The boundedness and spectral properties of multiplicative Toeplitz operators

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Abstract

The aim of this thesis is to study the properties of multiplicative Toeplitz operators with an emphasis on boundedness and spectral points. In particular, we consider these operators acting on the sequence space ℓ^p and the Besicovitch function space $\mathcal{B}^2_{\mathbb{N}}$, in which case the operator is denoted by M_f and M_F respectively.

First, we present conditions for M_f to be bounded when acting from ℓ^p to ℓ^q for $1 \leq p \leq q \leq \infty$. From this investigation, a surprising connection with multiplicative number theory emerges; namely, that for a particular class of f, the operator norm is attained at the multiplicative elements in ℓ^p. Furthermore, through the Bohr lift, we consider the implication of these results in the setting of classical Toeplitz operators.

Secondly, we seek to establish the spectral properties of $M_F : \mathcal{B}^2_{\mathbb{N}} \to \mathcal{B}^2_{\mathbb{N}}$. For a certain class of F, we present a new result which describes the spectrum (and point spectrum). In the case of general symbols, this is much more challenging. During the investigation we illustrate how, despite their similar construction, many of the mathematical tools used to establish the spectrum of Toeplitz operators cannot be used in this multiplicative setting.

Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Nicola Thorn

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List of Symbols

Commonly used symbols

Symbol	Description	Reference
$\mathcal{A}$	the set of arithmetic functions or sequences	page 5
$B(\mathcal{X})$	set of bounded linear operators on the space $\mathcal{X}$	page 13
$\mathcal{B}^2_{\mathbb{Q}^+}$	the $\mathbb{Q}^+$ restricted Besicovitch space	Def. 1.17
$\mathcal{B}^2_{\mathbb{N}}$	the $\mathbb{N}$ restricted Besicovitch space	Def. 1.18
C_ϕ	convolution operator acting on sequence spaces	Def. 2.4
C_Φ	convolution operator acting on function spaces	Def. 2.3
D_f	Dirichlet convolution operator acting on sequence spaces	Def. 2.6
D_F	Dirichlet convolution operator acting on function spaces	Def. 2.8
$\mathcal{D}^2$	the Hilbert space of Dirichlet series	Def. 2.11
$\mathcal{H}^p$	the p-Hardy space	Def. 1.15
$\overline{\mathcal{H}^p}$	the anti analytic p-Hardy space	page 9
$\mathcal{L}^p$	the p-Lebesgue space	Def. 1.13
ℓ^p	the usual sequence space	Def. 1.9
$\ell^p(S)$	the sequence space on a countable set S	Def. 1.12
$\operatorname{im}(L)$	the image of the operator L	page 14
$\ker L$	the kernel of the operator L	page 14
$\mathcal{M}$	the set of multiplicative functions	Def. 1.34
$\mathcal{M}_c$	the set of completely multiplicative functions	Def. 1.34
$\mathcal{M}^p$	the set of multiplicative functions in ℓ^p	Def. 1.38
$\mathcal{M}^p_c$	the set of completely multiplicative functions in ℓ^p	Def. 1.38
M_f	multiplicative Toeplitz operator acting on sequence spaces	Def. 2.5
M_F	multiplicative Toeplitz operator acting on function spaces	Def. 2.7

Symbol	Description	Page
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$	page 6
$\mathbb{N}_0^\infty$	infinite cross product of $\mathbb{N}_0$ with finite non-zero components	page 34
$\overline{\mathbb{N}}$	the set of reciprocals of natural numbers	page12
$\mathbb{P}$	the set of prime numbers	page 17
$\mathbb{Q}^+$	the set of positive rationals	page 7
$\sigma(L)$	the spectrum of the operator L	Def. 1.29
$\sigma_p(L)$	the point spectrum of the operator L	Def. 1.30
$\sigma_{\mathrm{e}}(L)$	the essential spectrum of the operator L	Def. 1.32
$\mathbb{T}^n$	the n-dimensional torus	page 35
$\mathbb{T}^\infty$	the infinite torus	page 35
T_ϕ	Toeplitz operator acting on sequence spaces	Def. 2.2
T_Φ	Toeplitz operator acting on function spaces	Def. 2.1
$\mathcal{W}$	the Wiener algebra	Def. 1.14
$\mathcal{W}_{\mathbb{Q}^+}$	the $\mathbb{Q}^+$ restricted Besicovitch Wiener algebra	Def. 1.19
$\mathcal{W}_{\mathbb{N}}$	the $\mathbb{N}$ restricted Besicovitch Wiener algebra	Def. 1.19
$\mathcal{W}_{\overline{\mathbb{N}}}$	the $\overline{\mathbb{N}}$ restricted Besicovitch Wiener algebra	Def. 1.20
$\mathbb{Z}^\infty$	infinite cross product of $\mathbb{Z}$ with finite non-zero components	page 34

Comparison between additive and multiplicative operators.

We provide the reader with a comparison between operators in the multiplicative and additive settings for easy reference.

Operator	Additive	Multiplicative
Toeplitz operator on sequence spaces	$T_\phi : \ell^p(\mathbb{N}_0) \to \ell^q(\mathbb{N}_0)$	$M_f : \ell^p \to \ell^q$
Toeplitz operator on function spaces	$T_\Phi : \mathcal{H}^2 \to \mathcal{H}^2$	$M_F : \mathcal{B}^2_{\mathbb{N}} \to \mathcal{B}^2_{\mathbb{N}}$
Convolution operator on sequence spaces	$C_\phi : \ell^p(\mathbb{N}_0) \to \ell^q(\mathbb{N}_0)$	$D_f : \ell^p \to \ell^q$
Convolution operator on function spaces	$C_\Phi : \mathcal{H}^2 \to \mathcal{H}^2$	$D_F : \mathcal{B}^2_{\mathbb{N}} \to \mathcal{B}^2_{\mathbb{N}}$

Contents

Introduction

Toeplitz operators and their corresponding matrices were notably studied in [52] by Otto Toeplitz in 1911 and have been extensively researched by many authors in the decades since. Their applications stretch far and wide, appearing in many mathematical areas such as control theory, differential operators, physics and probability theory to name a few, [11], [10]. Toeplitz operators are also interesting mathematical objects in their own right, and there is a vast literature on their properties as operators. The aim of this thesis is to study a class of operators that are a generalisation of Toeplitz operators, known as multiplicative Toeplitz operators. Although studied less than their classical counterparts, it is clear that many of the mathematical tools that are successfully used to study classical Toeplitz operators can no longer be applied in the generalised setting. As such, many open questions and connections to other areas of mathematics, such as analytic number theory, have fuelled recent research. In particular, the inspired observation of Harold Bohr, known as the Bohr lift, allows Dirichlet series to be written as functions on the infinite torus. This has allowed harmonic and functional analysis to be deployed in the study of Dirichlet series.

We primarily investigate the operator theoretic properties of multiplicative Toeplitz operators. First, we seek to establish a criterion for boundedness between the sequence spaces $\ell^p \to \ell^q$. Utilising the Bohr lift, we also state results regarding the boundedness of Toeplitz operators. A further connection to the field of multiplicative number theory is also observed and discussed. Secondly, we study the spectral properties of multiplicative Toeplitz operators on the Besicovitch function space, describing the spectrum for a class of multiplicative Toeplitz operators. We highlight comparisons with classical Toeplitz operators and present some consequences in the classical setting of the results proven.

Outline of thesis

Chapter 1 surveys relevant background mathematics from both functional analysis and number theory which we use throughout the thesis.

In Chapter 2, we primarily discuss two linear operators and their corresponding matrix mappings. First, we define Toeplitz operators and Toeplitz matrices. Secondly, multiplicative Toeplitz operators are introduced on both the sequence space ℓ^p and the Besicovitch space $\mathcal{B}^2_{\mathbb{N}}$, denoted by M_f and M_F respectively. We then describe the interplay between these two operators. Namely, we explore how, using the Bohr lift, multiplicative Toeplitz operators generalise Toeplitz operators to infinite dimensions.

A review of the literature regarding the boundedness and spectral behaviour of these operators follows. From this, the following open questions are formulated. These questions form the basis of Chapters 3 and 4 respectively. Lastly, the chapter concludes with the statement of a new result regarding matrix mappings which preserve multiplicativity.

Chapter 3 focuses on the boundedness of multiplicative Toeplitz operators. We present a new sufficient condition for the boundedness of a multiplicative Toeplitz operator between $\ell^p \to \ell^q$, and, for certain symbols, show that this is also necessary. In addition, we see how this can be used in the classical Toeplitz setting. To investigate whether this condition is also necessary in general, we consider a simpler class of multiplicative Toeplitz operators, known as Dirichlet convolution operators. A connection with the multiplicative elements of ℓ^p and Dirichlet convolution operators is discovered. This connection leads to a further new result which suggests that, in general, the sufficient condition may not be necessary. We conclude this chapter with some further speculation and open problems.

Chapter 4 investigates the spectral properties of multiplicative Toeplitz operators. We present a result which describes the spectrum of the simpler class of Dirichlet convolution operators. Thereafter, we attempt to establish the spectrum of general multiplicative Toeplitz operators. We present results echoing the surrounding literature which show that many of the mathematical tools used to describe the spectrum of a Toeplitz operator, cannot be utilised in the multiplicative setting. Indeed, throughout this chapter, we compare these findings and state results in the additive setting.

Chapter 1

Preliminaries

In this chapter, we cover the relevant background mathematics which are required throughout the thesis. Much of the material covered in this chapter can be found in [2] and [37]. We review theory from functional analysis in Sections 1.1 to 1.3, and concepts from number theory in Sections 1.4 and 1.5.

1.1 Banach and Hilbert Spaces

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. A set, $\mathcal{X}$ equipped with vector addition and scalar multiplication, which satisfy the following properties for all a, b in $\mathbb{F}$, and x, y, z in $\mathcal{X}$ is called a ***vector space***;

1. $x + y \in \mathcal{X}$ and $ax \in \mathcal{X}$
2. $(x + y) + z = x + (y + z)$ and $x + y = y + x$
3. $\exists\, 0 \in \mathcal{X}$ such that $\forall x \in \mathcal{X}, x + 0 = x$
4. $\forall x \in \mathcal{X}$, $\exists - x \in \mathcal{X}$ such that $x + (-x) = 0$
5. $a(x + y) = ax + ay$ and $(a + b)x = ax + bx$
6. $1x = x$, where 1 denotes the multiplicative identity in $\mathbb{F}$.

We call a vector space, $\mathcal{X}$, an ***algebra*** when $\mathcal{X}$ is also equipped with a bilinear [1] vector multiplication map from $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying $x(yz) = (xy)z$, for all $x, y, z \in \mathcal{X}$.

[1]In general $f : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a bilinear vector map if it satisfies $f(x + y, z) = f(x, z) + f(y, z)$, $f(z, x + y) = f(z, x) + f(z, y)$, $f(ax, y) = af(x, y)$ and $f(x, ay) = af(x, y)$ for all $a \in \mathbb{F}$ and $x, y, z \in \mathcal{X}$

Definition 1.1. A ***basis*** of a vector space $\mathcal{X}$ is a linearly independent subset of $\mathcal{X}$ which spans $\mathcal{X}$.

Definition 1.2. Let $\mathcal{X}$ be a vector space. The ***dimension*** of $\mathcal{X}$, denoted by $\dim \mathcal{X}$, is the cardinality of a basis of $\mathcal{X}$.

Note that all bases of a vector space will have the same cardinality, and therefore the dimension of $\mathcal{X}$ is well defined. In particular, we are interested in separable[2] spaces, in which case the basis is countable, say $\{\chi_n : n \in \mathbb{N}\}$.

Normed and Banach Spaces

We now proceed to consider vector spaces equipped with a norm.

Definition 1.3. Let $\mathcal{X}$ be a vector space. A ***normed space*** is a pair $(\mathcal{X}, \|\cdot\|)$, where $\|\cdot\|$, which is referred to as a norm, is a real-valued function which satisfies

1. $\|x\| \geq 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0 \iff x = 0$,
2. $\|cx\| = |c|\,\|x\|$ for any scalar $c \in \mathbb{F}$,
3. $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in \mathcal{X}$.

We say x_n in $\mathcal{X}$ is ***convergent*** to x with respect to $\|\cdot\|$, denoted by $x_n \to x$, if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Definition 1.4. Let $\mathcal{X}$ be a normed space. We say $\mathcal{X}$ is a ***Banach space*** if $\mathcal{X}$ is complete with respect to $\|\cdot\|$. That is, if every Cauchy sequence[3] is convergent in $\mathcal{X}$.

Definition 1.5. Let $\mathcal{X}$ be an algebra. A ***Banach algebra*** is a pair $(\mathcal{X}, \|\cdot\|)$ such that $\mathcal{X}$ is also a Banach space with respect to the norm and such that $\|xy\| \leq \|x\|\|y\|$ is satisfied for all $x, y \in \mathcal{X}$.

For convenience, we simply write $\mathcal{X}$ to denote either the normed space or Banach algebra $(\mathcal{X}, \|\cdot\|)$ which will be clarified to the reader in the case of ambiguity.

Definition 1.6. The ***dual*** of a Banach space $\mathcal{X}$ is the set of all bounded linear functionals $L : \mathcal{X} \to \mathbb{F}$, which we shall denote by $\mathcal{X}^*$.

[2] A separable space is one which contains a countable dense subset.

[3] We say a sequence $x_n \in \mathcal{X}$ is Cauchy if for every $\epsilon > 0$, there exists $N > 0$ such that for all $n, m > N$, $\|x_n - x_m\| < \epsilon$.

Through the dual space, the notion of weak convergence emerges. We say x_n in $\mathcal{X}$ is ***weakly convergent*** to x, denoted by $x_n \rightharpoonup x$, if for every $f \in \mathcal{X}^*$,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

Inner Product and Hilbert Spaces

We turn our attention to vector spaces equipped with an inner product.

Definition 1.7. Let $\mathcal{X}$ be a vector space. An ***inner product space*** is a pair $(\mathcal{X}, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is a complex-valued function which satisfies the following conditions for all $x, y, z \in \mathcal{X}$;

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$, where $\bar{\cdot}$ is the complex conjugate,
2. $\langle cx, y\rangle = c\langle x, y\rangle$ where $c \in \mathbb{C}$,
3. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,
4. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$.

Observe that any inner product gives rise to a norm defined by $\|x\| = \sqrt{\langle x, x\rangle}$ and, therefore, an inner product space is simultaneously a normed space.

Definition 1.8. Let $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ be a inner product space. We say $\mathcal{X}$ is a ***Hilbert*** space if $\mathcal{X}$ is complete with respect to the norm induced by the inner product.

Again for ease, we shall write $\mathcal{X}$ to denote $(\mathcal{X}, \langle\cdot,\cdot\rangle)$. We say two elements in $\mathcal{X}$, say x and y, are ***orthogonal*** if $\langle x, y\rangle = 0$. Furthermore, we say a basis for which $\langle \chi_n, \chi_n\rangle = 1$, and $\langle \chi_n, \chi_m\rangle = 0$ for all $n, m \in \mathbb{N}$ whenever $n \neq m$ is an ***orthonormal basis***.

For more information on Banach and Hilbert spaces, see Chapters 2 and 3 in [37]. We now review three examples of Banach spaces, which will play an important role in this thesis.

1.1.1 The ℓ^p sequence spaces

Let $\mathcal{A}$ be the ***set of arithmetical functions***, $f : \mathbb{N} \to \mathbb{C}$. We use the terms "sequences" and "arithmetical functions" interchangeably, as we can write any arithmetical function $f(n)$ as a sequence indexed by the natural numbers, which is written as $(f_n)_{n\in\mathbb{N}}$.

Definition 1.9. For $1 \leq p < \infty$ and $p = \infty$ respectively, the ***sequence space*** ℓ^p is given by

$$\ell^p = \left\{ f \in \mathcal{A} : \sum_{n\in\mathbb{N}} |f(n)|^p < \infty \right\}, \qquad \ell^\infty = \left\{ f \in \mathcal{A} : \sup_{n\in\mathbb{N}} |f(n)| < \infty \right\}.$$

When equipped with the norm,

$$\|f\|_p = \left(\sum_{n\in\mathbb{N}} |f(n)|^p \right)^{\frac{1}{p}} \text{ and } \|f\|_\infty = \sup_{n\in\mathbb{N}} |f(n)|,$$

respectively, the spaces ℓ^p are known to be Banach spaces; see Examples 1.5-2 and 1.5-4 in [37]. Moreover, in the case when $p = 2$, ℓ^2 is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{n\in\mathbb{N}} f(n)\overline{g(n)}.$$

We now discuss some properties of ℓ^p spaces. First, observe that the size of each ℓ^p space increases with p, i.e. $\ell^1 \subset \ell^p \subset \ell^q \subset \ell^\infty$ for $1 \leq p \leq q \leq \infty$. As a result for $p < q$, we have $\|f\|_q \leq \|f\|_p$. Secondly for $p < \infty$, as discussed in Section 2.3 in [37], the set $\{e_n : n \in \mathbb{N}\}$ forms a basis[4] in ℓ^p, where $e_n \in \mathcal{A}$ is defined by $e_n(n) = 1$, and $e_n(m) = 0$ when $m \neq n$. In the case when $p = 2$, the basis is orthonormal. In addition to the basis, the dual of ℓ^p is known.

Proposition 1.10. *The dual of ℓ^p (for $1 < p < \infty$), denoted by $(\ell^p)^*$, is ℓ^q where $1 = \frac{1}{p} + \frac{1}{q}$. Moreover, for any $h \in (\ell^p)^*$, there exists $y = (y_n)_{n\in\mathbb{N}} \in \ell^q$ such that for $x = (x_n)_{n\in\mathbb{N}} \in \ell^p$,*

$$h(x) = \sum_{k\in\mathbb{N}} x(k)y(k) \qquad \textit{where } y(k) = h(e_k).$$

Proof. See Example 2.10-7 in [37]. □

It is worth mentioning that the dual of ℓ^1 is ℓ^∞. Finally, we state a result that will be used throughout the thesis. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Theorem 1.11 (Hölder's Inequality)**.** *Let $a_1, \ldots, a_k$ be positive real numbers and $f_1, \ldots, f_k$*

[4] Note that this is not a basis in ℓ^∞.

be positive arithmetical functions in ℓ^1. If $a_1 + \cdots + a_k = 1$, then

$$\sum_{n\in\mathbb{N}} f_1(n)^{a_1} \dots f_k(n)^{a_k} \leq \left(\sum_{n\in\mathbb{N}} f_1(n)\right)^{a_1} \dots \left(\sum_{n\in\mathbb{N}} f_k(n)\right)^{a_k}.$$

Proof. See Section 1.2 in [37]. □

In the case when $k = 2$ and $a_1 = a_2 = 2$, we obtain the Cauchy-Schwarz inequality. That is, for arithmetical functions f and $g \in \ell^2$,

$$\left|\sum_{n\in\mathbb{N}} f(n)\overline{g(n)}\right|^2 \leq \sum_{n\in\mathbb{N}} |f(n)|^2 \sum_{n\in\mathbb{N}} |g(n)|^2.$$

We can also extend the notation of ℓ^p to those functions indexed by a countable set, say $\mathcal{S}$.

Definition 1.12. For $1 \leq p < \infty$ and $p = \infty$ respectively, the space $\ell^p(\mathcal{S})$ is given by

$$\ell^p(\mathcal{S}) = \left\{ f : \mathcal{S} \to \mathbb{C} : \sum_{n\in\mathcal{S}} |f(n)|^p < \infty \right\}, \qquad \ell^\infty(\mathcal{S}) = \left\{ f : \mathcal{S} \to \mathbb{C} : \sup_{n\in\mathcal{S}} |f(n)| < \infty \right\}.$$

$$\|f\|_{p,\mathcal{S}} = \left(\sum_{n\in\mathcal{S}} |f(n)|^p\right)^{\frac{1}{p}} \text{ and } \|f\|_{\infty,\mathcal{S}} = \sup_{n\in\mathcal{S}} |f(n)|.$$

In the case when $p = 2$, $\ell^2(\mathcal{S})$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \sum_{n\in\mathcal{S}} f(n)\overline{g(n)}.$$

Two examples we refer to within this thesis are $\mathcal{S} = \mathbb{Z}$ and $\mathcal{S} = \mathbb{Q}^+$, the set of positive rationals.

1.1.2 $\mathcal{L}^p$ and Hardy spaces

Let $\mathbb{T}$ denote the unit circle in the complex plane.

Definition 1.13. For $1 \leq p < \infty$, we define $\mathcal{L}^p(\mathbb{T})$ to be the space of measurable complex-valued functions on $\mathbb{T}$, $\Phi : \mathbb{T} \to \mathbb{C}$, such that

$$\int_{\mathbb{T}} |\Phi|^p = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\Phi(e^{i\theta})\right|^p d\theta < \infty.$$

In the case when $p = \infty$, $\mathcal{L}^\infty(\mathbb{T})$ is the space of all measurable complex-valued functions such that there exists a positive constant, C, with $|\Phi| \leq C$ almost everywhere (a.e.) [5]

For convenience, we shall simply use $\mathcal{L}^p$ to denote $\mathcal{L}^p(\mathbb{T})$. It is known that $\mathcal{L}^p$ is a Banach space [6] when equipped with

$$\|\Phi\|_{\mathcal{L}^p} = \left(\int_{\mathbb{T}} |\Phi|^p\right)^{\frac{1}{p}} \text{ or } \|\Phi\|_\infty = \operatorname{ess\,sup} \Phi,$$

respectively, where $\operatorname{ess\,sup} \Phi = \inf \{C : |\Phi| \leq C \text{ a.e.}\}$.

Of particular interest is the case when $p = 2$. In this case, as described in Section 3.1-5 in [37], $\mathcal{L}^2$ is a Hilbert space with the inner product and resulting norm,

$$\langle \Phi, \Gamma \rangle = \int_{\mathbb{T}} \Phi\overline{\Gamma} \text{ and } \|\Phi\|_{\mathcal{L}^2} = \left(\int_{\mathbb{T}} |\Phi|^2\right)^{\frac{1}{2}}.$$

Let $\chi_n(t) = t^n$ for $t \in \mathbb{T}$ and $n \in \mathbb{Z}$. The set $\{\chi_n : n \in \mathbb{Z}\}$ forms an orthonormal basis in $\mathcal{L}^2$. As shown in [16], every $\Phi \in \mathcal{L}^2$ can be represented by a ***Fourier series***. That is,

$$\Phi(t) \sim \sum_{n\in\mathbb{Z}} \phi(n)\chi_n(t),$$

where $\phi(n)$ are the ***Fourier coefficients*** of Φ given by

$$\phi(n) = \int_{\mathbb{T}} \Phi\overline{\chi_n}.$$

Functions in $\mathcal{L}^2$ can be associated with sequences in $\ell^2(\mathbb{Z})$ through the Fourier coefficients which we describe now. There exists an isometric isomorphism between $\mathcal{L}^2$

[5] A property P in a measure-space X holds almost everywhere if there exists $Y \subset X$ such that the measure of Y is 0, and P holds for all $x \in X \setminus Y$.

