1(\Omega)$.

(a)   (b)

Fig. 2 In this experiment we test the L_2 convergence of the interior penalty method to approximate the distributional solution (72). Notice the error converges approximately like $O(h^{1/2})$. **a** The IP approximation. **b** Error and the experimental order of convergence

(a)

(b)

Fig. 3 We test the L_2-norm convergence of the method (80) to approximate the fundamental solution of Laplace's problem (76). The error converges approximately like O(h). **a** The IP approximation. **b** Error and the experimental order of convergence

We define our approximation as seeking $u_h \in \mathbb{V}$ such that

$$\mathscr{A}_h(u_h, v_h) = E_0^{\mathscr{R}}(v_h)(\bar{x}) \quad \forall \, v_h \in \mathbb{V}, \tag{77}$$

and, under the assumptions of Theorem 3.1, we are able to show that this approximation satifies the a priori bound

$$\|u - u_h\|_{L_2(\Omega)} \le C h^{1-\epsilon} \quad \forall \, \epsilon > 0. \tag{78}$$

This is in agreement with the results of [2] for conforming finite elements applied to this problem.

We fix $k = 2$ and solve (80) over a sequence of unstructured, quasiuniform triangulations of $B(0, 1)$. For the coarsest mesh we have $h \approx 0.13$ and for the most fine $h \approx 0.0019$. In Fig. 3, we show the numerical approximation over the finest mesh along with the error measured in the L_2-norm and its convergence history. We remark, that the boundary approximation by straight-faced elements is not detrimental to the convergence rate in this case as the error is measured in the L_2-norm.

3.2 Test 3: $d = 2$, rough source terms defined over an one-dimensional manifold

We now test the proposed method on a more complicated problem, whereby, we set $\Omega = (0, 1)^2$ and consider the problem of finding u such that

$$\begin{aligned} -\Delta u &= \alpha \delta_{\mathscr{M}} + (1 - \alpha) \, \partial_x \delta_{\mathscr{M}} \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{79}$$

where $\alpha \in \{0, 1\}$ and $\mathscr{M} := \{(x, y) : |x - 1/2| < 1/4 \text{ and } y = 1/2 \text{ or } x = 1/2 \text{ and } |y - 1/2| < 1/4\}$ is an one-dimensional manifold. When $\alpha = 1$, we have $u \in H^1(\Omega)$, whereas when $\alpha = 0$, we have $u \in L_2(\Omega) \backslash H^1(\Omega)$.
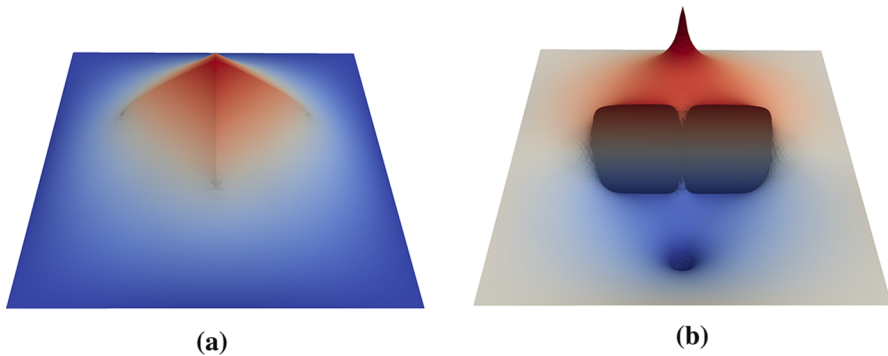
Fig. 4 The approximations to the solution of (79) produced by (80) for two different values of α. The two dark regions on the right plot depict jump discontinuity. **a** $\alpha = 1$. **b** $\alpha = 0$

We seek $u_h \in \mathbb{V}$ such that

$$\mathscr{A}_h(u_h, v_h) = \alpha E_0^{\mathscr{R}}(v_h)(\bar{x}) + (1 - \alpha) \, \partial_x E_0^{\mathscr{R}}(v_h)(\bar{x}) \quad \forall \, v_h \in \mathbb{V}, \bar{x} \in \mathscr{M}. \quad (80)$$

We fix $k = 2$ and solve (80) over a uniform, criss-cross triangulation of Ω with $h \approx 0.015$. In Fig. 4 we show the numerical approximation over this mesh for both values of α.

We conclude this exposition by noticing that, since the singularity in all tests were isolated, adaptive approximations should be able to recover best approximation. This motivates the extension of this analysis from the quite restrictive quasiuniform meshes to those that allow for some grading. Note the recent work [14] where, using a posteriori localisation techniques, analagous inf-sup results for the classical interior penalty dG scheme have been proven over meshes satisfying a *mesh variation* condition.

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