[6] Note that $\|\Phi\|_{\mathcal{L}^p} = 0$ does not imply $\Phi = 0$. Indeed, $\mathcal{L}^p$ is, strictly speaking, the set of equivalence classes, in which Φ and Γ are equivalent if $\Phi = \Gamma$ a.e.

and $\ell^2(\mathbb{Z})$ given by the mapping $\tau : \mathcal{L}^2 \to \ell^2(\mathbb{Z})$, where

$$\Phi \overset{\tau}{\mapsto} (\phi(n))_{n\in\mathbb{Z}}.$$

Moreover, due to the Parseval relation, it follows that for $\Phi, \Gamma \in \mathcal{L}^2$ (where ϕ(n) and $\gamma(n)$ are the Fourier coefficients respectively),

$$\|\Phi\|_{\mathcal{L}^2}^2 = \sum_{n\in\mathbb{Z}} |\phi(n)|^2 = \|\phi\|_2^2 \quad \text{and} \quad \langle \Phi, \Gamma \rangle = \sum_{n\in\mathbb{Z}} \phi(n)\overline{\gamma(n)} = \langle \phi, \gamma \rangle.$$

We now review two important subspaces of $\mathcal{L}^p$ which are used throughout this thesis. First, we consider the algebra formed of absolutely convergent Fourier series.

Definition 1.14. The ***Wiener*** algebra is defined by

$$\mathcal{W} = \left\{ \Phi = \sum_{n\in\mathbb{Z}} \phi(n)\chi_n : \phi \in \ell^1(\mathbb{Z}) \right\}.$$

Under point-wise multiplication, $\mathcal{W}$ forms a Banach algebra with the norm

$$\|\Phi\|_{\mathcal{W}} = \sum_{n\in\mathbb{Z}} |\phi(n)|.$$

Secondly, we consider the Hardy subspace of $\mathcal{L}^p$.

Definition 1.15. For $1 \leq p \leq \infty$, the ***Hardy*** space is given by

$$\mathcal{H}^p = \{\Phi \in \mathcal{L}^p : \phi(n) = 0 \text{ if } n < 0\}.$$

Furthermore, we define

$$\overline{\mathcal{H}^p} = \{\Phi \in \mathcal{L}^p : \phi(n) = 0 \text{ if } n \geq 0\}.$$

Functions in $\mathcal{H}^p$ are called ***analytic*** since the corresponding Fourier series

$$\Phi(t) \sim \sum_{n\in\mathbb{N}_0} \phi(n)\chi_n(t)$$

is holomorphic for all $|t| < 1$. We also say functions in $\overline{\mathcal{H}^p}$ are ***anti-analytic***. The space $\mathcal{H}^p$, for $1 \le p \le \infty$ is a Banach space with the norm

$$\|\Phi\|_{\mathcal{H}^p} = \left(\int_{\mathbb{T}} |\Phi|^p \right)^{\frac{1}{p}} \text{ or } \|\Phi\|_\infty = \operatorname{ess\,sup} \Phi.$$

In the case of $p = 2$, $\mathcal{H}^2$ assumes the Hilbert space properties of $\mathcal{L}^2$ with the inner product and norm

$$\langle \Phi, \Gamma \rangle = \int_{\mathbb{T}} \Phi \overline{\Gamma} = \sum_{n \in \mathbb{N}_0} \phi(n) \overline{\gamma(n)} \text{ and } \|\Phi\|_{\mathcal{L}^2} = \left(\int_{\mathbb{T}} |\Phi|^2 \right)^{\frac{1}{2}} = \left(\sum_{n \in \mathbb{N}_0} |\phi(n)|^2 \right)^{\frac{1}{2}}.$$

Moreover, the set $\{\chi_n : n \in \mathbb{N}_0\}$ is an orthonormal basis and there exists an isometric isomorphism between $\mathcal{H}^2$ and $\ell^2(\mathbb{N}_0)$ given by $\Phi \mapsto \phi(n)_{n \in \mathbb{N}_0}$.

Wiener's Lemma

One key theorem in the study of Fourier series is known as Wiener's Lemma and characterises the invertibility of functions in $\mathcal{W}$.

Theorem 1.16. *(Wiener's Lemma)*

1. *Let $\Phi \in \mathcal{W}$. If $\Phi(t) \neq 0$ for $t \in \mathbb{T}$ then $\Phi^{-1} \in \mathcal{W}$.*

2. *Let $\Phi \in \mathcal{W} \cap \mathcal{H}^2$. Then Φ is invertible in $\mathcal{W} \cap \mathcal{H}^2$ if and only if $\Phi(t) \neq 0$ for all $t \in \mathbb{D}$.*

Proof. See Lemma IIe in [55] for proof of 1. Item 2 follows from the properties of analytic functions on the complex plane. □

1.1.3 Besicovitch space

Let $\mathcal{S}$ be the set of Dirichlet polynomials, $F : \mathbb{R} \to \mathbb{C}$, of the form

$$F(t) = \sum_{k=1}^{n} a_k e^{i b_k t},$$

where $a_k \in \mathbb{C}$ and pairwise distinct $b_k \in \mathbb{R}$. The space $\mathcal{S}$ is an inner product space when equipped with the inner product

$$\langle F, G \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} F(t)\overline{G}(t)dt.$$

The resulting norm is given by

$$\|F\| = \left(\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |F(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

By taking the closure of $\mathcal{S}$ with respect to the above norm, we obtain a Hilbert space[7] known as the ***Besicovitch space*** which is denoted by $\mathcal{B}^2$. Let $\chi_\lambda(t) = \lambda^{it}$ for $t \in \mathbb{R}$ and $\lambda > 0$. As described in Section 3.2 in [30], we define the ***Dirichlet Fourier coefficients*** of F to be

$$f(\lambda) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} F(t)\overline{\chi_\lambda(t)}dt.$$

Given $F \in \mathcal{B}^2$, $f(\lambda)$ exists for all $\lambda > 0$, and the $f(\lambda)$ are non-zero for at most a countable set, say $(\lambda_k)_{k \in \mathbb{N}}$. Furthermore, Parseval's relation holds; that is, for $F, G \in \mathcal{B}^2$,

$$\langle F, G \rangle = \sum_{k \in \mathbb{N}} f(\lambda_k)\overline{g(\lambda_k)} = \langle f, g \rangle \text{ and } \|F\| = \left(\sum_{k \in \mathbb{N}} |f(\lambda_k)|^2 \right)^{\frac{1}{2}} = \|f\|_2.$$

Given $f \in \ell^2$ and $\lambda_k > 0$, there exists $F \in \mathcal{B}^2$, such that

$$F \sim \sum_{k \in \mathbb{N}} f(\lambda_k)\chi_{\lambda_k}$$

We call this series the ***Dirichlet Fourier series*** of F. By setting the support of λ_k to be $\mathbb{Q}^+$ and $\mathbb{N}$, we obtain two Hilbert spaces which are used within this thesis.

Definition 1.17. The $\mathbb{Q}^+$***-restricted Besicovitch space***, $\mathcal{B}^2_{\mathbb{Q}^+}$, is the space of all

[7]Strictly speaking $\mathcal{B}^2$ is the set of equivalence classes for which we say $F = G$ if $\|F - G\| = 0$.

$F \in \mathcal{B}^2$ whose Dirichlet Fourier coefficients are supported on $\mathbb{Q}^+$. In other words,

$$\mathcal{B}^2_{\mathbb{Q}^+} = \left\{ F \in \mathcal{B}^2 : f(q) = 0 \text{ if } q \notin \mathbb{Q}^+ \right\}.$$

Definition 1.18. The $\mathbb{N}$-***restricted Besicovitch space***, $\mathcal{B}^2_{\mathbb{N}}$, is the space of all $F \in \mathcal{B}^2$ whose Dirichlet Fourier coefficients are supported on $\mathbb{N}$. In other words,

$$\mathcal{B}^2_{\mathbb{N}} = \left\{ F \in \mathcal{B}^2 : f(n) = 0 \text{ if } n \notin \mathbb{N} \right\}.$$

We have the following induced norms respectively,

$$\|F\|_{\mathcal{B}^2_{\mathbb{Q}^+}} = \left(\sum_{q \in \mathbb{Q}^+} |f(q)|^2 \right)^{1/2} \text{ and } \|F\|_{\mathcal{B}^2_{\mathbb{N}}} = \left(\sum_{n \in \mathbb{N}} |f(n)|^2 \right)^{1/2}$$

The set $\{\chi_q : q \in \mathbb{Q}^+\}$ forms an orthonormal basis in $\mathcal{B}^2_{\mathbb{Q}^+}$, and for $\mathcal{B}^2_{\mathbb{N}}$, it is the set $\{\chi_n : n \in \mathbb{N}\}$ which forms an orthonormal basis.

We now consider some further subspaces which form Banach algebras.

Definition 1.19. Let the ***Besicovitch-Wiener algebras***, $\mathcal{W}_{\mathbb{Q}^+}$ and $\mathcal{W}_{\mathbb{N}}$ be defined respectively by

$$\mathcal{W}_{\mathbb{Q}^+} = \left\{ \sum_{q \in \mathbb{Q}^+} f(q) q^{it} : f \in \ell^1(\mathbb{Q}^+) \right\} \text{ and } \mathcal{W}_{\mathbb{N}} = \left\{ \sum_{n \in \mathbb{N}} f(n) n^{it} : f \in \ell^1 \right\}.$$

It is of use to define the algebra constructed from functions supported on the reciprocals of the natural numbers. Let $\overline{\mathbb{N}} = \left\{ q \in \mathbb{Q}^+ : q = \frac{1}{n} \text{ where } n \in \mathbb{N} \right\}$.

Definition 1.20. Let $\mathcal{W}_{\overline{\mathbb{N}}}$ be defined by

$$\mathcal{W}_{\overline{\mathbb{N}}} = \left\{ \sum_{n \in \overline{\mathbb{N}}} f(n) n^{it} : f \in \ell^1(\overline{\mathbb{N}}) \right\}.$$

Note here that given $F \in \mathcal{W}_{\mathbb{N}}$, the complex conjugate of F is

$$\sum_{n \in \mathbb{N}} \overline{f(n)} n^{-it} = \sum_{m \in \overline{\mathbb{N}}} \overline{f(1/m)} m^{it}.$$

Therefore, $\overline{F} \in \mathcal{W}_{\mathbb{N}} \iff F \in \mathcal{W}_{\overline{\mathbb{N}}}$.

For more information on Besicovitch spaces, [7] is recommended. We conclude this section on example Banach spaces with the observation that the space $\mathcal{B}^2_{\mathbb{N}}$ can be thought of as an analogous space of ℓ^2, and in turn $\mathcal{H}^2$. Indeed there exists an isometric isomorphism, τ, between $\mathcal{B}^2_{\mathbb{N}} \to \ell^2$, given by $F \overset{\tau}{\mapsto} (f(n))_{n\in\mathbb{N}}$. In particular, the basis of $\mathcal{B}^2_{\mathbb{N}}$ which is given by $\{\chi_n(t) = n^{it} : n \in \mathbb{N}\}$ corresponds with the basis of ℓ^2 $\{e_n : n \in \mathbb{N}\}$ since, for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{N}} e_n(k)\chi_k = \chi_n. \tag{1.1}$$

Similarly the basis of $\mathcal{H}^2$, $\{\chi_n(t) = t^n : n \in \mathbb{N}_0\}$ corresponds to that of $\ell^2(\mathbb{N}_0)$.

1.2 Linear operators on Banach Spaces

Definition 1.21. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces over $\mathbb{C}$. Then $L : \mathcal{X} \to \mathcal{Y}$ is said to be a ***linear operator*** from $\mathcal{X}$ to $\mathcal{Y}$ if for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and all scalars $a \in \mathbb{C}$,

$$L(x + y) = Lx + Ly$$
$$L(ax) = aLx.$$

Definition 1.22. A linear operator $L : \mathcal{X} \to \mathcal{Y}$ is called ***bounded*** if for some $c \geq 0$,

$$\|Lx\|_{\mathcal{Y}} \leq c\|x\|_{\mathcal{X}} \text{ for all } x \in \mathcal{X}.$$

Definition 1.23. The ***operator norm*** of a bounded linear operator is defined to be

$$\|L\|_{\mathcal{X},\mathcal{Y}} = \sup_{x\neq 0} \frac{\|Lx\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}}.$$

For the purposes of this thesis, linear operators acting from $\mathcal{X} \to \mathcal{X}$ are of interest. We denote the set of all bounded linear operators acting from $\mathcal{X} \to \mathcal{X}$ by $B(\mathcal{X})$. The space $B(\mathcal{X})$ is itself a Banach algebra where the norm is the operator norm given in Definition 1.23, and multiplication is given by the composition of the operators.

Definition 1.24. Let $L \in B(\mathcal{X})$. Then L is said to be ***compact*** if, for every bounded subset $\mathcal{S}$ of $\mathcal{X}$, the image of $L(\mathcal{S})$ is relatively compact, i.e. the closure, $\overline{L(\mathcal{S})}$ is compact.

We are particularly interested in operators acting upon Hilbert spaces. First, as described in [4], for any $L \in B(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, there exists a corresponding matrix representation with respect to a basis, denoted by A_L. Let $(\chi_n)_{n\in\mathbb{N}}$ be a basis[8] of $\mathcal{H}$. The i,jth entry of A_L, denoted as $a_{i,j}$, is given by

$$a_{i,j} = \langle L\chi_j, \chi_i\rangle.$$

Secondly, the adjoint and projection operators are two key mappings on Hilbert spaces.

Definition 1.25. For every $L \in B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, there exists a unique $L^* \in B(\mathcal{H})$ such that $\langle Lx, y\rangle = \langle x, L^*y\rangle$. The operator L^* is called the ***adjoint operator***.

Proposition 1.26. *Let $L \in B(\mathcal{H})$ with the adjoint L^*. Then,*

1. $L^{**} = L$
2. $(aL)^* = \overline{a}L^*$ *for* $a \in \mathbb{C}$
3. $(L+G)^* = L^* + G^*$, *and* $(LG)^* = G^*L^*$ *for* $G \in B(\mathcal{H})$.

Proof. See section 4.5 in [37]. $\square$

We say that an operator is ***self-adjoint*** if $L = L^*$. One important self-adjoint operator which we will encounter is a projection.

Definition 1.27. We say $P \in B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is a ***projection*** if $P^2 = P$ and P is self-adjoint.

1.3 Invertibility of bounded linear operators

Definition 1.28. An element of a Banach algebra, $x \in \mathcal{X}$ is called ***invertible*** if there exists $y \in \mathcal{X}$ such that $xy = yx = e$, where e is the multiplicative identity element of $\mathcal{X}$.

Let $\mathcal{X}$ be a Banach space. For $L \in B(\mathcal{X})$, we define the ***kernel*** and the ***image*** of L to be $\ker(L) = \{x \in \mathcal{X} : Lx = 0\}$ and $\mathrm{im}(L) = \{Lx \in \mathcal{X} : x \in \mathcal{X}\}$ respectively. Note here that if $L \in B(\mathcal{X})$ then the following are equivalent,

[8] We assume $\mathcal{H}$ to be separable

1. L is invertible

2. L has a bounded bijective inverse.

3. $\ker(L) = \{0\}$ and $\mathrm{im}(L) = \mathcal{X}$.

1.3.1 Spectrum of a bounded linear operator

Definition 1.29. Let $L \in B(\mathcal{X})$. The ***spectrum*** of L is the set

$$\sigma(L) = \{\lambda \in \mathbb{C} : L - \lambda I \text{ is not invertible }\}$$

where I is the identity operator.

The spectrum of a linear operator generalises the concept of eigenvalues to operators acting upon an infinite dimensional space (see section 1.1 in [3]).

Definition 1.30. The ***point spectrum***, $\sigma_p(L)$, is the set of eigenvalues of L.

The point spectrum is contained within the spectrum i.e. $\sigma_p(L) \subset \sigma(L)$. Indeed, given $\lambda \in \mathbb{C}$ that satisfies $\lambda x = Lx$ for some non-zero $x \in \mathcal{X}$, then $(L - \lambda I)x = 0$ for some $x \neq 0$. Hence, $\ker(L - \lambda I) \neq \{0\}$ and so, $L - \lambda I$ is not invertible.

For some classes of operators, the form of the spectrum is known. If, for example, $\mathcal{X}$ has finite dimension then $\sigma_p(L) = \sigma(L)$. As described in Theorem 8.3-1 in [37], on an infinite dimensional space, the spectrum of a compact operator, say K, is given by $\sigma(K) = \{\lambda_n : n \in \mathbb{N}\} \cup \{0\}$ where λ_n are eigenvalues and $\lambda_n \to 0$.

1.3.2 Fredholm operators

For this thesis, we shall consider Fredholmness of operators acting over separable Hilbert spaces. Let the ***co-kernel*** of $L \in B(\mathcal{H})$, denoted as $\operatorname{co}\ker(L)$, be the quotient space $\mathcal{H}/\mathrm{im}(L)$.

Definition 1.31. Let $L \in B(\mathcal{H})$. Then L is called ***Fredholm*** if $\dim \operatorname{co}\ker(L) < \infty$ and $\dim\ker(L) < \infty$. The ***index*** of a Fredholm operator is

$$\mathrm{ind}(L) = \dim\ker(L) - \dim \operatorname{co}\ker(L).$$

Fredholmness can be thought of as a measure of how "badly" an operator fails to be invertible. As explained in Section 3.2 in [3], $\dim\ker(L)$ and $\dim\operatorname{co}\ker(L)$ gives insight into the solutions, x, of $Lx = y$. Indeed, the dimension of the co-kernel is a measure of how many x fail to satisfy $Lx = y$. Moreover, the dimension of the kernel measures how many non-unique solutions there are.

Definition 1.32. Let $L \in B(\mathcal{H})$. The ***essential spectrum*** of L is the set $\sigma_{\mathrm{e}}(L) = \{\lambda \in \mathbb{C} : L - \lambda I \text{ is not Fredholm}\}$.

Observe that $\sigma_{\mathrm{e}}(L) \subset \sigma(L)$. For $\lambda \notin \sigma(L)$, we know

$$\ker(L - \lambda I) = \{0\} \text{ and } \operatorname{im}(L - \lambda I) = \mathcal{H}.$$

Therefore, $L - \lambda I$ is Fredholm by definition, and thus $\lambda \notin \sigma_{\mathrm{e}}(L)$. Moreover, if L is invertible, it follows that the index of L must be zero as both the kernel and co-kernel are the trivial set. In addition, from the identity

$$\overline{\operatorname{im}(L)} \oplus \ker(L^*) = \mathcal{H},$$

it follows that $\ker(L^*) = \{0\}$.

Equivalently, one can define Fredholm operators in terms of compact operators. As described in Section 1.3 in [10], L is Fredholm if and only if there exists $G \in B(\mathcal{H})$ such that $LG - I$ and $GL - I$ are both compact. In this case, we say L is ***invertible modulo compact operators***.

For an introduction to invertibility and spectral theory, [3] is recommended.

1.4 Arithmetical functions

We now turn our attention to concepts from number theory. Recall that $\mathcal{A}$ is the set of arithmetical functions.

Definition 1.33. We say that $f \in \mathcal{A}$ is of order $g \in \mathcal{A}$ and write $f = O(g)$ if, for some constant, $|f(n)| \leq C\,|g(n)|$ as $n \to \infty$. We also write $f \ll g$ to mean $f = O(g)$.

1.4.1 Multiplicative functions

Definition 1.34. Let $f \in \mathcal{A}$ be not identically 0. We say f is ***multiplicative***, and write $f \in \mathcal{M}$, if

$$f(mn) = f(m)f(n), \tag{1.2}$$

for every $m, n \in \mathbb{N}$ that are co-prime i.e. $(m, n) = 1$.
Further, we call f ***completely multiplicative*** and write $f \in \mathcal{M}_c$ if (1.2) holds for all $m, n \in \mathbb{N}$.
In addition, we call a sequence ***constant multiplicative*** if for some $c \in \mathbb{R}$,

$$f(n) = cg(n),$$

where $g(n)$ is multiplicative.

Example 1.35. Let $\mu(n)$ denote the ***Möbius function***, which is defined as $\mu(1) = 1$, $\mu(n) = 0$ if n is divisible by the square of a prime number, otherwise $\mu(n) = (-1)^k$, where k is the number of prime factors of n. As discussed in Section 2.9, Example 3 of [2], the Möbius function is multiplicative but not completely multiplicative. For example, $\mu(4) = 0$, but $\mu(2)\mu(2) = 1$.

We proceed by discussing some properties of sequences which have a multiplicative structure. First, it follows that for $f \in \mathcal{M}$, $f(1) = 1$, as $f(m) = f(m)f(1)$ and f cannot be identically zero. Secondly, multiplicative functions are governed by their behaviour on prime powers. Let $\mathbb{P}$ denote the ***set of prime numbers***. Let $f \in \mathcal{M}$ and suppose $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ is the prime decomposition of $n \in \mathbb{N}$ where $p_i \in \mathbb{P}$ and $\alpha_i \in \mathbb{N}$ for $1 \leq i \leq k$. Then, as each $p_i^{\alpha_i}$ is co-prime to $p_j^{\alpha_j}$ for all $1 \leq j \leq k$ where $j \neq i$, we have

$$f(n) = f(p_1^{\alpha_1}) \ldots f(p_k^{\alpha_k}).$$

Moreover, given $f \in \mathcal{M}_c$, we can write $f\left(p_i^{\alpha_i}\right) = f\left(p_i\right)^{\alpha_i}$ and thus

$$f(n) = f(p_1)^{\alpha_1} \ldots f(p_k)^{\alpha_k}.$$

From this, it follows that a series of an arithmetical function can be formulated as an infinite product ranging over the primes, see Theorem 11.6 in [2].

Proposition 1.36 (Euler Product)**.** *Let $f \in \mathcal{M}$ such that $f \in \ell^1$. Then*

$$\sum_{n\in\mathbb{N}} f(n) = \prod_{p\in\mathbb{P}} \left(\sum_{k\in\mathbb{N}_0} f\left(p^k\right) \right).$$

Furthermore, if $f \in \mathcal{M}_c$,

$$\sum_{n\in\mathbb{N}} f(n) = \prod_{p\in\mathbb{P}} \left(\frac{1}{1-f(p)} \right).$$

The notion of multiplicativity can be extended to sequences indexed by $\mathbb{Q}^+$.

Definition 1.37. A function $f : \mathbb{Q}^+ \to \mathbb{C}$ is ***multiplicative*** if $f(1) = 1$ and for all distinct primes, p_i,

$$f(p_1^{\alpha_1} \dots p_k^{\alpha_k}) = f(p_1^{\alpha_1}) \dots f(p_k^{\alpha_k})$$

(respectively), where $\alpha_i \in \mathbb{Z}$.

The Euler product also holds; that is, for $f \in \ell^1(\mathbb{Q}^+)$ multiplicative, we have

$$\sum_{q\in\mathbb{Q}^+} f(q) = \prod_{p\in\mathbb{P}} \left(\sum_{k\in\mathbb{Z}} f\left(p^k\right) \right).$$

1.4.2 Set of multiplicative functions

Definition 1.38. Let $\mathcal{M}^p$ and $\mathcal{M}_c^p$ denote the subsets of ℓ^p which consisting of multiplicative and completely multiplicative elements respectively. Namely,

$$\mathcal{M}^p = \ell^p \cap \mathcal{M} \text{ and } \mathcal{M}_c^p = \ell^p \cap \mathcal{M}_c.$$

Here we give two results relating to these multiplicative subsets. It is worth noting that to avoid confusion with ℓ^p, we sometimes use t to denote a prime. First, observe that if $f \in \mathcal{M}_c^p$, for $1 \leq p < \infty$, then by complete multiplicativity, we have $\sup_{t\in\mathbb{P}} |f(t)| < 1$. Moreover, $\sum_{t\in\mathbb{P}} |f(t)|^p < \infty$. Secondly, the linear span of $\mathcal{M}^p$ is dense in ℓ^p.

Proposition 1.39. *Let $\mathcal{S}_p = \left\{ \left(\frac{1}{n^\alpha}\right)_{n\geq 1} : \alpha > \frac{1}{p} \right\}$. The subspace of ℓ^p defined by* $\operatorname{span}(\mathcal{S}_p)$ *is dense in ℓ^p.*

Proof. From [37] Theorem 3.6-2, it is sufficient to show that for all $f \in (\ell^p)^*, f(x) = 0$ is satisfied for all $x \in \ell^p$ whenever[9] $f|_{\mathcal{S}_p} = 0$. We start by letting $f \in (\ell^p)^*$. By the description of the dual of ℓ^p, as given in Proposition 1.10, we have a unique representation,

$$f(x) = \sum_{k \in \mathbb{N}} x(k)\gamma(k), \qquad \text{where } \gamma(k) = f(e_k).$$

Now assume for all $x \in \mathcal{S}_p$ that

$$f(x) = \sum_{k \in \mathbb{N}} x(k)\gamma(k) = \sum_{k \in \mathbb{N}} \frac{\gamma(k)}{k^\alpha} = 0$$

As this holds for all $\alpha > \frac{1}{p}$, it follows from the properties of Dirichlet series (see Section 1.5) that $\gamma(k) = 0$ for all $k \in \mathbb{N}$. Therefore,

$$f(x) = \sum_{k \in \mathbb{N}} x(k)\gamma(k) = 0 \text{ as } \gamma(k) = 0 \text{ for all } k \in \mathbb{N}.$$

□

Corollary 1.40. *The subspace defined by* $\operatorname{span}(\mathcal{M}_c^p)$ *is dense in* ℓ^p.

Proof. As $\mathcal{S}_p \subset \mathcal{M}_c^p$, it follows that $\operatorname{span}(\mathcal{M}_c^p)$ is dense in ℓ^p. □

1.4.3 Divisibility

For $d, n \in \mathbb{N}$, we say d is a divisor of n, written as $d \mid n$, if there exists $m \in \mathbb{N}$ such that $n = md$. Furthermore, let (n, m) and $[n, m]$ denote the ***greatest common divisor*** and the ***least common multiple*** of n and m respectively. We say d is a ***unitary divisor*** of n, if d is a divisor of n such that d and n/d are co-prime i.e. $(d, n/d) = 1$.

Proposition 1.41. *For* $n, c, d \in \mathbb{N}$

1. $c, d \mid n \implies [c, d] \mid n$
2. $[c, d](c, d) = cd$
3. $d \mid (c, n) \implies d \mid c$ *and* $d \mid n$

[9] $f|_{\mathcal{S}_p}$ is the restriction of f to the set $\mathcal{S}_p$

4. $(c,d)=n \implies$ *there exists* $l,k\in\mathbb{N}$ *such that* $c=ln$ *and* $d=kn$ *where* $(l,k)=1$.

Proof. See Section 1 in [2]. □

The ***divisor counting function***, $\sum_{d|n} 1$, is denoted by $d(n)$. Note that this is a multiplicative function and as such

$$d(n)=\prod_{1\le i\le k}(\alpha_i+1) \tag{1.3}$$

where $n=p_1^{\alpha_1}\dots p_k^{\alpha_k}$ is the prime decomposition of n. The order of $d(n)$ is also known.

Proposition 1.42. *For every* $\epsilon>0$,

$$d(n)=O(n^\epsilon)$$

Proof. See Theorem 13.12 in [2]. □

In addition, we make use of the divisor series of the Möbius function, $\sum_{d|n}\mu(d)$ which has the following property as proven in Theorem 2.1 in [2]:

Theorem 1.43. *Let* $n\in\mathbb{N}$. *Then,*

$$\sum_{d|n}\mu(d)=\begin{cases}1 & \text{if } n=1\\ 0 & \text{if } n>1.\end{cases}$$

1.4.4 Dirichlet convolution

Definition 1.44. Let $f,g\in\mathcal{A}$. We define their ***Dirichlet convolution*** by

$$(f*g)(n)=\sum_{d|n} f\left(\frac{n}{d}\right)g(d).$$

Observe here that $\mathcal{A}$ is an algebra when equipped with $*$ and the usual point-wise addition and scalar multiplication. We proceed by reviewing some key properties of the Dirichlet convolution.

Proposition 1.45. *Let* $f,g\in\mathcal{A}$. *Then*

1. $f*g=0 \iff f=0$ *or* $g=0$.

2. *The Dirichlet convolution inverse of f exists if and only if $f(1) \neq 0$.*

Proof. See section 2.6 in [2]. $\square$

Proposition 1.46. *Let $f, g \in \mathcal{M}$. Then $f * g \in \mathcal{M}$.*

Proof. See Theorem 2.14 in [2]. $\square$

It is worth noting that $f, g \in \mathcal{M}_c$ does not imply $f*g \in \mathcal{M}_c$. Consider, for example, $f(n) = 1$ for all $n \in \mathbb{N}$, which is certainly in $\mathcal{M}_c$. However, $(f * f)(n) = \sum_{d|n} 1 = d(n)$ which is multiplicative but not completely multiplicative. In addition $f, g \in \mathcal{M}^2$ does not imply $f*g \in \mathcal{M}^2$, see for example, Example 2 in the appendix of [32]. The following theorem is a generalisation of Lemma 2.2 in [33].

Lemma 1.47. *Let f, g, h, j be arithmetical functions belonging to $\mathcal{M}_c^2$. Then,*

$$\langle f * g, h * j \rangle = \langle g, j \rangle \langle f, h \rangle \frac{\langle f, j \rangle \langle g, h \rangle}{\langle fg, hj \rangle} \tag{1.4}$$

Proof. We start by computing the LHS of (1.4):

$$\begin{aligned}
\langle f * g, h * j \rangle &= \sum_{n \in \mathbb{N}} (f * g)(n) \overline{(h * j)(n)} = \sum_{n \in \mathbb{N}} \sum_{c,d|n} f(c) g\left(\frac{n}{c}\right) \overline{h(d) j\left(\frac{n}{d}\right)} \\
&= \sum_{c,d \in \mathbb{N}} \sum_{\substack{n \in \mathbb{N} \\ c,d|n}} f(c) g\left(\frac{n}{c}\right) \overline{h(d) j\left(\frac{n}{d}\right)} = \sum_{c,d \in \mathbb{N}} \sum_{\substack{n \in \mathbb{N} \\ [c,d]|n}} f(c) g\left(\frac{n}{c}\right) \overline{h(d) j\left(\frac{n}{d}\right)},
\end{aligned}$$

since $c, d|n \iff [c,d]|n$ from Proposition 1.41. As $[c,d]|n \iff n = [c,d]m$ for some $m \in \mathbb{N}$,

$$\begin{aligned}
\sum_{c,d \in \mathbb{N}} \sum_{\substack{n \in \mathbb{N} \\ [c,d]|n}} f(c) g\left(\frac{n}{c}\right) \overline{h(d) j\left(\frac{n}{d}\right)} &= \sum_{c,d \in \mathbb{N}} \sum_{m \in \mathbb{N}} f(c) g\left(\frac{m[c,d]}{c}\right) \overline{h(d) j\left(\frac{m[c,d]}{d}\right)} \\
&= \sum_{m \in \mathbb{N}} g(m) \overline{j(m)} \sum_{c,d \in \mathbb{N}} f(c) g\left(\frac{[c,d]}{c}\right) \overline{h(d) j\left(\frac{[c,d]}{d}\right)},
\end{aligned}$$

since the functions are completely multiplicative. From Proposition 1.41, we have

$[c,d](c,d) = cd$ and therefore, the above is equal to

$$\langle g, j\rangle \sum_{c,d\in\mathbb{N}} f(c)g\left(\frac{d}{(c,d)}\right)\overline{h(d)j\left(\frac{c}{(c,d)}\right)} = \langle g, j\rangle \sum_{k\in\mathbb{N}} \sum_{\substack{c,d\in\mathbb{N}\\(c,d)=k}} f(c)g\left(\frac{d}{(c,d)}\right)\overline{h(d)j\left(\frac{c}{(c,d)}\right)}.$$

Note that if $(c,d) = k$, then $c = c'k, d = d'k$ where $(c', d') = 1$. Therefore,

$$\langle f * g, h * j\rangle = \langle g, j\rangle \sum_{k\in\mathbb{N}} \sum_{\substack{c',d'\in\mathbb{N}\\(c',d')=1}} f(c'k)g(d')\overline{h(d'k)j(c')},$$

and, by complete multiplicativity, is equal to

$$\langle g, j\rangle\langle f, h\rangle \sum_{\substack{c',d'\in\mathbb{N}\\(c',d')=1}} f(c')g(d')\overline{h(d')j(c')}. \tag{1.5}$$

We now consider the RHS of (1.4). Observe

$$\begin{aligned}\langle f, j\rangle\langle g, h\rangle &= \sum_{c,d\in\mathbb{N}} f(c)\overline{j(c)}g(d)\overline{h(d)} = \sum_{k\in\mathbb{N}} \sum_{\substack{c,d\in\mathbb{N}\\(c,d)=k}} f(c)\overline{j(c)}g(d)\overline{h(d)}\\ &= \sum_{k\in\mathbb{N}} \sum_{\substack{c',d'\in\mathbb{N}\\(c',d')=1}} f(c'k)\overline{j(c'k)}g(d'k)\overline{h(d'k)}.\end{aligned}$$

By complete multiplicativity, we have

$$\begin{aligned}\sum_{k\in\mathbb{N}} \sum_{\substack{c',d'\in\mathbb{N}\\(c',d')=1}} f(c'k)\overline{j(c'k)}g(d'k)\overline{h(d'k)} &= \sum_{k\in\mathbb{N}} f(k)\overline{j(k)}g(k)\overline{h(k)} \sum_{\substack{c',d'\in\mathbb{N}\\(c',d')=1}} f(c')\overline{j(c')}g(d')\overline{h(d')}\\ &= \langle fg, hj\rangle \sum_{\substack{c',d'\in\mathbb{N}\\(c',d')=1}} f(c')\overline{j(c')}g(d')\overline{h(d')}. \qquad (1.6)\end{aligned}$$

Hence, by comparing (1.5) with (1.6) we obtain (1.4). □

Corollary 1.48. *Let f, g be arithmetical functions belonging to $\mathcal{M}_c^2$. Then,*

$$\|f * g\|_2^2 = \frac{\|g\|_2^2\|f\|_2^2\,|\langle f, g\rangle|^2}{\|fg\|_2^2}$$

Proof. The result follows immediately by taking $h = f$ and $j = g$. $\square$

For more information on arithmetical functions and Dirichlet convolution, see [2], [24] .

1.5 Dirichlet series

Definition 1.49. Let $f \in \mathcal{A}$. For $s \in \mathbb{C}$, a ***Dirichlet series*** is of the form,

$$F(s) = \sum_{n\in\mathbb{N}} \frac{f(n)}{n^s}.$$

Observe that if $\Re s > \sigma$, then $|f(n)n^{-s}| = |f(n)|\, n^{-\Re s} < |f(n)|\, n^{-\sigma}$. Therefore, if $F(s)$ converges absolutely for $\Re s = \sigma$ then $F(s)$ converges absolutely for all s such that $\Re s > \sigma$. This observation leads to the existence of a half-plane of absolute convergence, see Theorem 11.1 in [2].

Theorem 1.50. *There exists σ_a such that $F(s)$ converges absolutely for $\Re s > \sigma_a$ and diverges for $\Re s \leq \sigma_a$. In the case when F converges absolutely everywhere, we define $\sigma_a = -\infty$. Similarly, in the case where F does not converge absolutely anywhere, we define $\sigma_a = \infty$.*

In addition, as given in Theorem 11.8 in [2], there also exists a half-plane of convergence i.e. there exists σ_c such that $F(s)$ converges if $\Re s > \sigma_c$ but does not if $\Re s \leq \sigma_c$.

Proposition 1.51. *Let $f \in \mathcal{A}$ such that $f(1) \neq 0$ and let $g = f^{-1}$, the Dirichlet inverse of f. Then in any half-plane where both series $F(s) = \sum_{n\in\mathbb{N}} f(n)n^s$ and $G(s) = \sum_{n\in\mathbb{N}} g(n)n^s$ converge absolutely, we have $F(s) \neq 0$ and $G(s) = F(s)^{-1}$.*

Proof. See [2] Example 2, page 229. $\square$

Some properties

An absolutely convergent Dirichlet series with $f \in \mathcal{M}$ can be reformulated through Euler products. Namely, if $\sum_{n\in\mathbb{N}} f(n)n^{-s}$ converges absolutely for $\Re s > \sigma_a$ then for $f \in \mathcal{M}$ and for $f \in \mathcal{M}_c$ respectively,

$$F(s) = \prod_{p\in\mathbb{P}} \sum_{k\in\mathbb{N}_0} \frac{f(p^k)}{p^{ks}} \text{ for } \Re s > \sigma_a, \quad \text{and} \quad F(s) = \prod_{p\in\mathbb{P}} \frac{1}{1 - f(p)p^{-s}} \text{ for } \Re s > \sigma_a.$$

Proposition 1.52. *Let two Dirichlet series $F(s)$ and $G(s)$ be absolutely convergent for $\Re s > \sigma_a$. If $F(s) = G(s)$ for all s with sufficiently large $\Re s$, then $f(n) = g(n)$ for all $n \in \mathbb{N}$.*

Proof. See Theorem 11.3 in [2]. $\square$

This above result leads to the existence of a half-plane in which F is never zero.

Proposition 1.53. *Suppose that $F(s) \neq 0$ for some s with $\Re s > \sigma_a$. Then there is a half-plane $\alpha > \sigma_a$ in which $F(s) \neq 0$ for $\Re s \geq \alpha$.*

Proof. See Theorem 11.4 in [2]. $\square$

Furthermore, when $F(s)$ is zero for all s such that $\Re s > \sigma_a$, f must be identically zero.

Proposition 1.54. *If $F(s) = \sum_{n\in\mathbb{N}} \frac{f(n)}{n^s} = 0$ for all s with $\Re s > \sigma_a$, then $f(n) = 0$ for all $n \in \mathbb{N}$.*

Proof. See (3) in Section 17.1 in [24]. $\square$

1.5.1 Riemann zeta function

Perhaps the most notable Dirichlet series is the Riemann zeta function.

Definition 1.55. For $s \in \mathbb{C}$, the ***Riemann zeta*** function is given by

$$\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s}, \text{ for } \Re s > 1.$$

Alternatively, by using Euler products, we can view the Riemann zeta function as a product over primes given by

$$\zeta(s) = \prod_{p\in\mathbb{P}} \frac{1}{1 - p^{-s}}.$$

It is clear that the series $\zeta(s)$ is absolutely convergent in the half-plane $\Re s > 1$ i.e. $\sigma_a = 1$. Furthermore, as described in Chapter 12 in [2], $\zeta(s)$ has an analytic continuation to the whole complex plane apart from a simple pole at $s = 1$. The Riemann zeta function also satisfies a functional equation which reveals that the behaviour of $\zeta(s)$ is symmetric about the line $\Re s = \frac{1}{2}$, known as the critical line. Named after Bernhard Riemann, the function has been heavily studied for over 200 years, and perhaps most notably on this line, as the Riemann hypothesis states that the (non-trivial) zeros of $\zeta(s)$ all lie on it.

For the purposes of our discussions, we are interested in the maximal order of the Riemann zeta function for $1/2 < \Re s < 1$, a region referred to as the critical strip. Let

$$Z_\alpha(T) = \max_{t\in[0,T]} \left|\zeta(\alpha + it)\right|.$$

In [41], the author shows that

$$\log Z_\alpha(T) \geq c_\alpha \frac{(\log T)^{1-\alpha}}{(\log\log T)^\alpha} \text{ for } \frac{1}{2} < \alpha < 1, \tag{1.7}$$

where c_α is an explicit constant dependent on α. In the case when $\alpha = 1/2$, the above holds if the Riemann Hypothesis is assumed to hold. Moreover, the author conjectures, apart from the constant term which may differ, this is the correct order of $\log Z_\alpha(T)$. In the more recent work [38], it is shown that

$$\log Z_\alpha(T) \geq C_\alpha \frac{(\log T)^{1-\alpha}}{(\log\log T)^\alpha} \text{ for } \frac{1}{2} < \alpha < 1,$$

where C_α is improved constant. It is also conjectured that this is in fact the order of $\log Z_\alpha(T)$. There are many texts available on the Riemann zeta function. See for example [2], [24], [36] and [51] to name a few.

Chapter 2

Two linear operators and matrix mappings

In this chapter, we primarily discuss two closely related linear operators, classical Toeplitz operators and multiplicative Toeplitz operators. We explore the relationship between these two classes of operators, exploring how, due to the Bohr lift, multiplicative Toeplitz operators can be seen as Toeplitz operators on the infinite torus. This is followed by a discussion on the literature regarding boundedness, invertibility and spectral theory of these operators from which open questions (that form the focus of Chapters 3 and 4) are established. The chapter is then concluded with the statement of a new result regarding linear operators that preserve multiplicativity.

2.1 Toeplitz operators

As described in the introduction, Toeplitz operators have been extensively studied over many decades, and have a wide range of applications throughout mathematics, engineering, physics, and many other areas. In the field of operator theory, for example, Toeplitz operators have been studied on a variety of spaces such as the Hardy space $\mathcal{H}^p$, the Bergman and the Fock spaces (see [11], [58] and [57]). For the purpose of this discussion, we shall consider Toeplitz operators acting on the Hardy space $\mathcal{H}^2$ and on complex-valued ℓ^p sequence spaces.

Toeplitz operators acting on $\mathcal{H}^2$.

As described in Section 1.3 in [10], let P denote the projection from $\mathcal{L}^2 \to \mathcal{H}^2$ given by

$$P\left(\sum_{n\in\mathbb{Z}} \phi(n)\chi_n\right) = \sum_{n\in\mathbb{N}_0} \phi(n)\chi_n.$$

Definition 2.1. Let $\Phi \in \mathcal{L}^\infty$. The ***Toeplitz operator***, denoted by T_Φ, is the linear mapping from $\mathcal{H}^2 \to \mathcal{H}^2$, which sends X to $P(\Phi X)$. The function Φ is called the ***symbol***.

The corresponding ***Toeplitz matrix***, with respect to the basis $\chi_n(t) = t^n$ for $t \in \mathbb{T}$ and $n \in \mathbb{N}_0$, is characterised by constant diagonals and is of the form

$$\begin{pmatrix} \phi(0) & \phi(-1) & \phi(-2) & \phi(-3) & \cdots \\ \phi(1) & \phi(0) & \phi(-1) & \phi(-2) & \cdots \\ \phi(2) & \phi(1) & \phi(0) & \phi(-1) & \cdots \\ \phi(3) & \phi(2) & \phi(1) & \phi(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.1}$$

Indeed, for $\Phi \in \mathcal{L}^\infty$ and $j \in \mathbb{N}$, we have

$$T_\Phi\chi_j = P(\Phi\chi_j) = P\left(\sum_{n\in\mathbb{Z}} \phi(n)\chi_{n+j}\right) = P\left(\sum_{n\in\mathbb{Z}} \phi(n-j)\chi_n\right) = \sum_{n\in\mathbb{N}_0} \phi(n-j)\chi_n.$$

Therefore, by using the fact that the set of $\chi_n(t)$ forms an orthonormal basis in $\mathcal{H}^2$, the matrix entries are

$$a_{i,j} = \langle T_\Phi\chi_j, \chi_i\rangle = \int_{\mathbb{T}} \sum_{n\in\mathbb{N}_0} \phi(n-j)\chi_n\overline{\chi_i} = \sum_{n\in\mathbb{N}_0} \phi(n-j)\langle\chi_n, \chi_i\rangle = \phi(i-j),$$

as required.

Toeplitz operators acting on ℓ^p.

Equivalently, we can define the Toeplitz operator acting on the sequence space. Note that T_ϕ is used to denote Toeplitz operators acting on sequences, rather than T_Φ which is used to denote Toeplitz operators acting on $\mathcal{H}^2$.

Definition 2.2. Let ϕ be a function from $\mathbb{Z} \to \mathbb{C}$. The ***Toeplitz operator***, denoted by T_ϕ, is the linear mapping from $\mathcal{A}_0 \to \mathcal{A}_0$ which sends $(x_n)_{n\in\mathbb{N}_0}$ to $(y_n)_{n\in\mathbb{N}_0}$ where

$$y_n = \sum_{k\in\mathbb{N}_0} \phi(n-k)x_k. \tag{2.2}$$

For the purposes of this thesis, we are interested in the case when $T_\phi : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ as this is the same operator as $T_\Phi : \mathcal{H}^2 \to \mathcal{H}^2$. To see this, firstly recall that the spaces $\mathcal{H}^2$ and $\ell^2(\mathbb{N}_0)$ are isometrically isomorphic and that the basis $\{\chi_n : n \in \mathbb{N}\}$ corresponds with the basis $\{e_n : n \in \mathbb{N}_0\}$ by (1.1). Secondly, we consider the matrix of T_ϕ with respect to the basis $\{e_n : n \in \mathbb{N}_0\}$,

$$a_{i,j} = \langle T_\phi e_j, e_i\rangle = \sum_{n\in\mathbb{N}_0} \phi(n-j)e_i(n) = \phi(i-j),$$

since,

$$T_\phi e_j = \sum_{k\in\mathbb{N}_0} \phi(n-k)e_j(k) = \phi(n-j).$$

Convolution operators.

Of particular interest will be one class of Toeplitz operators constructed by restricting the support of the symbols. Namely, we only consider Φ analytic, i.e. those functions with Fourier coefficients supported on $\mathbb{N}_0$. Observe that in this case, the projection P has no effect on the operator T_Φ,

$$P(\Phi\chi_j) = P\left(\sum_{n\in\mathbb{N}_0} \phi(n)\chi_{n+j}\right) = \sum_{n\in\mathbb{N}_0} \phi(n)\chi_{n+j}.$$

Therefore, the operator T_Φ is given by the multiplication of ΦX.

Definition 2.3. Let $\Phi \in \mathcal{H}^\infty$. The ***convolution operator***, denoted by C_Φ, is the linear mapping from $\mathcal{H}^2 \to \mathcal{H}^2$, which sends X to ΦX.

Equivalently, for T_ϕ, we consider sequences $\phi(n)$ supported on $\mathbb{N}_0$, which means (2.2) is given by the standard convolution $\sum_{k=0}^n \phi(n-k)x_k$.

Definition 2.4. Let ϕ be a function from $\mathbb{N}_0 \to \mathbb{C}$. The ***convolution operator***, denoted by C_ϕ, is the linear mapping from $\mathcal{A}_0 \to \mathcal{A}_0$ which sends $(x_n)_{n\in\mathbb{N}_0}$ to $(y_n)_{n\in\mathbb{N}_0}$

where

$$y_n = \sum_{k=0}^{n} \phi(n-k) x_k.$$

The matrix representation of C_Φ and C_ϕ is given by the following lower triangular matrix,

$$\begin{pmatrix} \phi(0) & 0 & 0 & 0 & \cdots \\ \phi(1) & \phi(0) & 0 & 0 & \cdots \\ \phi(2) & \phi(1) & \phi(0) & 0 & \cdots \\ \phi(3) & \phi(2) & \phi(1) & \phi(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To see this, consider

$$a_{i,j} = \langle C_\Phi \chi_j, \chi_i \rangle = \int_{\mathbb{T}} \sum_{n \in \mathbb{N}_0} \phi(n) \chi_{n+j} \overline{\chi_i} = \sum_{n \geq j} \phi(n-j) \langle \chi_n, \chi_i \rangle = \phi(i-j).$$

Equivalently,

$$\langle C_\phi e_j, e_i \rangle = \sum_{n \in \mathbb{N}_0} \left(\sum_{k \leq n} \phi(n-k) e_j(k) \right) e_i(n) = \sum_{n \geq j} \phi(n-k) e_i(n) = \phi(i-j).$$

2.2 Multiplicative Toeplitz operators

In this section, we introduce the reader to multiplicative Toeplitz operators. First, we define multiplicative Toeplitz operators acting on $\mathcal{A}$, and secondly over the function space $\mathcal{B}^2_{\mathbb{N}}$.

Multiplicative Toeplitz operators on ℓ^p.

Definition 2.5. Let f be a function from $\mathbb{Q}^+ \to \mathbb{C}$. The ***multiplicative Toeplitz operator***, denoted by M_f, is the linear mapping from $\mathcal{A} \to \mathcal{A}$ which sends $(x_n)_{n \in \mathbb{N}}$ to $(y_n)_{n \in \mathbb{N}}$ where

$$y_n = \sum_{k \in \mathbb{N}} f\left(\frac{n}{k}\right) x_k. \tag{2.3}$$

We are interested in studying M_f as a mapping between the sequence spaces, $\ell^p \to \ell^q$ where $1 \leq p \leq q \leq \infty$, and, in particular, when $p = q = 2$. In this case, the ***multiplicative Toeplitz matrix*** representation, with respect to the basis $(e_n)_{n\in\mathbb{N}}$, is characterised by constants on skewed diagonals and is of the form

$$\begin{pmatrix} f(1) & f(1/2) & f(1/3) & f(1/4) & \cdots \\ f(2) & f(1) & f(2/3) & f(1/2) & \cdots \\ f(3) & f(3/2) & f(1) & f(3/4) & \cdots \\ f(4) & f(2) & f(4/3) & f(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For $j \in \mathbb{N}$, $M_f e_j$ is given by $y_n = \sum_{k\in\mathbb{N}} f(n/k)e_j(k) = f(n/j)$. Hence, the matrix representation of M_f is indeed

$$a_{i,j} = \langle M_f e_j, e_i\rangle = \sum_{n\in\mathbb{N}} f(n/j)e_i(n) = f(i/j).$$

Of notable interest is the class of operators obtained by considering f supported on $\mathbb{N}$ i.e. $f(n/k) = 0$ if k does not divide n. In this case, $y_n = \sum_{k|n} f(n/k)\, x_k = (f * x)(n)$, where $*$ is Dirichlet convolution.

Definition 2.6. Let f be a function from $\mathbb{N} \to \mathbb{C}$. The ***Dirichlet convolution operator***, denoted as D_f, is the linear mapping which sends $(x_n)_{n\in\mathbb{N}} \to (y_n)_{n\in\mathbb{N}}$, where $x \mapsto f * x$; that is,

$$y_n = \sum_{d|n} f\left(\frac{n}{d}\right) x_d.$$

The matrix representation is now given by $a_{i,j} = f(i/j)$ if $j \mid i$ and 0 otherwise. This leads to a lower triangular matrix, which is of the form

$$\begin{pmatrix} f(1) & 0 & 0 & 0 & \cdots \\ f(2) & f(1) & 0 & 0 & \cdots \\ f(3) & 0 & f(1) & 0 & \cdots \\ f(4) & f(2) & 0 & f(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.4}$$

As the name may suggest, the operators M_f and D_f can be thought of as the multiplicative analogue of Toeplitz and convolution operators. In particular, the entries of the corresponding matrices are given by $f(i/j)$ as opposed to those given in (2.2) and (2.1) which have an additive structure $\phi(i-j)$.

Multiplicative Toeplitz operators on $\mathcal{B}^2_{\mathbb{N}}$.

In this thesis, we shall consider the function space $\mathcal{B}^2_{\mathbb{N}}$. Let f be the Dirichlet Fourier coefficients of $F \in \mathcal{B}^2_{\mathbb{N}}$, and recall that the functions $\chi_q(t) = q^{it}$, for $t \in \mathbb{R}$ and $q \in \mathbb{Q}^+$, form an orthonormal basis in $\mathcal{B}^2_{\mathbb{Q}^+}$. We define P to be the projection from $\mathcal{B}^2_{\mathbb{Q}^+} \to \mathcal{B}^2_{\mathbb{N}}$ given by

$$P\left(\sum_{q\in\mathbb{Q}^+} f(q)\chi_q\right) = \sum_{n\in\mathbb{N}} f(n)\chi_n.$$

Definition 2.7. Let $F \in \mathcal{W}_{\mathbb{Q}^+}$. The ***multiplicative Toeplitz operator***, denoted by M_F, is the linear mapping from $\mathcal{B}^2_{\mathbb{N}} \to \mathcal{B}^2_{\mathbb{N}}$ which sends $X \mapsto P(FX)$. We call F the ***symbol*** of M_F.

Note that M_F is used to denote the operator acting on $\mathcal{B}^2_{\mathbb{N}}$ as opposed to M_f which is used in the sequence space case. To see how M_F equates with M_f, first recall that there exists an isometric isomorphism which identifies $X \in \mathcal{B}^2_{\mathbb{N}}$ with the sequence $x \in \ell^2$. In particular, note that $\{\chi_n : n \in \mathbb{N}\}$ corresponds to the basis $\{e_n : n \in \mathbb{N}\}$. Secondly, observe that $M_F : \mathcal{B}^2_{\mathbb{N}} \to \mathcal{B}^2_{\mathbb{N}}$ has the same matrix representation with respect to $(\chi_n)_{n\in\mathbb{N}}$ as the matrix representation of $M_f : \ell^2 \to \ell^2$ with respect to $(e_n)_{n\in\mathbb{N}}$. Indeed, for $F \in \mathcal{W}_{\mathbb{Q}^+}$ and $j \in \mathbb{N}$, we have

$$\begin{aligned} M_F\chi_j = P(F\chi_j) = P\left(\sum_{q\in\mathbb{Q}^+} f(q)\chi_q\chi_j\right) &= P\left(\sum_{q\in\mathbb{Q}^+} f(q)\chi_{jq}\right) \\ = P\left(\sum_{q\in\mathbb{Q}^+} f(q/j)\chi_q\right) &= \sum_{n\in\mathbb{N}} f(n/j)\chi_n. \end{aligned}$$

Therefore, the matrix representation of M_F is given by

$$a_{i,j} = \langle M_F \chi_j, \chi_i \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{n\in\mathbb{N}} f(n/j)\chi_n(t)\overline{\chi_i(t)}dt = \sum_{n\in\mathbb{N}} f(n/j)\langle \chi_n, \chi_i \rangle = f(i/j),$$

if $i, j \in \mathbb{N}$.

Now, note that the projection P has no effect on functions in $\mathcal{B}^2_{\mathbb{N}}$, that is for $F \in \mathcal{B}^2_{\mathbb{N}}$

$$P\left(\sum_{n\in\mathbb{N}} f(n)\chi_n\right) = \sum_{n\in\mathbb{N}} f(n)\chi_n.$$

Therefore, by restricting the symbol of M_F to functions in $\mathcal{W}_{\mathbb{N}}$, we obtain a multiplication operator leading to the following definition.

Definition 2.8. Let $F \in \mathcal{W}_{\mathbb{N}}$. The ***Dirichlet convolution operator***, denoted by D_F, is the linear mapping from $\mathcal{B}^2_{\mathbb{N}} \to \mathcal{B}^2_{\mathbb{N}}$, which sends X to XF.

This mapping is the same as $D_f : \ell^2 \to \ell^2$, as the entries of the matrix representation of $D_F : \mathcal{B}^2_{\mathbb{N}} \to \mathcal{B}^2_{\mathbb{N}}$ are equal to those given in (2.4). This follows from computing

$$\langle D_F \chi_j, \chi_i \rangle = \sum_{j|n} f(n/j)\langle \chi_n, \chi_i \rangle = \begin{cases} f(i/j) \text{ if } j/i, \\ 0 \text{ otherwise,} \end{cases}$$

as

$$D_F \chi_j = F\chi_j = \sum_{n\in\mathbb{N}} f(n)\chi_n\chi_j = \sum_{n\in\mathbb{N}} f(n)\chi_{jn} = \sum_{j|n} f(n/j)\chi_n.$$

2.3 From Toeplitz operators to multiplicative Toeplitz operators; the Bohr lift

We start this section by making the observation that the mapping M_F is defined by the same operation as T_Φ but over a different function space. As such, one can indeed think of M_F and D_F as the multiplicative versions of the classical T_Φ and C_Φ, and view the space $\mathcal{B}^2_{\mathbb{N}}$ as the multiplicative analogue of $\mathcal{H}^2$.

In this section, we explore the connection between the two classes of operators further by considering how, in fact, multiplicative Toeplitz operators provide a generalisation of Toeplitz operators and, moreover, that they are Toeplitz operators acting on an infinite number of variables.

Generalisation of classical Toeplitz operators.

Assume that the Dirichlet Fourier coefficients of a symbol F are supported only on powers of a fixed prime p i.e. $f(q) = 0$ if $q \neq p^k$ for $k \in \mathbb{Z}$. Then

$$F(t) = \sum_{k\in\mathbb{Z}} f(p^k)p^{kit} = \sum_{k\in\mathbb{Z}} f(p^k)(e^{it\log p})^k$$

By defining $\phi(n) = f(p^n)$ and writing $\theta = t\log p$, the above can be written as a classical Fourier series as follows

$$\Phi(\theta) = \sum_{k\in\mathbb{Z}} \phi(k)e^{ik\theta} \tag{2.5}$$

Therefore, any Fourier series can be associated with a Dirichlet Fourier series whose coefficients are supported on the powers of a prime. Moreover, Toeplitz operators are unitarily equivalent to multiplicative Toeplitz operators through the mapping $\tau : \ell^2(\mathbb{N}_0) \to \ell^2$, where, for $i \in \mathbb{Z}$

$$\phi(i) \longmapsto \begin{cases} \phi(p^i) \\ 0 \text{ otherwise,} \end{cases}$$

where now we can write $M_F = \tau T_\Phi \tau^{-1}$.

The Bohr lift.

The two operators are in fact much more connected than a reduction from multiplicative to additive Toeplitz operators. Harald Bohr made the inspired observation in [8] that one could utilise the fundamental theorem of arithmetic to write any Dirichlet series as a function acting upon infinitely many variables. In particular, let $p = (p_k)_{k\in\mathbb{N}}$ denote the sequence such that p_k is the k-th prime, and define

$$\begin{aligned} \mathbb{N}_0^\infty &:= \{\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \dots \text{ s.t. } \exists\, K \in \mathbb{N} \text{ with } \alpha_k = 0, \forall k \geq K\} \\ \mathbb{Z}^\infty &:= \{\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{Z} \times \mathbb{Z} \times \dots \text{ s.t. } \exists\, K \in \mathbb{Z} \text{ with } \alpha_k = 0, \forall\, |k| \geq K\} \end{aligned}$$

Now for any $n \in \mathbb{N}$, we may write

$$n = p^\alpha = \prod_{k\in\mathbb{N}} p_k^{\alpha_k}, \text{ where } \alpha \in \mathbb{N}_0^\infty.$$

With this notation, it follows that

$$\sum_{n\in\mathbb{N}} f(n)n^{-s} = \sum_{\alpha\in\mathbb{N}_0^\infty} f_p(\alpha)p^{-\alpha s},$$

where f_p is the corresponding coefficient to f.

The process described above is referred to as the ***Bohr lift***. Now consider the two numbers, $n, m \in \mathbb{N}$; Bohr's lift allows us to write $n = p^\alpha$ and $m = p^\beta$, where $\alpha, \beta \in \mathbb{N}_0^\infty$. It follows that

$$\frac{n}{m} = \frac{p^\alpha}{p^\beta} = p^{\alpha-\beta}.$$

Therefore, by applying the Bohr lift to (2.3), we obtain

$$y_n = y_{p^\alpha} = \sum_{\beta\in\mathbb{N}_0^\infty} \phi\left(p^{\alpha-\beta}\right) x_{p^\beta} = \sum_{\beta\in\mathbb{N}_0^\infty} \phi_p\left(\alpha-\beta\right) x_\beta,$$

where now $\phi_p : \mathbb{Z}^\infty \to \mathbb{C}$ and $x \in \ell^p(\mathbb{N}_0^\infty)$. This is of the form of a Toeplitz operator, and therefore one can transform a multiplicative Toeplitz operator into a Toeplitz operator acting upon infinitely many variables. The discussion given in the previous section is in fact the Bohr lift on a just a single prime i.e. $\alpha_k \neq 0$ for only one $k \in N$.

The spaces on the infinite torus and the space of Dirichlet series.

Let $\mathbb{T}^\infty := \{t = (z_1, z_2, \dots) \in \mathbb{T} \times \mathbb{T} \times \dots\}$. We call $\mathbb{T}^\infty$ the ***infinite torus***. Occasionally, we will refer to the n***-dimensional torus***, by which we mean
$\mathbb{T}^n := \{t = (z_1, \dots, z_n) \in \mathbb{T} \times \cdots \times \mathbb{T}\}$.

Definition 2.9. For $t \in \mathbb{T}^\infty$, let

$$\Phi(t) = \sum_{\alpha\in\mathbb{N}_0^\infty} \phi(\alpha)t^\alpha.$$

We define the ***Hardy space of the infinite torus*** by

$$\mathcal{H}^2(\mathbb{T}^\infty) = \left\{ \Phi : \sum_{\alpha \in \mathbb{N}_0^\infty} |\phi(\alpha)|^2 < \infty \right\}.$$

We can also consider those functions with absolutely convergent coefficients.

Definition 2.10. The ***Wiener algebra on the infinite torus*** is defined as

$$W(\mathbb{T}^\infty) = \left\{ \Phi : \sum_{\alpha \in \mathbb{N}_0^\infty} |\phi(\alpha)| < \infty \right\}.$$

The seminal publication [25] has sparked a surge in the study of these spaces. Within the article, the authors introduce the space $\mathcal{D}^2$.

Definition 2.11. Let $\mathcal{D}^2$ denote the space of Dirichlet series for which

$$\sum_{n \in \mathbb{N}} |f(n)|^2 < \infty.$$

We observe here that functions in $\mathcal{D}^2$ are analytic in the half plane $\Re s > 1/2$. Indeed, by the Cauchy-Schwarz inequality, we have

$$\sum_{n \in \mathbb{N}} \left| \frac{f(n)}{n^s} \right| \leq \left(\sum_{n \in \mathbb{N}} |f(n)|^2 \right)^{1/2} \left(\sum_{n \in \mathbb{N}} \frac{1}{n^{2\Re s}} \right)^{1/2},$$

which converges absolutely for all $\Re s > 1/2$. The recent increase in interest is primarily due to the observation that by utilising the Bohr lift, the space of $\mathcal{D}^2$ can be seen as the same as $\mathcal{H}^2(\mathbb{T}^\infty)$. This allows Dirichlet series to be studied in a new way, through techniques and concepts from functional and harmonic analysis, and vice versa.

In addition, we note that there also exists an isometric isomorphism τ, between $\mathcal{B}^2_{\mathbb{N}}$ and $\mathcal{D}^2$, given by

$$F(t) \sim \sum_{n \in \mathbb{N}} f(n) n^{it} \xrightarrow{\tau} \tilde{F}(s) = \sum_{n \in \mathbb{N}} f(n) n^{-s}. \tag{2.6}$$

Another key finding from [25], which we shall require later within this thesis, concerns the set of multipliers of $\mathcal{D}^2$.

Theorem 2.12. *Let $F(s)$ be a function which is analytic in the half-plane $\Re s > 1/2$. If $FG \in \mathcal{D}^2$ for all $G \in \mathcal{D}^2$, then*

$$F(s) = \sum_{n\in\mathbb{N}} f(n)n^{-s},$$

is a convergent Dirichlet series for $\Re s > \sigma$ for some $\sigma > 0$, which extends to a bounded analytic function of $\mathbb{C}_+$.

Proof. See Theorem 3.1 in [25]. □

We refer the reader to [46] and [48] for an overview of the field including an exploration of these connections, open questions and, most relevant for this thesis, the operators which act upon these spaces. One such example is the Hankel operator. A ***Hankel matrix*** is of the form

$$\begin{pmatrix} \phi(0) & \phi(1) & \phi(2) & \phi(3) & \cdots \\ \phi(1) & \phi(2) & \phi(3) & \phi(4) & \cdots \\ \phi(2) & \phi(3) & \phi(4) & \phi(5) & \cdots \\ \phi(3) & \phi(4) & \phi(5) & \phi(6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Definition 2.13. Let $\phi : \mathbb{N}_0 \to \mathbb{C}$. The ***Hankel operator***, denoted by H_ϕ, is the linear mapping from $\mathcal{A}_0 \to \mathcal{A}_0$ which sends $(x_n)_{n\in\mathbb{N}_0}$ to $(y_n)_{n\in\mathbb{N}_0}$ where

$$y_n = \sum_{k\in\mathbb{N}_0} \phi(n+k)x(k)$$

As described in Section 2.10 in [11], one can equivalently define the Hankel operator on the Hardy space. Let Q be the operator defined by $I - P$ and let J denote the operator given by $(JX)(t) = \frac{1}{t}X\left(\frac{1}{t}\right)$ for $t \in \mathbb{T}$.

Definition 2.14. Let $\Phi \in \mathcal{L}^\infty$. The ***Hankel operator***, denoted by H_Φ, is the mapping from $\mathcal{H}^2 \to \mathcal{H}^2$ which sends $X \mapsto P(\Phi\hat{X})$, where $\hat{X} = QJX$.

Much like we explored for Toeplitz operators, one can consider multiplicative Hankel operators.

Definition 2.15. Let $\phi : \mathbb{N}_0 \to \mathbb{C}$. The ***multiplicative Hankel operator***, denoted by H_ϕ, is the linear mapping from $\mathcal{A} \to \mathcal{A}$ which sends $(x_n)_{n\in\mathbb{N}}$ to $(y_n)_{n\in\mathbb{N}}$ where

$$y_n = \sum_{k\in\mathbb{N}_0} \phi(nk)x(k) \tag{2.7}$$

Observe that by the Bohr lift, we can write (2.7) as

$$y_n = y_{p^\alpha} = \sum_{\beta\in\mathbb{N}_0^\infty} \phi\left(p^{\alpha+\beta}\right) x_{p^\beta} = \sum_{\beta\in\mathbb{N}_0^\infty} \phi_p\left(\alpha+\beta\right) x_\beta,$$

which is a Hankel operator acting on infinitely many variables. Sometimes referred to as Helson operators, multiplicative Hankel operators were initially considered in [26],[28], and [27]. Several publications followed, and considered key operator properties such as compactness, boundedness, spectrum and eigenvalues, [46], [9], [12], [13], [14], [40], [43], and [45] to name a few. Another notable operator is the composition operator, see for example [21], [47], [39], [5], [6]. In this series of papers, properties of the operator such as invertibility, compactness, boundedness and normality are considered.

Through the Bohr lift, our investigation within this thesis into multiplicative Toeplitz operators and Dirichlet convolution operators will also yield research into Toeplitz operators and the multiplication operator acting on $\mathcal{H}^2(\mathbb{T}^\infty)$. Many of the publications listed above highlight how the mathematical tools fail when moving from single dimensions (i.e. the classical case) to infinite dimensions. We explore in Section 2.4 how some well-known results for Toeplitz operator fail even when considering the two dimensional case, i.e. on the space $\mathcal{H}^2(\mathbb{T}^2)$.

Tensor products.

In the special case where f is multiplicative, we note that the function f can be written as a product over prime powers. As such, it is also possible to formulate multiplicative Toeplitz operators as products of Toeplitz operators in this case. The following example highlights this on two primes.

Example 2.16. Let $f \in \mathcal{A}$ be such that $f(1) = f(2) = f(3) = f(6) = 1$ and 0 otherwise, which is a multiplicative sequence. Then $F(t) = 1 + 2^{it} + 3^{it} + 6^{it} = (1 + 2^{it})(1 + 3^{it})$. By letting Φ and Γ be of the form given in (2.5) for $p = 2$ and $p = 3$ respectively, we have $F = \Phi\Gamma$. Therefore, we can write M_F as a Toeplitz operator acting on two dimensions where symbols are given by Φ and Γ.

This is further explored in [35], where f is assumed to be multiplicative on the positive rationals. In this case, the author shows that a multiplicative Toeplitz matrix[10] can be written as an infinite tensor product of Toeplitz matrices.

Namely, let $F_p(t) = \sum_{k\in\mathbb{Z}} f(p^k)p^{kit} \in \mathcal{W}_{\mathbb{Q}^+}$ and let $\Phi_p : \mathbb{T} \to \mathbb{C}$ denote the function $\Phi_p(e^{i\theta}) = F_p\left(\frac{\theta}{\log p}\right)$. Observe that

$$F_p\left(\frac{\theta}{\log p}\right) = \sum_{k\in\mathbb{Z}} f(p^k)p^{\frac{ik\theta}{\log p}} = \sum_{k\in\mathbb{Z}} f(p^k)e^{ik\theta},$$

i.e. $\Phi_p \in \mathcal{W}$. As such, the matrix representation of M_{F_p} is the same as that of T_{Φ_p}. Namely, $a_{i,j} = f(p^{i-j})$ for $i, j \in \mathbb{N}_0$. From this the author shows that, for $f \in \ell^1(\mathbb{Q}^+)$ such that f is multiplicative,

$$M_F = \bigotimes_{p\in\mathbb{P}} T_{\Phi_p},$$

where $\bigotimes$ denotes the tensor product.

2.4 Literature Review

In its infancy in comparison to classical Toeplitz operators, the field of multiplicative versions of well-known operators is a growing area. In recent years, research into multiplicative Toeplitz operators has been fueled by connections to number theory and in particular to the Riemann zeta function, see for example [20], [32], [30], [31], [33] and [50].

In this section, we shall review known literature regarding multiplicative Toeplitz operators, exploring the fascinating connection between these operators with multiplicative number theory. We also state theorems on Toeplitz operators on $\mathcal{H}^2$ in both higher and infinite dimensions. These discussions lead to two topics of interest and open questions which form the primary focus of Chapters 3 and 4.

[10] Recall a multiplicative Toeplitz matrix is given by $a_{i,j} = f(i/j)$

2.4.1 Boundedness

Boundedness of Toeplitz operators

We start our discussion by considering the following theorem attributed to Otto Toeplitz, after who the operators are named. The theorem was also independently proved in [15] some decades later.

Theorem 2.17. *(Toeplitz) The Toeplitz matrix generates a bounded operator on ℓ^2 if and only if there is a function in $\Phi \in L^\infty$ whose sequence of Fourier coefficients is the sequence $\phi(n)$.*

Proof. See footnote in [52]. □

The boundedness of Toeplitz operators is also well understood in the case $T_\Phi : \mathcal{H}^p \to \mathcal{H}^p$, where $p \in (1, \infty)$.

Theorem 2.18. *Let $p \in (1\infty)$. The operator $T_\Phi : \mathcal{H}^p \to \mathcal{H}^p$ is bounded if and only if $\Phi \in L^\infty$. Moreover,*

$$\|T_\Phi\| = \|\Phi\|_\infty.$$

However, it is difficult to establish when T_ϕ is bounded on ℓ^p, and in general, no sufficient and necessary condition is known. The boundedness of Toeplitz operators on weighted sequence spaces ℓ^p_μ is considered in Chapter 6 of [11]. Let ℓ^p_μ denote the space of $x \in \mathcal{A}$ such that

$$\|x\|_{\ell^p_\mu} = \left(\sum_{n \in \mathbb{N}} (n+1)^{p\mu} |a_n|^p \right)^{\frac{1}{p}} < \infty.$$

Note that the usual ℓ^p space we defined in Chapter 1 is contained with the weighted space by letting $\mu = 0$. In [11], the authors state that if f lies within a set of multipliers then $T_\phi : \ell^p \to \ell^p$ is bounded. It remains an open question if this is also a necessary condition.

Boundedness of multiplicative Toeplitz operators.

As well as the additive operator, Toeplitz, in [53], also considered matrices with multiplicative form. Namely, the author investigates properties, such as the limiting behaviour and asymptotic growth, of the $n \times n$ truncation of the matrix $a_{i,j} = f(j/i)$ if $i \mid j$ and 0 otherwise. That is, an upper triangular matrix of the form

$$\begin{pmatrix} f(1) & f(2) & f(3) & f(4) & f(5) & 0 & \cdots \\ 0 & f(1) & 0 & f(2) & 0 & f(3) & \cdots \\ 0 & 0 & f(1) & 0 & 0 & f(2) & \cdots \\ 0 & 0 & 0 & f(1) & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

More recent works have considered the boundedness of M_f and D_f. We state a result from [32] which examines the boundedness of $D_f : \ell^p \to \ell^q$ for $f(n) = n^{-\alpha}$, $\alpha > 0$, which we shall denote as D_α.

Theorem 2.19. *Let $1 \leq p \leq q \leq \infty$. Define $r \in [1, \infty]$ such that $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$ where $\frac{1}{\infty} = 0$. If $\alpha > \frac{1}{r}$ then D_α is bounded between $\ell^p \to \ell^q$. Moreover, if $p = 1$ with any q, or $q = \infty$ with any p, or $p = q$, then D_α is bounded if and only if $\alpha > \frac{1}{r}$, in which case*

$$\|D_\alpha\|_{p,q} = \zeta(\alpha r)^{\frac{1}{r}}.$$

Proof. See Theorem 1.1 in [32]. □

The boundedness of $D_f : \ell^p \to \ell^p$ has also been considered. The following theorem, which is given in [20], utilises smooth numbers, a number which has small prime factors[11].

Theorem 2.20. *If $f \in \ell^1$ is such that $f = gh$, where $g \in \mathcal{M}_c$ and $h \in \mathcal{A}$ is non-negative. Then $\|D_f\|_{p,p} = \|f\|_1$.*

Proof. See Theorem 1 in [20]. □

More generally, the boundedness of M_f is examined in [30], though only on ℓ^2.

Theorem 2.21. *Let $f \in \ell^1(\mathbb{Q}^+)$, then $M_f : \ell^2 \to \ell^2$ is bounded and the operator norm is given by*

$$\|M_f\|_{2,2} = \sup_{t \in \mathbb{R}} \left| \sum_{q \in \mathbb{Q}^+} f(q) q^{it} \right|. \tag{2.8}$$

[11] We say a positive integer is n-smooth if its prime factors are all less than or equal to n.

Proof. See Theorem 3.1 in [30] □

Unlike Theorems 2.19 and 2.20, there is no condition on the positivity of f in the above theorem. Observe by taking $g(n) = 1$ for all $n \in \mathbb{N}$ in Theorem 2.20, we obtain the operator norm of D_f for any non-negative f. Additionally, in Theorem 2.19, $f(n) = n^{-\alpha}$ is, of course, positive for all $n \in \mathbb{N}$. An example of how the operator norm differs for f not strictly positive, is given in [20] as follows. Let $f(1) = 1, f(2) = -1, f(4) = -1$ and $f(n) = 0$ otherwise. Then $\|D_f\|_{2,2} = \sqrt{5}$, whereas $\|f\|_1 = 3$. Theorem 2.21 removes any restriction on f, and reveals a more general operator norm. By assuming $f : \mathbb{N} \to \mathbb{C}$ is non-negative, the supremum of (2.8) is attained when $t = 0$ and so the operator norm coincides with that given in Theorem 2.19 and Theorem 2.20.

Although the boundedness of M_f has been discussed in existing literature, there is not a fully known criterion for boundedness. Even the boundedness of D_α is not fully understood.

2.4.2 Spectral theory, invertibility and factorisation.

Spectrum of Toeplitz Operators

The spectrum of Toeplitz operators has been extensively studied over many spaces, however describing the spectrum of T_Φ is a challenging problem and is unknown for general symbols as discussed in [3] and [10].

As previously mentioned in Section 1.3.1, the spectrum of compact operators can be characterised. However, as stated in Theorem 4.2.4 in [3], T_Φ is never compact except in the trivial case when $\Phi = 0$. The main difficulty of computing $\sigma(T_\Phi)$ stems from the fact that the product of two Toeplitz operators, $T_\Phi T_\Gamma$, is not in general equal to another Toeplitz operator. In particular, $T_{\Phi\Gamma} - T_\Phi T_\Gamma$ is not zero. This operator is called the ***semi-commutator***. For analytic symbols, so T_Φ is given by C_Φ, observe that

$$C_\Phi C_\Gamma X = C_\Phi \Gamma X = \Phi \Gamma X = C_{\Phi\Gamma} X.$$

Hence, in this case, the semi-commutator is zero. As a result, the spectrum of C_Φ is known.

Theorem 2.22. *Let $\Phi \in \mathcal{H}^\infty$. Then $\sigma(C_\Phi) = \overline{\tilde{\Phi}(\mathbb{D})}$, where $\mathbb{D}$ is the unit disc, and $\tilde{\Phi}$ is the harmonic extension of Φ to the unit disc.*

Proof. See Theorem 7 in [54]. $\square$

To establish the spectrum of T_Φ where the symbol is not analytic, more work is required. As shown in Proposition 1.13 in [10], if $\Phi \in \overline{\mathcal{H}^\infty}$, $\Gamma \in \mathcal{L}^\infty$ and $\Psi \in \mathcal{H}^\infty$, then

$$T_{\Phi\Gamma\Psi} = T_\Phi T_\Gamma T_\Psi. \tag{2.9}$$

Moreover, let $\Phi_- \in \overline{\mathcal{H}^\infty}$ and $\Phi_+ \in \mathcal{H}^\infty$ be invertible. If $\Phi = \Phi_-\Phi_+$, then $T_\Phi = T_{\Phi_-\Phi_+}$ is invertible since T_{Φ_-} and T_{Φ_+} are invertible as, by[12] (2.9),

$$T_{\Phi_+}T_{\Phi_+^{-1}} = T_{\Phi_+\Phi_+^{-1}} = I = T_{\Phi_+^{-1}\Phi_+} = T_{\Phi_+^{-1}}T_{\Phi_+},$$

and similarly for Φ_-. We ask therefore which symbols in $\mathcal{L}^\infty$ can be factorised in terms of analytic and anti-analytic functions. The proceeding theorem, referred to as the Wiener-Hopf factorisation, states that many symbols in the Wiener algebra can in fact be factorised in this way. Before we state the theorem, we require a few definitions. Let $\mathcal{W}_+$ and $\mathcal{W}_-$ denote the subspaces of $\mathcal{H}^\infty$ and $\overline{\mathcal{H}^\infty}$ for which $\sum_{n=1}^\infty |\phi(n)| < \infty$. For Φ continuous[13] on $\mathbb{T}$, such that $\Phi(t) \neq 0$ for all $t \in \mathbb{T}$, the ***winding number*** of Φ around the point $a \in \mathbb{T}$ is defined by

$$\text{wind}(\Phi, a) = \frac{\theta(1) - \theta(0)}{2\pi},$$

where $a + r(t)e^{i\theta(t)}$ is the polar coordinates system of Φ. One can think of the winding number as the total number of times that a closed curve winds around a given point.

Theorem 2.23 (Wiener-Hopf factorisation)**.** *Let $\Phi \in \mathcal{W}$ such that $\Phi(t) \neq 0$ for all $t \in \mathbb{R}$ and $\text{wind}(\Phi, 0) = k$. Then, there exists $\Phi_- \in \mathcal{W}_-$, and $\Phi_+ \in \mathcal{W}_+$ such that*

$$\Phi = \Phi_-\chi_k\Phi_+.$$

Proof. See Theorem 1.14 in [10]. $\square$

The Wiener-Hopf factorisation leads to the description of the Fredholmness and index of Toeplitz operators with symbols in the Wiener algebra.

[12]Note that the inverse of a function in $\mathcal{H}^\infty$ need not be in $\mathcal{H}^\infty$ and similarly for functions in $\overline{\mathcal{H}^\infty}$

[13]Observe that symbols in the Wiener algebra are continuous, and so the winding number can be defined for $\Phi \in \mathcal{W}$.

Theorem 2.24. *Let $\Phi \in \mathcal{W}$. Then T_Φ is Fredholm $\iff$ $\Phi \in \mathcal{W}$ such that $\Phi(t) \neq 0$ for all $t \in \mathbb{R}$. In that case,* $\operatorname{ind}(T_\Phi) = -\operatorname{wind}(\Phi, 0)$.

Proof. See Theorem 1.15 in [10]. $\square$

From this, we have that $\sigma_{\mathrm{e}}(T_\Phi) = \Phi(\mathbb{T})$, since $T_\Phi - \lambda I = T_{\Phi-\lambda}$ is Fredholm if and only if $\Phi - \lambda = 0$ i.e. $\lambda \neq \Phi(t)$ for any $t \in \mathbb{T}$. One key result, known as Coburn's lemma which we state below, connects the problem of invertibility to that of Fredholmness. Note that the adjoint of T_Φ is given by $T_{\overline{\Phi}}$. This follows from writing[14] $T_\Phi = P\Phi = P\Phi P$. By Proposition 1.26,

$$(P\Phi P)^* = (\Phi P)^* P^* = P^*\Phi^* P^* = P(\Phi)^* P = P\Phi^*.$$

Thus, we consider the adjoint of multiplication by[15] Φ,

$$\langle \Phi X, Y\rangle = \int_{\mathbb{T}} \Phi X\overline{Y} = \int_{\mathbb{T}} X\overline{\overline{\Phi}Y} = \langle X, \overline{\Phi}Y\rangle.$$

Theorem 2.25 (Coburn's Lemma)**.** *For $\Phi \in \mathcal{L}^\infty$ not identically zero, either*

$$\ker(T_\Phi) = \{0\} \;\; \textit{or} \;\; \ker(T_{\overline{\Phi}}) = \{0\}.$$

Proof. See [18] $\square$

We have already seen in Section 1.3.2, how invertible operators are Fredholm operators with index zero. Now suppose T_Φ is Fredholm with index zero. Then, $\dim\ker(T_\Phi) = \dim\operatorname{co}\ker(T_\Phi)$ where both are finite. By Coburn's lemma, it follows that $\ker(T_\Phi) = \{0\}$ and $\operatorname{im}(T_\Phi) = \mathcal{H}^2$, i.e. T_Φ is invertible. This leads to the following result.

Theorem 2.26. *Let $\Phi \in \mathcal{W}$. Then T_Φ is invertible $\iff$ T_Φ is Fredholm of index zero. Therefore,*

$$\sigma(T_\Phi) = \sigma_e(T_\Phi) \cup \{\lambda \in \mathbb{C} \setminus \sigma_e(T_\Phi) : \operatorname{ind}(T_\Phi - \lambda I) \neq 0\}.$$

Proof. See Theorem 1.15 in [10] $\square$

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[14]By this we mean, $T_\Phi X = P(\Phi X)$. It is, however, convenient to consider solely the operators in this case. Therefore, we briefly use Φ to denote the multiplication by Φ.

[15]We now revert back to Φ as a function.

From Theorem 2.24, the spectrum of T_Φ can be formulated.

Theorem 2.27. *Let $\Phi \in \mathcal{W}$. Then*

$$\sigma\left(T_\Phi\right) = \Phi(\mathbb{T}) \cup \left\{\lambda \in \mathbb{C} \setminus \Phi(\mathbb{T}) : \mathrm{wind}(\Phi - \lambda, 0) \neq 0\right\}.$$

Coburn's Lemma allows for the description of the spectrum of more general symbols. Recall, from Section 1.3.2, that an operator is Fredholm if and only if it is invertible modulo compact operators. Therefore, one can ask when $T_\Phi T_\Gamma = I + K$, where K is a compact operator. The answer comes in the form of the semi-commutator.

Theorem 2.28. *If $\Phi, \Gamma \in \mathcal{L}^\infty$, then*

$$T_{\Phi\Gamma} - T_\Phi T_\Gamma = H_\Phi H_{\tilde{\Gamma}},$$

where H is the Hankel operator, and $\tilde{\Gamma}(t) = \Gamma\left(\frac{1}{t}\right)$.

Proof. See Proposition 2.14 in [11]. $\square$

For continuous symbols, H is a compact operator, see Theorem 1.16 in [10]. Therefore, for some compact operator K, $T_\Phi T_{\Phi^{-1}} = I + K$ and $T_{\Phi^{-1}} T_\Phi = I + K$ i.e T_Φ is a Fredholm operator. As a result, the essential spectrum and index of T_Φ is described. Moreover, by applying Coburn's Lemma, the spectrum emerges.

Theorem 2.29. *For Φ continuous on $\mathbb{T}$, $T_\Phi - \lambda I$ is Fredholm $\iff$ $\lambda \notin \Phi(\mathbb{T})$. Moreover, when $T_\Phi - \lambda I$ is Fredholm, we have*

$$\mathrm{ind}(T_\Phi - \lambda I) = -\mathrm{wind}(\Phi, \lambda),$$

Furthermore, for Φ continuous on $\mathbb{T}$, we have that

$$\sigma\left(T_\Phi\right) = \Phi(\mathbb{T}) \cup \left\{\lambda \in \mathbb{C} \setminus \Phi(\mathbb{T}) : \mathrm{wind}(\Phi, \lambda) \neq 0\right\}.$$

Proof. See Theorem 1.17 in [10]. $\square$

Spectrum of multiplicative Toeplitz operators

In comparison, literature surrounding the spectral properties of multiplicative Toeplitz operators is sparse.

First, an analogous factorisation of (2.9) was given in [30].

Theorem 2.30. *Let $F \in \mathcal{W}_{\overline{\mathbb{N}}}, G \in \mathcal{W}_{\mathbb{Q}^+}$ and $H \in \mathcal{W}_{\mathbb{N}}$. Then,*

$$M_{FGH} = M_F M_G M_H.$$

Proof. See Theorem 3.3 in [30]. □

Corollary 2.31. *If $F \in \mathcal{W}_{\overline{\mathbb{N}}}$ and $G \in \mathcal{W}_{\mathbb{Q}^+}$, or if $F \in \mathcal{W}_{\mathbb{Q}^+}$ and $G \in \mathcal{W}_{\mathbb{N}}$, then $M_F M_G = M_{FG}$.*

Proof. Take $H = 1$ and $F = 1$ respectively in Theorem 2.30. □

Secondly, in [30] the invertibility of M_F is characterised when the symbol is ***factorisable*** by which we mean

$$F = F_- \chi_q F_+,$$

where $F_- \in \mathcal{W}_{\overline{\mathbb{N}}}$ and $F_+ \in \mathcal{W}_{\mathbb{N}}$ are invertible and $\chi_q(t) = q^{it}$ for $t \in \mathbb{R}$ and $q \in \mathbb{Q}^+$. We shall denote the set of factorisable $F \in \mathcal{W}_{\mathbb{Q}^+}$ by $\mathcal{F}\mathcal{W}_{\mathbb{Q}^+}$. Let $w(F)$ denote the ***average winding number*** of F which is defined by

$$w(F) = \lim_{T \to \infty} \frac{\theta(T) - \theta(-T)}{2T},$$

where $\theta(T)$ is the argument of $F(T)$ in polar coordinates.

Theorem 2.32. *Let $F \in \mathcal{F}\mathcal{W}_{\mathbb{Q}^+}$. Then M_F is invertible $\iff$ $w(F) = 0$.*

Proof. See Theorem 3.4 in [30]. □

Moreover, symbols whose Dirichlet Fourier coefficients are multiplicative (see Definition 1.37) are also factorisable. By above, it then follows that the invertibility of M_F is also known.

Theorem 2.33. *Let $F \in \mathcal{W}_{\mathbb{Q}^+}$ with multiplicative coefficients i.e. f is multiplicative on $\mathbb{Q}^+$. If F has no zeros and $w(F) = 0$ then M_F is invertible.*

Proof. Theorem 3.5 in [30]. □

However, given a general $F \in \mathcal{W}_{\mathbb{Q}^+}$, it remains an open question to establish when F is factorisable.

Spectrum of Toeplitz operators on higher dimensions

We now turn our attention to the spectral properties of Toeplitz operators on higher dimensions. Echoing the literature surrounding operators on the infinite torus, there are significant differences between the classical case and Toeplitz operators on higher dimensions.

In [22] for example, the product of Toeplitz operators on the Hardy space of the n-dimensional disc $\mathbb{D}^n$ is studied. The authors in [22] show that for T_Φ acting on $\mathcal{H}^2(\mathbb{D}^n)$,

$$T_\Phi T_\Gamma = 0 \text{ if and only if } \Phi = 0 \text{ or } \Gamma = 0.$$

Moreover, a criterion for when the product of two Toeplitz operators is itself a Toeplitz operator, is given.

The semi-commutator of Toeplitz operators on the two-dimensional case is discussed in [23]. Although not compact in general, the authors find a necessary condition for the semi-commutator to be compact. By doing so, they show that for a large class of symbols, the semi-commutator is compact if and only if it is zero. Recall this contrasts with Theorem 2.28 in the one dimensional case for which there are symbols that yield a non-zero, compact, semi-commutator. It remains an open problem to find which, if any, symbols yield a compact semi-commutator for Toeplitz operators on infinite dimensions.

In [17], Coburn's Lemma in two dimensions is also considered. The authors show how, in contrast to the classical case, Coburn's Lemma fails in general on the bidisc by finding a symbol for which simultaneously

$$\ker T_\Phi \neq \{0\} \text{ and } \ker T_\Phi^* \neq \{0\}.$$

However, they do prove a Coburn's Lemma type theorem for Toeplitz operators with particular symbols. We refer the reader to [11] for further discussions and details on the two dimensional case.

2.4.3 Application in number theory.

We previously eluded to the interplay between multiplicative Toeplitz operators and the field of number theory. In particular, how harmonic and functional analysis can be utilised to study mathematical objects arising in number theory. Primarily, motivation for this is to understand the behaviour of Dirichlet series, and in particular the Riemann zeta function, which we discuss now.

The study of the Riemann Zeta function

Recall from Section 1.5.1 that the behaviour of the Riemann zeta function is well-known in the half-plane $\Re s > 1$. The behaviour of $\zeta(s)$ in the critical strip ($\Re s \leq 1$) is less understood. Observe from Theorem 2.19, in the case where $p = q = 2$, it follows that D_α is unbounded $\iff \alpha \leq 1$. Remarkably, the author of [32] relates the unbounded mapping D_α for $\alpha \leq 1$, to ζ in the critical strip. This is achieved by restricting the range of the mapping when $\alpha \in (\frac{1}{2}, 1]$ and considering the truncated norm

$$Y_\alpha(N) = \sup_{\|x\|_2=1} \left(\sum_{n=1}^{N} |y_n|^2 \right)^{\frac{1}{2}}.$$

It is shown that $Y_\alpha(N)$ is a lower bound for the maximal order of the Riemann zeta function. Specifically, for $\alpha \in (\frac{1}{2}, 1)$

$$Z_\alpha(T) := \max_{t\in[0,T]} |\zeta(\alpha + it)| \geq Y_\alpha \left(T^{2/3(\alpha-1/2)-\epsilon}\right),$$

for all $\epsilon > 0$ and for sufficiently large T. Moreover, the author gives an approximation of $Y_\alpha(N)$ which leads to

$$\log Z_\alpha(T) \gg \frac{(\log T)^{1-\alpha}}{\log\log T} \text{ for } \frac{1}{2} < \alpha < 1,$$

a known estimate for the maximal order of ζ in the critical strip (see Section 1.5.1). Interestingly, similar methods have since been utilised to give improvements upon this, and also for the case $\alpha = 1/2$, new approximations have been found, see [1] and [9] respectively.

As discussed in [33], instead of restricting the range of the unbounded operator, one can instead consider D_α acting upon a restricted domain. Specifically, the domain of the operator is chosen to be the set of multiplicative sequences in ℓ^2. By doing so, the author evaluates the following “quasi”-norm over $\mathcal{M}^2$,

$$\sup_{\substack{x\in\mathcal{M}^2\\ \|x\|=T}} \frac{\|D_\alpha x\|}{\|x\|},$$

where T is constant. For large T, this is shown to be equal to

$$\exp\left(\frac{c_\alpha (\log T)^{1-\alpha}}{(\log\log T)^\alpha}\right),$$

for $\frac{1}{2} < \alpha < 1$, where c_α is a constant dependent on α. This clearly has a strong resemblance to the conjectured maximal order of ζ as given in (1.7). However, the author notes, that despite the similarity, this norm has not been shown to be a lower bound of $Z_\alpha(T)$.

We refer the reader to [48] for an overview of the connections between analytic number theory and these operators.

2.4.4 Open Questions

We end this section by highlighting some of open questions which have emerged from the literature that will be addressed in the latter chapters of this thesis.

Boundedness

1. Let $1 \leq p \leq q \leq \infty$. For which $f : \mathbb{Q}^+ \to \mathbb{C}$, is $M_f : \ell^p \to \ell^q$ a bounded operator? Even for the case when $f(n) = n^{-\alpha}$, this is an open problem for $1 < p < q < \infty$.

2. Is there a sufficient and necessary condition for $T_\phi : \ell^p \to \ell^p$ to be bounded for $1 \leq p \leq \infty$ where $p \neq 2$?

Spectral Theory

1. For which F, if any, is the operator M_F compact?

2. What is the spectrum of M_F? Can we describe the spectrum for a class of M_F? For example, can we find the spectrum of D_F?

3. Given general $F \in \mathcal{W}_{\mathbb{Q}^+}$ when is F factorisable?

4. When is the semi-commutator of T_Φ on the infinite torus compact? Can we find a class of symbols for which it is compact?

2.5 Operators acting on $\mathcal{M}$.

As previously mentioned, in [33] for example, it can be fruitful to study mappings on the multiplicative subset of ℓ^p; we later present findings on how $D_f : \mathcal{M}^p \to \mathcal{M}^q$ are closely connected to the boundedness of $D_f : \ell^p \to \ell^q$ in Chapter 3. We ask therefore which linear mappings preserve multiplicativity; that is, given a linear operator L, when is $Lx = y \in \mathcal{M}$ given $x \in \mathcal{M}$?

Theorem 2.34. *Let $L : \mathcal{A} \to \mathcal{A}$ be a linear operator. We denote the matrix representation[16] of L, $A_L = (a_{i,j})$. Then L preserves multiplicativity if and only if the following conditions hold for all $(n, m) = 1$:*

$$a_{nm,r} = a_{n,d}a_{m,\frac{r}{d}} \text{ for some } d \text{ which is a unitary divisor of } r \tag{2.10}$$

$$a_{n,r}a_{m,s} = 0 \quad \forall r, s \in \mathbb{N} \text{ such that } (r, s) \geq 2. \tag{2.11}$$

Proof. We write for convenience $a_{i,j} = a_i(j)$. First, assuming that L maps $\mathcal{M}$ to $\mathcal{M}$, we show that (2.10) and (2.11) hold for $(n, m) = 1$. Fix $n, m \in \mathbb{N}$ such that $(n, m) = 1$. As $y \in \mathcal{M}$, we know $y(nm) = y(n)y(m)$, and so

$$\sum_{r\in\mathbb{N}} a_{nm}(r)x(r) = \sum_{s\in\mathbb{N}} a_n(s)x(s) \sum_{t\in\mathbb{N}} a_m(t)x(t). \tag{2.12}$$

Let $K \in \mathbb{N}$ and suppose x is supported on a fixed, arbitrary, finite product of prime powers,

$$x(n) = \begin{cases} x(p_1^{\alpha_1} \dots p_k^{\alpha_k}), & \text{if } n = p_1^{\alpha_1} \dots p_k^{\alpha_k} \text{ for } \alpha_1, \dots, \alpha_k \leq K. \\ 0, & \text{otherwise} \end{cases}$$

Since $x \in \mathcal{M}$, we have

$$x\left(p_1^{\alpha_1} \dots p_k^{\alpha_k}\right) = x\left(p_1^{\alpha_1}\right) \dots x\left(p_k^{\alpha_k}\right).$$

[16] with respect to the basis $(e_n)_{n\in\mathbb{N}}$

For convenience, we shall denote $\lambda_i(\alpha_i) = x\left(p_i^{\alpha_i}\right)$ where $\lambda_i(0) = 1$ and $\lambda_i(\alpha_i) = 0$ if $\alpha_i > K$, for each $i = 1, \ldots, k$. By writing $r = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$, $s = p_1^{\beta_1} \ldots p_k^{\beta_k}$ and $t = p_1^{\delta_1} \ldots p_k^{\delta_k}$, (2.12) can be reformulated as

$$\sum_{\alpha_1,\ldots,\alpha_k=0}^{K} a_{nm}\left(p_1^{\alpha_1} \ldots p_k^{\alpha_k}\right) \lambda_1(\alpha_1) \ldots \lambda_k(\alpha_k) =$$
$$\sum_{\beta_1,\ldots,\beta_k=0}^{K} a_n\left(p_1^{\beta_1} \ldots p_k^{\beta_k}\right) \lambda_1(\beta_1) \ldots \lambda_k(\beta_k) \sum_{\delta_1,\ldots,\delta_k=0}^{K} a_m\left(p_1^{\delta_1} \ldots p_k^{\delta_k}\right) \lambda_1(\delta_1) \ldots \lambda_k(\delta_k). \tag{2.13}$$

The proof proceeds by equating coefficients of the above equation, noting that $\lambda_i(\alpha_i)$ are free variables.

• Choose $\beta_i = \delta_i \geq 1$, for all $i \leq k$. Then the RHS of (2.13) is given by

$$\sum_{\beta_1,\ldots,\beta_k=0}^{K} a_n\left(p_1^{\beta_1} \ldots p_k^{\beta_k}\right) a_m\left(p_1^{\beta_1} \ldots p_k^{\beta_k}\right) \lambda_1(\beta_1)^2 \ldots \lambda_k(\beta_k)^2,$$

while on the LHS, there are no square terms. Thus, we equate coefficients of the form $\lambda_1(\beta_1)^2 \ldots \lambda_k(\beta_k)^2$, giving

$$a_n\left(p_1^{\beta_1} \ldots p_k^{\beta_k}\right) a_m\left(p_1^{\beta_1} \ldots p_k^{\beta_k}\right) = 0. \tag{2.14}$$

• Choose $\beta_i \neq \delta_i$, with $\beta_i, \delta_i \geq 1$ for all $i \leq k$. Now, we equate coefficients of the form $\lambda_1(\beta_1)\lambda_1(\delta_1) \ldots \lambda_k(\beta_k)\lambda_k(\delta_k)$. This gives

$$a_n\left(p_1^{\beta_1} \ldots p_k^{\beta_k}\right) a_m\left(p_1^{\delta_1} \ldots p_k^{\delta_k}\right) + a_n\left(p_1^{\delta_1} \ldots p_k^{\delta_k}\right) a_m\left(p_1^{\beta_1} \ldots p_k^{\beta_k}\right) = 0,$$

By multiplying through by $a_m\left(p_1^{\beta_1} \ldots p_k^{\beta_k}\right)$, it follows from (2.14) that,

$$0 + a_n\left(p_1^{\delta_1} \ldots p_k^{\delta_k}\right) a_m\left(p_1^{\beta_1} \ldots p_k^{\beta_k}\right)^2 = 0,$$

and therefore

$$a_n\left(p_1^{\delta_1} \ldots p_k^{\delta_k}\right) a_m\left(p_1^{\beta_1} \ldots p_k^{\beta_k}\right) = 0.$$

As this is true for any arbitrary set of primes, where $\beta_i, \delta_i \geq 1$ for any $i \in \mathbb{N}$ (i.e. they share a common factor greater than 1), we must have

$$a_n(r)a_m(s) = 0 \quad \forall (n,m) = 1 \text{ and } (r,s) \geq 2,$$

giving us (2.11).

• Now choose β_i and δ_i such $\beta_i \neq 0 \implies \delta_i = 0$ for all $i \leq k$, i.e. $(s,t) = 1$. We equate coefficients of the form $\lambda(\alpha_1) \dots \lambda(\alpha_k)$, which yields

$$\begin{aligned} a_{nm}\left(p_1^{\alpha_1} \dots p_k^{\alpha_k}\right) &= a_n(1)a_m\left(p_1^{\alpha_1} \dots p_k^{\alpha_k}\right) + a_n\left(p_1^{\beta_1}\right) a_m\left(p_2^{\alpha_2} \dots p_k^{\alpha_k}\right) + \dots \\ &+ a_n\left(p_1^{\beta_1} \dots p_{k-1}^{\beta_{k-1}}\right) a_m\left(p_k^{\alpha_k}\right) + a_n\left(p_1^{\beta_1} \dots p_k^{\beta_1}\right) a_m(1) \\ &= \sum_{\substack{d \mid p_1^{\alpha_1} \dots p_k^{\alpha_k} \\ d \text{ unitary}}} a_n(d)a_m\left(\frac{p_1^{\alpha_1} \dots p_k^{\alpha_k}}{d}\right). \end{aligned}$$

As this is true for an arbitrary set of prime powers, we have that

$$a_{nm}(r) = \sum_{\substack{d \mid r \\ d \text{ unitary}}} a_n(d)a_m\left(\frac{r}{d}\right) \quad \forall (n,m) = 1, r \in \mathbb{N}.$$

We now show that this summation is only ever one term. In other words, given a unitary divisor d such that $a_n(d)\, a_m\left(\frac{r}{d}\right) \neq 0$, then for all other unitary divisors $a_n(d)a_m\left(\frac{r}{d}\right) = 0$. We proceed by contradiction. Fix $r \in \mathbb{N}$, and let r have the prime decomposition $r = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ for $\alpha_1, \dots, \alpha_k \in \mathbb{N}$. Suppose that for two unitary divisors, of r, d_1 and d_2 where $d_1 \neq d_2$, we have

$$a_n(d_1)\, a_m\left(\frac{r}{d_1}\right) \neq 0 \text{ and } a_n(d_2)\, a_m\left(\frac{r}{d_2}\right) \neq 0.$$

Therefore,

$$a_n(d_1)\, a_m\left(\frac{r}{d_1}\right) a_n(d_2)\, a_m\left(\frac{r}{d_2}\right) \neq 0, \tag{2.15}$$

and hence,

$$a_n(d_1)\, a_m\left(\frac{r}{d_2}\right) \neq 0 \text{ and } a_n(d_2)\, a_m\left(\frac{r}{d_1}\right) \neq 0.$$

Without loss of generality, let $d_1 = p_1^{\alpha_1} \dots p_l^{\alpha_l}$ for some $l \leq k$ and as such $\frac{r}{d_1} = p_{l+1}^{\alpha_{l+1}} \dots p_k^{\alpha_k}$. From (2.11), we must have $(d_2, \frac{r}{d_1}) = 1$. Hence, $d_2 = p_1^{\beta_1} \dots p_j^{\beta_j}$ for some $j \leq l$ and $\beta_j = \alpha_j$ or 0. Therefore $d_2 \mid d_1$. In the same manner, we can show that $d_1 \mid d_2$ and hence, $d_1 = d_2$, which is a contradiction. This shows that (2.10) and (2.11) hold for $(n, m) = 1$ as required.

We now show that the converse is true. Fix $n, m \in \mathbb{N}$ such that $(n, m) = 1$. We start by using (2.15)[17] to obtain

$$y(nm) = \sum_{r \in \mathbb{N}} a_{nm}(r)x(r) = \sum_{r \in \mathbb{N}} \sum_{\substack{d \mid r \\ \left(d, \frac{r}{d}\right)=1}} a_n(d)a_m\left(\frac{r}{d}\right)x(r) = \sum_{\substack{c,d \in \mathbb{N} \\ (d,c)=1}} a_n(d)a_m(c)\, x(dc),$$

since $d \mid r \iff dc = r$ for $c \in \mathbb{N}$. Now,

$$\begin{aligned} y(n)y(m) &= \sum_{r,s \in \mathbb{N}} a_n(r)a_m(s)x(r)x(s) \\ &= \sum_{\substack{r,s \in \mathbb{N} \\ (r,s)=1}} a_n(r)a_m(s)x(r)x(s) + \sum_{\substack{r,s \in \mathbb{N} \\ (r,s)\geq 2}} a_n(r)a_m(s)x(r)x(s). \end{aligned}$$

It follows from (2.11), the above is equal to

$$\sum_{\substack{r,s \in \mathbb{N} \\ (r,s)=1}} a_n(r)a_m(s)x(r)x(s) = \sum_{\substack{r,s \in \mathbb{N} \\ (r,s)=1}} a_n(r)a_m(s)x(rs),$$

as $x \in \mathcal{M}$. Hence, $y(nm) = y(n)y(m)$ for all $(n, m) = 1$, as required. $\square$

We make some further observations from Theorem 2.34. First, by taking $r = 1$ in (2.10), we obtain $a_{nm}(1) = a_n(1)a_m(1)$. In other words, the first column of A is multiplicative and gives $a_1(1) = 1$. Secondly, the first row of A must be equal to $e_1 = (1, 0, 0, \dots)$. Indeed, it follows from (2.11) that $a_1(r) = 0$ for all $r \geq 2$ by taking $n = m = 1$ and $r = s \geq 2$.

[17] Observe that we have already shown this is equivalent to (2.10) but as the unitary divisors may change for each $r \in \mathbb{N}$, we use (2.15)

Example 2.35. Fix $p \in \mathbb{P}$. Let $L : \mathcal{A} \to \mathcal{A}$ be the operator defined by $Lx = y$, where $y = y(n)$ is given by

$$y(n) = \begin{cases} x(1) & \text{if } n = 1 \\ \sum_{i=1}^{\infty} a_n(i)x(i) & \text{if } n = p^k, k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

By setting $p = 2$, for example, we obtain the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ a_2(1) & a_2(2) & a_2(3) & \cdots \\ 0 & 0 & 0 & \cdots \\ a_4(1) & a_4(2) & a_4(3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ a_8(1) & a_8(2) & a_8(3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example 2.36. Let $f \in \mathcal{M}$. Let $L : \mathcal{A} \to \mathcal{A}$ be the operator defined by $Lx = y$, where $y = y(n)$ is given by

$$y(n) = \begin{cases} x(1) & \text{if } n = 1 \\ f(n)x(1) & \text{otherwise} \end{cases}$$

gives the following matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ f(2) & 0 & 0 & \cdots \\ f(3) & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

From these two examples, we make the observation that matrices which satisfy (2.10) and (2.11) may even map non-multiplicative elements to $\mathcal{M}$. We have already seen in Section 1.4.4 that D_f preserves multiplicativity. Moreover, D_f is an example of an operator which maps only multiplicative elements to $\mathcal{M}$. It remains an open problem to establish when a linear operator maps only $\mathcal{M}$ to $\mathcal{M}$.

Chapter 3

Bounded multiplicative Toeplitz operators on sequence spaces

In this chapter, we investigate the first of the topics discussed in Chapter 2. That is, we seek to determine when M_f is a bounded operator.

In section 3.1, the main results of this chapter are presented. We give a sufficient condition for M_f to be bounded in Theorem 3.1, and Theorem 3.3 shows that, for some values of p and q, this is also a necessary condition. This is followed by observations on how these theorems relate to the additive setting.

In an attempt to establish whether the condition given in Theorem 3.1 is necessary for all values of p and q, we consider the simpler case of D_f. In doing so, a relationship with multiplicative number theory emerges. Namely, we see that for certain values of p and q, the operator norm of D_f is attained at multiplicative sequences in ℓ^p for f multiplicative.

As such, we firstly investigate the n-dimensional additive convolution operator with multiplicative symbols. Secondly, the infinite dimensional case, that is, $D_f : \mathcal{M}_c^p \to \mathcal{M}_c^q$ is considered. This is followed by a discussion on how the operator on multiplicative sequences reflects the behaviour of the operator on ℓ^p and we consider the possible existence of a counterexample.

We end the chapter with a summary of the open problems that arise within. The majority of the results proven in this chapter have been published in [50].

3.1 Criterion for boundedness

In this section, we present two new results which extend theorems from [20] and [31] as stated in Theorems 2.19 and 2.20.

Theorem 3.1. *For $1 \le p \le q \le \infty$, define $r \in [1, \infty]$ by $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$ where $\frac{1}{\infty} = 0$. If $f \in \ell^r(\mathbb{Q}^+)$ then $M_f : \ell^p \to \ell^q$ is bounded. More precisely, we have*

$$\|M_f x\|_q \le \|x\|_p \|f\|_{r,\mathbb{Q}^+} \text{ for all } x \in \ell^p.$$

Proof of Theorem 3.1. Let $y = M_f x$. The proof proceeds by considering separate cases.

- $1 \le p \le q < \infty$

By Hölder's inequality,

$$\begin{aligned}
|y(n)| &\le \sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{k}\right) x(k) \right| = \sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{k}\right) \right|^{r\left(1-\frac{1}{p}\right)} \left| f\left(\frac{n}{k}\right) \right|^{\frac{r}{q}} |x(k)|^{\frac{p}{q}} |x(k)|^{1-\frac{p}{q}} \\
&\le \left(\sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{k}\right) \right|^r \right)^{1-\frac{1}{p}} \left(\sum_{k\in\mathbb{N}} |x(k)|^p \right)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{k}\right) \right|^r |x(k)|^p \right)^{\frac{1}{q}} \\
&\le \|f\|_{r,\mathbb{Q}^+}^{r\left(1-\frac{1}{p}\right)} \|x\|_p^{1-\frac{p}{q}} \left(\sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{k}\right) \right|^r |x(k)|^p \right)^{\frac{1}{q}}.
\end{aligned}$$

Hence,

$$\sum_{n\in\mathbb{N}} |y(n)|^q \le \|f\|_{r,\mathbb{Q}^+}^{rq\left(1-\frac{1}{p}\right)} \|x\|_p^{q-p} \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{k}\right) \right|^r |x(k)|^p .$$

Considering only the summation on the RHS of above,

$$\sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{k}\right) \right|^r |x(k)|^p \le \sum_{s\in\mathbb{Q}^+} |f(s)|^r \sum_{k\in\mathbb{N}} |x(k)|^p = \|f\|_{r,\mathbb{Q}^+}^r \|x\|_p^p.$$

Therefore,

$$\|M_f x\|_q^q = \sum_{n\in\mathbb{N}} |y(n)|^q \le \|f\|_{r,\mathbb{Q}^+}^{qr\left(1-\frac{1}{p}\right)+r} \|x\|_p^{q-p+p} = \|f\|_{r,\mathbb{Q}^+}^q \|x\|_p^q.$$

- $p = 1$ and $q = \infty$ (so $r = \infty$)

By the triangle inequality,

$$|y(n)| \leq \sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{d}\right) x(k) \right| \leq \|f\|_{\infty,\mathbb{Q}^+} \sum_{k\in\mathbb{N}} |x(k)| \leq \|f\|_{\infty,\mathbb{Q}^+} \|x\|_1.$$

Hence, $\|M_f x\|_\infty \leq \|f\|_{\infty,\mathbb{Q}^+} \|x\|_1$.

• $q = \infty$ with $1 < p < \infty$ (so $r = \frac{p}{p-1}$)

By Hölder's inequality, we have

$$|y(n)| \leq \sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{k}\right) x(k) \right| \leq \left(\sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{k}\right) \right|^r \right)^{\frac{1}{r}} \left(\sum_{k\in\mathbb{N}} |x(k)|^p \right)^{\frac{1}{p}} \leq \|f\|_{r,\mathbb{Q}^+} \|x\|_p.$$

Thus, $\|M_f x\|_\infty \leq \|f\|_{r,\mathbb{Q}^+} \|x\|_p$.

• $p = q = \infty$ (so $r = 1$)

We now have $|y(n)| \leq \|x\|_\infty \sum_{k\in\mathbb{N}} \left| f\left(\frac{n}{k}\right) \right| \leq \|x\|_\infty \|f\|_{1,\mathbb{Q}^+}$, which gives the desired inequality $\|M_f x\|_\infty \leq \|x\|_\infty \|f\|_{1,\mathbb{Q}^+}$. □

By taking the support of f on positive powers of 2, Theorem 3.1 reduces to the well-known Young's convolution inequality, see [56], which we state below.

Theorem 3.2 (Young's convolution inequality)**.** *For $1 \leq p \leq q \leq \infty$, define $r \in [1, \infty]$ by $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$ where $\frac{1}{\infty} = 0$. If $f \in \ell^r$ then $C_\phi : \ell^p \to \ell^q$ is bounded. More precisely, we have*

$$\|C_\phi x\|_q \leq \|x\|_p \|f\|_r \text{ for all } x \in \ell^p.$$

Theorem 3.1 can be equivalently written in terms of the sets ℓ^p as $\ell^r * \ell^p \subset \ell^q$ for r defined as in Theorem 3.1. Further to Young's inequality, in the classic setting it is also known that $\ell^p * \ell^r \not\subset \ell^s$ for any $s < p$. The same is true for Dirichlet convolution. That is, there exists $f \in \ell^r$ and $x \in \ell^p$ such that $f * x$ does not lie within ℓ^s for $s < q$. However, this does not reveal that, if given $x * f \in \ell^p$ where $x \in \ell^p$, whether $f \in \ell^r$ is also a necessary condition. In other words, Theorem 3.1 gives only a partial criterion for the boundedness of M_f between ℓ^p and ℓ^q; partial in the sense that $f \in \ell^r(\mathbb{Q}^+)$ is a sufficient condition. It is natural to ask whether this is also a necessary condition. Moreover, can we find the operator norm, $\|M_f\|_{p,q}$? Theorem 3.1 gives the upper bound $\|M_f\|_{p,q} \leq \|f\|_{r,\mathbb{Q}^+}$. Therefore, we can ask when is this a sharp bound?

Necessary condition for boundedness.

For certain f, these questions can be answered by Theorem 3.3 for the cases where $p = q$, or $p = 1$ with any q, or $q = \infty$ with any p. We refer to these as the "edge" cases as they be can visualised as the edges of a right-angled triangle.

Theorem 3.3. *Let us define r as in Theorem 3.1. For the following cases:*

1. $p = 1$ *with any* q*,*

2. $p = q$ *with* $f \in \ell^r(\mathbb{Q}^+)$ *non-negative,*

3. $q = \infty$ *with any* p *and with* $f \in \ell^r(\mathbb{Q}^+)$ *non-negative,*

$$\|M_f\|_{p,q} = \|f\|_{r,\mathbb{Q}^+}.$$

Proof. Observe that $\|M_f\|_{p,q} \leq \|f\|_{r,\mathbb{Q}^+}$ follows from Theorem 3.1. We wish to show, therefore, that this bound is sharp. We consider each edge case separately.

1. We first embark on the case when $p = 1$ with any q.

• Let $q \in [1, \infty)$, so that $r = q$.

Fix $c \in \mathbb{N}$ and let $x(n) = 1$ if $n = c$ and 0 otherwise. Then $\|x\|_1 = 1$ and so,

$$|y(n)|^q = \left|\sum_{k\in\mathbb{N}} f\left(\frac{n}{k}\right) x(k)\right|^q = \left|f\left(\frac{n}{c}\right)\right|^q.$$

Therefore,

$$\begin{aligned}\|M_f x\|_q^q = \sum_{n\in\mathbb{N}} |y(n)|^q = \sum_{n\in\mathbb{N}} \left|f\left(\frac{n}{c}\right)\right|^q = \sum_{d|c} \sum_{\substack{n\in\mathbb{N}\\(n,c)=d}} \left|f\left(\frac{n}{c}\right)\right|^q \\ = \sum_{d|c} \sum_{\substack{m\in\mathbb{N}\\(m,\frac{c}{d})=1}} \left|f\left(\frac{md}{c}\right)\right|^q,\end{aligned}$$

where $n = md$. Now, by writing $\frac{c}{d} \mapsto d$, the above is equal to

$$\sum_{d|c} \sum_{\substack{m\in\mathbb{N}\\(m,d)=1}} \left|f\left(\frac{m}{d}\right)\right|^q. \tag{3.1}$$

Note that we can write

$$\|f\|_{q,\mathbb{Q}^+}^q = \sum_{s\in\mathbb{Q}^+} |f(s)|^q = \sum_{v\in\mathbb{N}} \sum_{\substack{u\in\mathbb{N}\\(u,v)=1}} \left|f\left(\frac{u}{v}\right)\right|^q. \tag{3.2}$$

By computing the difference between (3.2) and (3.1), we shall show that $\|M_f x\|_q$ can be made arbitrarily close to $\|f\|_{q,\mathbb{Q}^+}$. We have

$$\sum_{v\in\mathbb{N}} \sum_{\substack{u\in\mathbb{N}\\(u,v)=1}} \left|f\left(\frac{u}{v}\right)\right|^q - \sum_{d|c} \sum_{\substack{m\in\mathbb{N}\\(m,d)=1}} \left|f\left(\frac{m}{d}\right)\right|^q = \sum_{\substack{u,v\in\mathbb{N}\\(u,v)=1\\v\nmid c}} \left|f\left(\frac{u}{v}\right)\right|^q.$$

Now, choose $c = (2\cdot 3\cdot 5\cdots T)^k$ where $k\in\mathbb{N}$ and T is prime. If $v\nmid c$ then $v > T$ for k large enough. Therefore, for every $\epsilon > 0$, we can choose T such that

$$\|f\|_{q,\mathbb{Q}^+}^q - \|M_f x\|_q^q = \sum_{\substack{u,v\in\mathbb{N}\\(u,v)=1\\v\nmid c}} \left|f\left(\frac{u}{v}\right)\right|^q = \sum_{\substack{u,v\in\mathbb{N}\\(u,v)=1\\v>T}} \left|f\left(\frac{u}{v}\right)\right|^q < \epsilon.$$

Hence, $\|M_f\|_{1,q} = \|f\|_{q,\mathbb{Q}^+}$ as required.

• Let $q=\infty$, so $r = q = \infty$.

Fix $c\in\mathbb{N}$. Like before, choose $x(n) = 1$ if $n = c$ and 0 otherwise. Again $\|x\|_1 = 1$. Now,

$$\|M_f x\|_\infty = \sup_{n\in\mathbb{N}} |y(n)| = \sup_{n\in\mathbb{N}} \left|f\left(\frac{n}{c}\right)\right|.$$

Note here that for every $\epsilon > 0$, there exists $u, v\in\mathbb{N}$ with $(u,v) = 1$ such that

$$\|f\|_{\infty,\mathbb{Q}^+} - \epsilon < \left|f\left(\frac{u}{v}\right)\right|.$$

Simply choose $n = u$ and $c = v$. Then

$$\|f\|_{\infty,\mathbb{Q}^+} - \|M_f x\|_\infty < \epsilon.$$

2. Now consider the edge case where $p = q$.

• Let $1 < p = q < \infty$ so $r = 1$.

Fix $c \in \mathbb{N}$. Choose $x(n) = \frac{1}{d(c)^{\frac{1}{q}}}$ if $n \mid c$ and 0 otherwise, where $d(n)$ is the divisor counting function. Hence, $\|x\|_q^q = \frac{1}{d(c)} \sum_{d|c} 1 = 1$. Observe by Hölder's inequality,

$$\sum_{n\in\mathbb{N}} x(n)^{q-1} y(n) \leq \left(\sum_{n\in\mathbb{N}} |x(n)|^q\right)^{1-\frac{1}{q}} \left(\sum_{n\in\mathbb{N}} |y(n)|^q\right)^{\frac{1}{q}} = \left(\sum_{n\in\mathbb{N}} |y(n)|^q\right)^{\frac{1}{q}} = \|M_f x\|_q.$$

Consequently, it suffices to show that $\sum_{n\in\mathbb{N}} x(n)^{q-1} y(n)$ can be made arbitrarily close to $\|f\|_{1,\mathbb{Q}^+}$. To do this, consider

$$\begin{aligned}
\sum_{n\in\mathbb{N}} x(n)^{q-1} y(n) &= \frac{1}{d(c)^{\frac{q-1}{q}}} \sum_{n|c} y(n) = \frac{1}{d(c)^{\frac{q-1}{q}}} \sum_{n|c} \sum_{k|c} f\left(\frac{n}{k}\right) x(k) \\
&= \frac{1}{d(c)} \sum_{n,k|c} f\left(\frac{n}{k}\right) \\
&= \frac{1}{d(c)} \sum_{s\in\mathbb{Q}^+} f(s) \sum_{\substack{n,k|c \\ s=\frac{n}{k}}} 1.
\end{aligned}$$

We now follow the argument given in [30] (page 87). Let $s = \frac{u}{v} \in \mathbb{Q}^+$, where $(u,v) = 1$, then the above is equal to

$$\frac{1}{d(c)} \sum_{\substack{u,v\in\mathbb{N} \\ (u,v)=1}} f\left(\frac{u}{v}\right) \sum_{\substack{n,k|c \\ nv=uk}} 1,$$

where we used the fact that $\frac{n}{k} = \frac{u}{v}$ if and only if $nv = uk$. Since $(u,v) = 1$, it follows that $u \mid n$ and $v \mid k$, and for any contribution to the above summation, we must have $u, v \mid c$, i.e., $uv \mid c$. Assume, therefore, that $uv \mid c$. Then

$$\begin{aligned}
\frac{1}{d(c)} \sum_{\substack{uv|c \\ (u,v)=1}} f\left(\frac{u}{v}\right) \sum_{\substack{n,k|c \\ nv=uk}} 1 &= \frac{1}{d(c)} \sum_{\substack{uv|c \\ (u,v)=1}} f\left(\frac{u}{v}\right) \sum_{lu,lv|c} 1 \\
= \frac{1}{d(c)} \sum_{\substack{uv|c \\ (u,v)=1}} f\left(\frac{u}{v}\right) \sum_{l|\frac{c}{uv}} 1 &= \sum_{\substack{uv|c \\ (u,v)=1}} f\left(\frac{u}{v}\right) \frac{d\left(c/uv\right)}{d(c)}.
\end{aligned}$$

Now, by choosing c appropriately, we can show that $\frac{d(c/uv)}{d(c)}$ can be made close to 1 for all u, v less than some large constant. Fix $T \in \mathbb{P}$ and choose c to be

$$c = \prod_{\substack{t \leq T \\ t \in \mathbb{P}}} t^{\alpha_t} \quad \text{where} \quad \alpha_t = \left[\frac{\log T}{\log t}\right].$$

If $uv \mid c$, then $uv = \prod_{t \leq T} t^{\beta_t}$ where $\beta_t \in [0, \alpha_t]$. Hence, by (1.3), we have

$$\frac{d\left(c/uv\right)}{d(c)} = \prod_{t \leq T}\left(\frac{\alpha_t - \beta_t + 1}{\alpha_t + 1}\right) = \prod_{t \leq T}\left(1 - \frac{\beta_t}{\alpha_t + 1}\right).$$

If we take $uv \leq \sqrt{\log T}$, then $t^{\beta_t} \leq \sqrt{\log T}$ for every prime divisor t of uv. Therefore, $\beta_t \leq \frac{\log \log T}{2 \log t}$ and $\beta_t = 0$ if $t > \sqrt{\log T}$. It follows that

$$\begin{aligned}
\frac{d\left(c/uv\right)}{d(c)} &= \prod_{t \leq \sqrt{\log T}}\left(1 - \frac{\beta_t}{\alpha_t + 1}\right) \geq \prod_{t \leq \sqrt{\log T}}\left(1 - \frac{\log \log T}{2 \log T}\right) \\
&= \left(1 - \frac{\log \log T}{2 \log T}\right)^{\pi\left(\sqrt{\log T}\right)},
\end{aligned}$$

where $\pi\left(x\right)$ is the prime counting function up to x. Note that in general for $|a| < 1$ and $b > 0$, we have

$$(1 - a)^b = e^{b \log(1-a)} \sim e^{-ba} \geq 1 - ba,$$

where we have made use for the fact that $\log(1 - a) = -a + O(a^2)$. Therefore, for sufficiently large T, it follows that

$$\left(1 - \frac{\log \log T}{2 \log T}\right)^{\pi\left(\sqrt{\log T}\right)} \geq 1 - \pi\left(\sqrt{\log T}\right) \frac{\log \log T}{2 \log T}.$$

Now, as $\pi(x) \ll \frac{x}{\log x}$ for sufficiently large x, we have

$$\begin{aligned}
\left(1 - \frac{\log \log T}{2 \log T}\right)^{\pi\left(\sqrt{\log T}\right)} &\geq 1 - \frac{C\sqrt{\log T} \log \log T}{\log\left(\sqrt{\log T}\right) 2 \log T} \\
&\geq 1 - \frac{C}{\sqrt{\log T}},
\end{aligned}$$

for some constant C. Therefore,

$$\begin{aligned}\sum_{\substack{uv|c\\(u,v)=1}} f\left(\frac{u}{v}\right)\frac{d\left(c/uv\right)}{d(c)} \geq \sum_{\substack{uv|c\\uv\leq\sqrt{\log T}}} f\left(\frac{u}{v}\right)\frac{d\left(c/uv\right)}{d(c)} &> \sum_{\substack{uv|c\\uv\leq\sqrt{\log T}}} f\left(\frac{u}{v}\right)\left(1-\frac{C}{\sqrt{\log T}}\right)\\ &\geq \sum_{\substack{uv|c\\uv\leq\sqrt{\log T}}} f\left(\frac{u}{v}\right) - \frac{C}{\sqrt{\log T}}\sum_{s\in\mathbb{Q}^+} f(s)\\ &\geq \sum_{\substack{uv|c\\uv\leq\sqrt{\log T}}} f\left(\frac{u}{v}\right) - \frac{C_1}{\sqrt{\log T}}\end{aligned}$$

as $f \in \ell^1(\mathbb{Q}^+)$. Note that if $uv \not| c$, then there exists k for which $t^k \mid uv$ such that

$$t^k > t^{\left(\frac{\log T}{\log t}\right)},$$

and therefore $k \geq \frac{\log T}{\log t}$, i.e. $t^k \geq T$. This further implies that $uv \geq T$ as $t^k \mid uv$. It follows that

$$\begin{aligned}\sum_{\substack{uv|c\\uv\leq\sqrt{\log T}}} f\left(\frac{u}{v}\right) &= \sum_{\substack{(u,v)=1\\uv\leq\sqrt{\log T}}} f\left(\frac{u}{v}\right) - \sum_{\substack{uv\not| c\\uv\leq\sqrt{\log T}}} f\left(\frac{u}{v}\right)\\ &= \sum_{\substack{(u,v)=1\\uv\leq\sqrt{\log T}}} f\left(\frac{u}{v}\right) - \sum_{uv>T} f\left(\frac{u}{v}\right)\end{aligned}$$

By choosing T to be arbitrarily large, for every $\epsilon > 0$, we have

$$\sum_{uv|c} f\left(\frac{u}{v}\right)\frac{d\left(c/uv\right)}{d(c)} > \sum_{s\in\mathbb{Q}^+} f(s) - \epsilon.$$

Moreover,

$$\|f\|_{1,\mathbb{Q}^+} - \|M_f x\|_q \leq \|f\|_{1,\mathbb{Q}^+} - \sum_{n\in\mathbb{N}} x(n)^{q-1}y(n) < \epsilon.$$

- We now consider the case where $p = q = \infty$, and so $r = 1$.

Let $x(n) = 1$ for all $n \in \mathbb{N}$ so that $\|x\|_\infty = 1$. Moreover, for a fixed $c \in \mathbb{N}$, we have

$$|y(c)| = \sum_{k\in\mathbb{N}} f\left(\frac{c}{k}\right) x(k) = \sum_{k\in\mathbb{N}} f\left(\frac{c}{k}\right).$$

By applying the same method as given in case (1), we conclude that $y(c)$ can be arbitrarily close to $\|f\|_{1,\mathbb{Q}^+}$. Hence, $\|M_f\|_{\infty,\infty} = \|f\|_{1,\mathbb{Q}^+}$.

3. Finally, we consider the case when $q = \infty$ with any p. We have already dealt with the cases when $p = 1$ or $p = \infty$. So let $p \in (1, \infty)$, giving $r = \frac{p}{p-1}$.

Fix $c \in \mathbb{N}$, and let

$$x(n) = f\left(\frac{c}{n}\right)^{\frac{r}{p}} F(c)^{-\frac{1}{p}} \text{ where } F(c) = \sum_{n\in\mathbb{N}} f\left(\frac{c}{n}\right)^r \text{ exists as } f \in \ell^r(\mathbb{Q}^+).$$

In the case when $F(c) = 0$ for all $c \in \mathbb{N}$, it follows from the non-negativity of f that $f = 0$, i.e. the trivial case. Thus, we assume $F(c) \neq 0$. Now,

$$\|x\|_p = \frac{1}{F(c)} \sum_{n\in\mathbb{N}} f\left(\frac{c}{n}\right)^r = \frac{F(c)}{F(c)} = 1.$$

Now consider just the term $y(c)$,

$$y(c) = F(c)^{-\frac{1}{p}} \sum_{k\in\mathbb{N}} f\left(\frac{c}{k}\right) f\left(\frac{c}{k}\right)^{\frac{r}{p}} = F(c)^{-\frac{1}{p}} \sum_{k\in\mathbb{N}} f\left(\frac{c}{k}\right)^r,$$

as $1 + \frac{r}{p} = \frac{p-1+1}{p-1} = r$. Therefore,

$$y(c) = F(c)^{1-\frac{1}{p}} = F(c)^{\frac{1}{r}} = \left(\sum_{k\in\mathbb{N}} f\left(\frac{c}{k}\right)^r\right)^{\frac{1}{r}}.$$

We can apply the same argument as case (1) to show that for every $\epsilon > 0$, we can choose $c = (2 \cdot 3 \cdot 5 \cdots T)^k$ where T is prime such that $y(c)$ can be made arbitrarily close to $\|f\|_{r,\mathbb{Q}^+}$. Hence, $\|M_f\|_{p,\infty} = \|f\|_{r,\mathbb{Q}^+}$. $\square$

Corollary 3.4. *Let us define r and M_f as above. For the following cases,*

1. $p = 1$ *with any* q,
2. $p = q$ *with* $f \in \ell^r(\mathbb{Q}^+)$ *non-negative,*

3. $q = \infty$ *with any* p *and with* $f \in \ell^r(\mathbb{Q}^+)$ *non-negative,*

$$M_f : \ell^p \to \ell^q \text{ is bounded} \iff f \in \ell^r(\mathbb{Q}^+).$$

Recall that Theorems 2.19 and 2.20 require the positivity of f. We remark here that Theorem 3.3 also requires positivity. Determining the operator norm, $\|M_f\|_{p,q}$, for any f, not strictly positive (i.e. a generalisation of Theorem 2.21 to all p and q) remains an open question. In this thesis however, we focus on establishing a necessary condition for boundedness and the connection this reveals to multiplicative number theory.

Implication for Toeplitz operators.

As discussed in Section 2.4, the boundedness of $T_\phi : \ell^p \to \ell^p$ is not well understood. We can utilise Theorems 3.1 and 3.8 to give some further results in this case. The following corollary gives a sufficient condition for the boundedness of Toeplitz operators between $\ell^p \to \ell^q$. Moreover, in certain cases, this is also necessary.

Corollary 3.5. *For* $1 \leq p \leq q \leq \infty$, *define* $r \in [1, \infty]$ *by* $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$ *where* $\frac{1}{\infty} = 0$. *If* $\phi \in \ell^r(\mathbb{Z})$ *then* $T_\phi : \ell^p \to \ell^q$ *is bounded. More precisely, we have*

$$\|T_\phi x\|_q \leq \|x\|_p \|\phi\|_r \text{ for all } x \in \ell^p.$$

For the following cases,

1. $p = 1$ *with any* q,

2. $p = q$ *with* $\phi \in \ell^r(\mathbb{Z})$ *non-negative,*

3. $q = \infty$ *with any* p *and with* $\phi \in \ell^r(\mathbb{Z})$ *non-negative,*

$$T_\phi : \ell^p \to \ell^q \text{ is bounded} \iff \phi \in \ell^r(\mathbb{Z}).$$

Proof. Given $\phi \in \ell^r(\mathbb{Z})$, the operator T_ϕ can be equivalently constructed as a multiplicative Toeplitz operator, M_f where $f(2^k) = \phi(k)$ for $k \in \mathbb{Z}$ and 0 otherwise. The result then follows from Theorem 3.1 and Theorem 3.3. $\square$

3.2 Connection to multiplicative number theory

Theorem 3.3 gives a sufficient condition for boundedness. However, establishing whether this is also a necessary condition and finding the operator norm for all other p and q (which we will refer to as the interior cases) is challenging.

We shall consider the Dirichlet convolution operator D_f i.e. when f is supported on $\mathbb{N}$. In order to better understand the boundedness of D_f in the interior cases, we seek to establish the maximisers of the D_f i.e. the sequences in ℓ^p which yield the supreme value of $\|D_f x\|_q$. There are several publications regarding similar optimisation problems involving multiplicative structures, see for example, [19], [44], [34] and [35]. Interestingly, multiplicative sequences are highlighted in all as playing a crucial role.

In [35] the author investigates, for $p \in (1, 2)$, the supremum of

$$\sum_{c,d|k} f(c,d) x_c x_d \quad \text{subject to } \|x\|_p = 1,$$

where $f(c,d) = \frac{[c,d]}{(c,d)}$ and k is square-free. It is shown that for certain ranges of p within $(1, 2)$, the supremum is attained when x is either a constant or multiplicative sequence.

In addition, the authors in [44] consider the minimisation of

$$\sum_{c,d|k} f(c,d) x_c x_d \quad \text{subject to } \|x\|_2 = 1.$$

again where k is square-free. They showed that for some fixed f, the minimum is achieved at the point $x_n = \mu(n) d(k)^{-1/2}$ which is constant multiplicative.

Our investigation into establishing the maximisers begins with the edge cases where the boundedness of D_f is fully understood. We consider the maximiser in each edge case, as presented in the proof of Theorem 3.3, separately.

- For $1 = p \leq q < \infty$ in case 1, the supremum of $\|D_f x\|_q$ is attained when $x = e_1$ i.e. $x(n) = 1$ if $n = 1$ and 0 otherwise, so x is completely multiplicative.
- For $1 < p = q < \infty$ in case 2, we choose $x(n) = \frac{1}{d(c)^{1/p}}$, whenever $n \mid c$ for some fixed $c \in \mathbb{N}$, and 0 otherwise. This is constant multiplicative as $x(n) = \frac{1}{d(c)^{1/p}} g(n)$ where $g(n) = 1$ if $n \mid c$ and 0 otherwise. Observe that g is multiplicative as $n \mid c$ and $m \mid c$ if and only if $nm \mid c$, , for $n, m \in \mathbb{N}$ such that $(n, m) = 1$,.
- Moreover, for $p = q = \infty$ in case 2, the completely multiplicative sequence $x(n) = 1$ (for all $n \in \mathbb{N}$) attains the operator norm.

• Finally, for $1 < p < q = \infty$ in case 3, $x(n) = Cf(n/c)^{r/p}$ is constant multiplicative, if f is multiplicative.

Corollary 3.6. *For $p = 1$ with any q, $q = \infty$ with any p, and $p = q$, the operator norm is attained at $x \in \ell^p$ which have multiplicative structure.*

It appears that a similar phenomenon, highlighted in the above discussion, is occurring here too, and perhaps, for f multiplicative, one could expect $D_f : \ell^p \to \ell^q$ to be largest on the multiplicative elements of ℓ^p as well. This is also reminiscent of the reproducing kernel thesis for Toeplitz operators, which says that the operator is bounded on the whole space if and only if it is bounded on the reproducing kernel, see Lemma 4.1.9 in [42]. Similarly, the behaviour of the completely multiplicative elements in ℓ^p could determine the boundedness of the operator on the whole space. Indeed, we have seen from Proposition 1.39 that the $\mathrm{span}(\mathcal{M}_c^p)$ is dense in ℓ^p, and therefore the behaviour of $\mathcal{M}_c^p$ should indicate how the operator acts on ℓ^p. However, the theory of the reproducing kernel thesis is considered within a Hilbert space setting and as such can not be considered when $1 < p < q < \infty$.

We focus our investigation into the boundedness[18] of $D_f : \mathcal{M}_c^p \to \ell^q$ for $1 < p < q < \infty$ for $f \in \mathcal{M}_c$.

3.2.1 Boundedness of $D_f : \mathcal{M}_c^p \to \ell^q$.

Recall, from Theorem 3.1, that if $f \in \ell^r$ where $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$ then $D_f : \mathcal{M}_c^p \to \ell^q$ is bounded. We ask if this is also a necessary condition.

In light of the Bohr lift, we first consider what happens in the case of the n-dimensional convolution operator with a completely multiplicative symbol. Recall this is equivalent to the Dirichlet convolution operator whose symbol is completely multiplicative and supported on a finite number of prime powers.

Proposition 3.7. *Fix $n \in \mathbb{N}$ and let $1 \leq p \leq q < \infty$. Suppose $f \in \mathcal{M}_c$ such that f is supported on the set*

$$S = \{t_1^{\alpha_1} \dots t_n^{\alpha_n} : t_i \in \mathbb{P}, \alpha_i \in \mathbb{N}\}$$

[18] $\mathcal{M}_c^p$ and $\mathcal{M}^p$ are subsets, not subspaces of ℓ^p. For example, they are not closed under addition. Given X, Y which are subsets of some Banach space, we say $L : X \to Y$ is bounded $\iff \|Lx\| \leq C\|x\|$ for all $x \in X$.

. *Then* $D_f : \ell^p \to \ell^q$ *is bounded if and only if* $f \in \mathcal{M}_c^r$ *for* $r \in [1, \infty)$.

Proof. Let $f \in \mathcal{M}_c$ such that f is supported on S. We start with the observation that if $f \in \mathcal{M}_c^s$ for some $s \in [1, \infty)$, then f is automatically in $\mathcal{M}_c^r$ for all $r \in (1, \infty)$. Indeed, by using Euler products, we can write

$$\sum_{m \in \mathbb{N}} |f(m)|^r = \prod_{i \leq n} \frac{1}{1 - |f(p_i)|^r},$$

which is finite. Therefore, $f \in \mathcal{M}_c^r$ for all $r > 1$ and in particular when $1/r = 1 + 1/p - 1/q$. Thus by applying Theorem 3.1, it follows that D_f is bounded. Conversely, if $D_f : \ell^p \to \ell^q$ is bounded, then $f * e_1 = f$ must lie within ℓ^q, and by the argument above, must also lie in all other ℓ^r for $r \in (1, \infty]$. □

Restricting the support of f to S shows that $f \in \mathcal{M}_c^r$ is not a necessary condition for boundedness. In a sense this is a trivial case however, as $f \in \mathcal{M}_c$ is automatically in all other ℓ^r for $r \in [1, \infty)$. The problem is far more delicate in the infinite dimensional case i.e. when considering D_f without restriction on the support of f. For simplicity, we consider the case when $p \in (1, 2)$ and $q = 2$, so $\frac{1}{r} = \frac{3}{2} - \frac{1}{p}$.

Theorem 3.8. *Let* $1 < p < 2$. *If* $f \in \mathcal{M}_c^2$, *the mapping* $D_f : \mathcal{M}_c^p \to \mathcal{M}^2$ *is bounded*[19].

Proof. By taking $h = f$ and $g = j = x$ in (1.4), we have

$$\|D_f x\|_2 = \frac{\|f\|_2 \|x\|_2 \, |\langle f, x \rangle|}{\|fx\|_2} \leq \|f\|_2 \|x\|_2 \, |\langle f, x \rangle| \, ,$$

as f and x are completely multiplicative, and as such we have $x_1 = 1$ and $f(1) = 1$, giving

$$\|fx\|_2 = \sum_{n=1}^{\infty} |f(n) x_n|^2 \geq 1.$$

Now,

$$\frac{\|D_f x\|_2}{\|x\|_p} \leq \frac{\|f\|_2 \|x\|_2 \, |\langle f, x \rangle|}{\|x\|_p} \leq \|f\|_2 \prod_{t \in \mathbb{P}} \frac{(1 - |x(t)|^p)^{\frac{1}{p}}}{\left(1 - |x(t)|^2\right)^{\frac{1}{2}} (1 - |x(t) f(t)|)}, \tag{3.3}$$

[19] The convolution of two multiplicative sequences is also multiplicative, so we can consider $y \in \mathcal{M}^2$.

where we made use of Euler products. Therefore, it remains to show that the product over primes is bounded independently of $x(t)$. As $0 \leq |x(t)| < 1$, we can say that $|x(t)|^2 < |x(t)|^p$ and so,

$$\frac{1}{1-|x(t)|^2} < \frac{1}{1-|x(t)|^p}.$$

Hence, the product of (3.3) is at most

$$\prod_{t\in\mathbb{P}} \frac{(1-|x(t)|^p)^{\frac{1}{p}}}{(1-|x(t)|^p)^{\frac{1}{2}}(1-|x(t)f(t)|)} = \prod_{t\in\mathbb{P}} \frac{(1-|x(t)|^p)^{\frac{2-p}{2p}}}{(1-|x(t)f(t)|)}.$$

By taking logarithms, we arrive at the equality

$$\log\left(\prod_{t\in\mathbb{P}} \frac{(1-|x(t)|^p)^{\frac{2-p}{2p}}}{(1-|x(t)f(t)|)}\right) = \sum_{t\in\mathbb{P}}\left(\log\frac{1}{1-|x(t)f(t)|} - \frac{2-p}{2p}\log\frac{1}{1-|x(t)|^p}\right).$$

Note in general for $0 < a < 1$, we have $a \leq \log\left(\frac{1}{1-a}\right) = a + O\left(a^2\right)$. Hence,

$$\sum_{t\in\mathbb{P}} \log\left(\frac{1}{1-|x(t)|^p}\right) \geq \sum_{t\in\mathbb{P}} |x(t)|^p,$$

and,

$$\sum_{t\in\mathbb{P}} \log\left(\frac{1}{1-|x(t)f(t)|}\right) = \sum_{t\in\mathbb{P}} |x(t)f(t)| + O(1),$$

where the $O(1)$ constant term is independent of the sequence $x(t)$. Therefore, we obtain

$$\begin{aligned}&\sum_{t\in\mathbb{P}}\left(\log\frac{1}{1-|x(t)f(t)|} - \frac{2-p}{2p}\log\frac{1}{1-|x(t)|^p}\right)\\ &\leq \sum_{t\in\mathbb{P}}\left(|x(t)f(t)| - \frac{2-p}{2p}|x(t)|^p\right) + O(1).\end{aligned}$$

Now, we consider the case when the terms of the above series are non-negative. In other words,

$$|x(t)f(t)| \geq \frac{2-p}{2p}|x(t)|^p \iff \left(\frac{2p}{2-p}|f(t)|\right)^\beta \geq |x(t)|,$$

where $\beta = \frac{1}{p-1}$. Hence, by only summing over the t which yield non-negative terms, we have

$$\begin{aligned}\sum_{t\in\mathbb{P}}\left(|x(t)f(t)| - \frac{2-p}{2p}|x(t)|^p\right) &\leq \sum_{\substack{t \text{ s.t} \\ |x(t)|\leq\left(\frac{2p}{2-p}|f(t)|\right)^\beta}}\left(|x(t)f(t)| - \frac{2-p}{2p}|x(t)|^p\right) \\ &\leq \sum_{\substack{t \text{ s.t} \\ |x(t)|\leq\left(\frac{2p}{2-p}|f(t)|\right)^\beta}}|x(t)f(t)| \leq \left(\frac{2p}{2-p}\right)^\beta \sum_{t\in\mathbb{P}}|f(t)|^\beta|f(t)|.\end{aligned}$$

As $\beta + 1 = \frac{p}{p-1} > 2$, we see that

$$\sum_{t\in\mathbb{P}}|f(t)|^{\beta+1} \leq \sum_{t\in\mathbb{P}}|f(t)|^2 < \infty,$$

as $f \in \mathcal{M}_c^2$. Hence, the product in (3.3) is bounded, which implies that the mapping $D_f : \mathcal{M}_c^p \to \mathcal{M}^2$ is bounded.

□

Theorem 3.8 tells us that the condition $f \in \mathcal{M}_c^r$ from Theorem 3.1 is not necessary, and instead $f \in \mathcal{M}_c^2$ is sufficient. To highlight this difference more clearly, we consider the following example.

Example 3.9. Let $f(n) = \frac{1}{n^\alpha}$ with $\alpha > \frac{1}{2}$ and let $p = \frac{3}{2}$ giving $\frac{1}{r} = 1 - \frac{2}{3} + \frac{1}{2} = \frac{5}{6}$. Theorem 2.19 states that if $\alpha > \frac{5}{6}$, then $D_\alpha : \ell^{3/2} \to \ell^2$ is bounded. In contrast, Theorem 3.8 shows that only $\alpha > \frac{1}{2}$ is required for boundedness on $\mathcal{M}_c^{3/2}$.

As suggested by Corollary 3.6, we might expect $D_f : \ell^p \to \ell^q$ to be largest when restricted to $\mathcal{M}_c^p$, and therefore, one could speculate, at least in the case when $q = 2$, that $f \in \mathcal{M}_c^2$ is a sufficient condition for $D_f : \ell^p \to \ell^2$ to be bounded.

To understand further how the behaviour of D_f on $\mathcal{M}_c^p$ reflects the behaviour of D_f on ℓ^p, we can consider how "big" the subset $\mathcal{M}_c^p$ is in ℓ^p. Recall from Corollary 1.40, that the span$(\mathcal{M}_c^p)$ is a dense subset of ℓ^p. Therefore, we wish to determine if, for

$f \in \mathcal{M}_c^2$,

$$\|D_f x\|_2 \leq C\|x\|_p \text{ for all } x \in \operatorname{span}(\mathcal{M}_c^p). \tag{3.4}$$

From (3.4) it would follow, by Corollary 1.40, that $f \in \mathcal{M}_c^r$ is not a necessary condition. In the case when $x = x_1 + x_2$ for $x_1, x_2 \in \mathcal{M}_c^p$, we present a new result on the continuity of D_f on $\mathcal{M}_c^2$ to show $\|D_f x\|_2 \leq C\|x\|_p$. We, in fact, state a more general theorem which considers the continuity of $\mathcal{M}^p$.

Theorem 3.10. *Let $f \in \mathcal{M}_c^2$. For $p \in (1,2)$, $D_f : \mathcal{M}^p \to \mathcal{M}^2$ is continuous*[20]*. Moreover, $\|D_f x - D_f z\|_2^2 \leq C_z \|x - z\|_p$, where C_z is a constant dependent on z.*

Proof. Let $\epsilon > 0$ and let $x \in \mathcal{M}^p$ be close to a fixed $z \in \mathcal{M}^p$, i.e. $x = z + h$ where $\|h\|_p = \epsilon$. For convenience, we assume x, z, f are non-negative, although the same argument works for complex-valued functions. Note we may also assume that h is positive since $D_f(z+h)$ is largest when h is positive. We start by writing

$$\|D_f x - D_f z\|_2^2 = \|D_f x\|_2^2 + \|D_f z\|_2^2 - 2\langle D_f x, D_f z\rangle. \tag{3.5}$$

As z and x are multiplicative, by Euler products we can write

$$\|D_f x\|_2^2 = \prod_{t\in\mathbb{P}}(1+\chi(t)),\ \|D_f z\|_2^2 = \prod_{t\in\mathbb{P}}(1+\gamma(t)),\ \text{and } \langle D_f x, D_f z\rangle = \prod_{t\in\mathbb{P}}(1+\beta(t))$$

where for a prime t,

$$\chi(t) = \sum_{k\in\mathbb{N}} \left(\sum_{d|t^k} f\left(\frac{t^k}{d}\right) x(d) \right)^2, \quad \gamma(t) = \sum_{k\in\mathbb{N}} \left(\sum_{d|t^k} f\left(\frac{t^k}{d}\right) z(d) \right)^2,$$
$$\text{and } \beta(t) = \sum_{k\in\mathbb{N}} \left(\sum_{d|t^k} f\left(\frac{t^k}{d}\right) x(d) \right) \left(\sum_{d|t^k} f\left(\frac{t^k}{d}\right) z(d) \right).$$

Now, since $d|t^k \implies d = t^r$ where $0 \leq r \leq k$, it follows that

$$\gamma(t) = \sum_{k\in\mathbb{N}} f\left(t\right)^{2k} + 2\sum_{k\in\mathbb{N}} f\left(t^k\right) \sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r) + \sum_{k\in\mathbb{N}} \left(\sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r) \right)^2,$$

[20] Continuity is equivalent to boundedness for linear operators acting on Banach spaces. However, here we act on sets.

and

$$\chi(t) = \sum_{k\in\mathbb{N}} \left(f\left(t^k\right) + \sum_{r=1}^{k} f\left(t^{k-r}\right) x(t^r) \right)^2$$
$$= \sum_{k\in\mathbb{N}} f\left(t\right)^{2k} + 2\sum_{k\in\mathbb{N}} f\left(t^k\right) \sum_{r=1}^{k} f\left(t^{k-r}\right) x(t^r) + \sum_{k\in\mathbb{N}} \left(\sum_{r=1}^{k} f\left(t^{k-r}\right) x(t^r) \right)^2$$

By setting $x = z + h$, the above is equal to

$$\sum_{k\in\mathbb{N}} f\left(t\right)^{2k} + 2\sum_{k\in\mathbb{N}} f\left(t^k\right) \sum_{r=1}^{k} f\left(t^{k-r}\right) (z+h)(t^r) + \sum_{k\in\mathbb{N}} \left(\sum_{r=1}^{k} f\left(t^{k-r}\right) (z+h)(t^r) \right)^2$$
$$= \gamma(t) + \eta(t),$$

where

$$\eta(t) = 2\sum_{k\in\mathbb{N}} f\left(t^k\right) \sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r) + \sum_{k\in\mathbb{N}} \left(\sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r) \right)^2$$
$$+ 2\sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r) \sum_{s=1}^{k} f\left(t^{k-s}\right) h(t^s).$$

Similarly,

$$\beta(t) = \sum_{k\in\mathbb{N}} f\left(t\right)^{2k} + \sum_{k\in\mathbb{N}} f\left(t^k\right) \sum_{r=1}^{k} f\left(t^{k-r}\right) x(t^r) + \sum_{k\in\mathbb{N}} f\left(t^k\right) \sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r)$$
$$+ \sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t^{k-r}\right) x(t^r) \sum_{s=1}^{k} f\left(t^{k-s}\right) z(t^s),$$
$$= \sum_{k\in\mathbb{N}} f\left(t\right)^{2k} + 2\sum_{k\in\mathbb{N}} f\left(t^k\right) \sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r) + \sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r)^2$$
$$+ \sum_{k\in\mathbb{N}} f\left(t^k\right) \sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r) + \sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r) \sum_{s=1}^{k} f\left(t^{k-s}\right) z(t^s)$$
$$= \gamma(t) + \nu(t),$$

where

$$\nu(t) = \sum_{k\in\mathbb{N}} f\left(t^k\right) \sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r) + \sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r) \sum_{s=1}^{k} f\left(t^{k-s}\right) z(t^s).$$

Putting these together, we can rewrite (3.5) as

$$\|D_f x - D_f z\|_2^2 = \prod_{t\in\mathbb{P}} (1+\gamma(t)+\eta(t)) + \prod_{t\in\mathbb{P}} (1+\gamma(t)) - 2\prod_{t\in\mathbb{P}} (1+\gamma(t)+\nu(t)).$$

We shall proceed to prove $\|D_f x - D_f z\|_2^2 \to 0$ as $x \to z$ by showing that

$$\prod_{t\in\mathbb{P}} (1+\gamma(t)+\eta(t)) \to \prod_{t\in\mathbb{P}} (1+\gamma(t)) \text{ and } \prod_{t\in\mathbb{P}} (1+\gamma(t)+\nu(t)) \to \prod_{t\in\mathbb{P}} (1+\gamma(t)).$$

Since $1+a+b = (1+a)\left(1+\frac{b}{1+a}\right)$, we can rewrite the above as

$$\prod_{t\in\mathbb{P}} \left(1+\frac{\eta(t)}{1+\gamma(t)}\right) \to 1 \quad \text{and} \quad \prod_{t\in\mathbb{P}} \left(1+\frac{\nu(t)}{1+\gamma(t)}\right) \to 1 \qquad \text{as } x \to z.$$

By taking logarithms, this is equivalent to

$$\sum_{t\in\mathbb{P}} \log\left(1+\frac{\eta(t)}{1+\gamma(t)}\right) \to 0 \quad \text{and} \quad \sum_{t\in\mathbb{P}} \log\left(1+\frac{\nu(t)}{1+\gamma(t)}\right) \to 0 \qquad \text{as } x \to z.$$

Observe that $1+\gamma(t) \geq 1$, and as such, it suffices to show

$$\sum_{t\in\mathbb{P}} \log(1+\eta(t)) \to 0 \quad \text{and} \quad \sum_{t\in\mathbb{P}} \log(1+\nu(t)) \to 0 \qquad \text{as } x \to z.$$

Moreover, it is sufficient in fact to show that $\sum_{t\in\mathbb{P}} \eta(t) \to 0$ for the following reasons. First, since $\log(1+a) = a + O(a^2)$, we have

$$\sum_{t\in\mathbb{P}} \log(1+\eta(t)) = \sum_{t\in\mathbb{P}} \left(\eta(t) + O\left(\eta(t)^2\right)\right) \ll \sum_{t\in\mathbb{P}} \eta(t)$$

as $\sum_{t\in\mathbb{P}} \eta(t)^2 \ll \sum_{t\in\mathbb{P}} \eta(t)$ if $\sum_{t\in\mathbb{P}} \eta(t) \to 0$. Secondly, note that

$$0 \leq \nu(t) \leq \eta(t) \text{ for all } t \in \mathbb{P},$$

and as such $\sum_{t\in\mathbb{P}} \nu(t) \to 0$ if $\sum_{t\in\mathbb{P}} \eta(t) \to 0$. Thus, we proceed by separately considering the three terms given in (3.6),

$$\eta(t) \leq 2\sum_{k\in\mathbb{N}} f\left(t^k\right) \sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r) + \sum_{k\in\mathbb{N}} \left(\sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r)\right)^2$$
$$+ 2\sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r) \sum_{s=1}^{k} f\left(t^{k-s}\right) h(t^s). \tag{3.6}$$

• We consider the first terms of the above equation. Note that since f is completely multiplicative, we have

$$\sum_{k\in\mathbb{N}} f(t^k) \sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r) = \sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t^{2k-r}\right) h(t^r).$$

Now,

$$\sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t^{2k-r}\right) h(t^r) = \sum_{r\in\mathbb{N}} \sum_{k=r}^{\infty} f\left(t^{2k-r}\right) h(t^r) = \sum_{k\in\mathbb{N}_0} \sum_{r\in\mathbb{N}} f(t^{2k+r})h(t^r)$$
$$= \sum_{k\in\mathbb{N}_0} f(t^k)^2 \sum_{r\in\mathbb{N}} f(t)^r h(t^r) \leq \|f\|_2^2 \sum_{r\in\mathbb{N}} f\left(t\right)^r h(t^r)$$
$$\ll \sum_{r\in\mathbb{N}} f\left(t\right)^r h(t^r).$$

Now summing over all $t \in \mathbb{P}$ yields

$$\sum_{t\in\mathbb{P}} \sum_{r\in\mathbb{N}} f\left(t\right)^r h(t^r) \leq \sum_{n\in\mathbb{N}} f(n)h(n) \leq \|f\|_2 \|h\|_2 \ll \|h\|_p = \epsilon$$

where we have again applied the Cauchy-Schwarz inequality.

• Next, we consider the second term of (3.6). Since f is completely multiplicative, we can write

$$\left(\sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r)\right)^2 = \left(\sum_{r=1}^{k} f\left(t\right)^{\frac{k-r}{2}} f\left(t\right)^{\frac{k-r}{2}} h(t^r)\right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{r=1}^{k} f\left(t\right)^{\frac{k-r}{2}} f\left(t\right)^{\frac{k-r}{2}} h(t^r)\right)^2 \leq \sum_{r=1}^{k} f\left(t\right)^{k-r} \sum_{r=1}^{k} f\left(t\right)^{k-r} h(t^r)^2.$$

Now,

$$\sum_{r=1}^{k} f\left(t\right)^{k-r} \sum_{r=1}^{k} f\left(t\right)^{k-r} h(t^r)^2 \leq \sum_{r\in\mathbb{N}} f\left(t\right)^{r} \sum_{r=1}^{k} f\left(t\right)^{k-r} h(t^r)^2 \ll \sum_{r=1}^{k} f\left(t^{k-r}\right) h(t^r)^2,$$

as $\frac{1}{1-f(t)} \ll 1$. Therefore,

$$\sum_{k\in\mathbb{N}} \left(\sum_{r=1}^{k} f\left(t\right)^{k-r} h(t^r)\right)^2 \ll \sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t\right)^{k-r} h(t^r)^2 = \sum_{k\in\mathbb{N}_0} f(t)^k \sum_{r\in\mathbb{N}} h(t^r)^2 \ll \sum_{r\in\mathbb{N}} h(t^r)^2$$

By summing over $t \in \mathbb{P}$, we obtain

$$\sum_{t\in\mathbb{P}} \sum_{r\in\mathbb{N}} h(t^r)^2 \leq \|h\|_2^2 \leq \|h\|_p^2 = \epsilon^2$$

• Finally, we consider the third term of (3.6), and we start by noting that for $a, b \in \mathcal{A}$, we can write

$$\sum_{n=1}^{k} a(n) \sum_{m=1}^{k} b(m) = \sum_{n=1}^{k} a(n) \sum_{m=1}^{n} b(m) + \sum_{m=1}^{k} b(m) \sum_{n=1}^{m-1} a(n) \leq \sum_{n=1}^{k} \sum_{m=1}^{n} a(n)b(m) + \sum_{m=1}^{k} \sum_{n=1}^{m} b(m)a(n)$$

Applying this to the third term of (3.6) gives

$$\begin{aligned}\sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r) \sum_{s=1}^{k} f\left(t^{k-s}\right) h(t^s) \leq & \sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r) \sum_{s=1}^{r} f\left(t^{k-s}\right) h(t^s) \\ & + \sum_{s=1}^{k} f\left(t^{k-s}\right) h(t^s) \sum_{r=1}^{s} f\left(t^{k-r}\right) z(t^r). \qquad (3.7)\end{aligned}$$

We concentrate on the first term of (3.7) on the RHS. For ease of notation, we will briefly denote $g(r,k) = \sum_{s=1}^{r} f\left(t^{k-s}\right) h(t^s)$. We can rearrange to give

$$\sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r) g(r,k) = \sum_{r\in\mathbb{N}} \sum_{k=r}^{\infty} f\left(t^{k-r}\right) z(t^r) g(r,k) = \sum_{k\in\mathbb{N}_0} f\left(t^{k}\right) \sum_{r\in\mathbb{N}} z(t^r) g(r,k+r).$$

Furthermore,

$$\begin{aligned}\sum_{r\in\mathbb{N}} z(t^r) g(r,k+r) &= \sum_{r\in\mathbb{N}} z(t^r) \sum_{s=1}^{r} f\left(t^{k+r-s}\right) h(t^s) = \sum_{s\in\mathbb{N}} \sum_{r=s}^{\infty} z(t^r) f\left(t^{k-s+r}\right) h(t^s) \\ &= \sum_{r\in\mathbb{N}_0} \sum_{s\in\mathbb{N}} z(t^{r+s}) f\left(t^{k+r}\right) h(t^s).\end{aligned}$$

Therefore, by putting these together, we have

$$\sum_{k\in\mathbb{N}} \sum_{r=1}^{k} f\left(t^{k-r}\right) z(t^r) \sum_{s=1}^{r} f\left(t^{k-s}\right) h(t^s) = \sum_{k\in\mathbb{N}_0} f\left(t\right)^{2k} \sum_{r\in\mathbb{N}_0} f\left(t^r\right) \sum_{s\in\mathbb{N}} z(t^{r+s}) h(t^s)$$

As $f \in \mathcal{M}_c^2$, it follows that

$$\sum_{k\in\mathbb{N}_0} f\left(t\right)^{2k} \sum_{r\in\mathbb{N}_0} f\left(t^r\right) \sum_{s\in\mathbb{N}} z(t^{r+s}) h(t^s) \ll \sum_{r\in\mathbb{N}_0} f\left(t\right)^{r} \sum_{s\in\mathbb{N}} z(t^{r+s}) h(t^s).$$

Summing over $t \in \mathbb{P}$ and then isolating the term when $r = 0$ yields

$$\sum_{t\in\mathbb{P}} \sum_{r\in\mathbb{N}_0} f\left(t\right)^{r} \sum_{s\in\mathbb{N}} z(t^{r+s}) h(t^s) = \sum_{t\in\mathbb{P}} \sum_{s\in\mathbb{N}} z(t^s) h(t^s) + \sum_{r\in\mathbb{N}} \sum_{t\in\mathbb{P}} f\left(t\right)^{r} \sum_{s\in\mathbb{N}} z(t^{r+s}) h(t^s) \tag{3.8}$$

We note here that by the Cauchy-Schwarz inequality,

$$\sum_{t\in\mathbb{P}} \sum_{s\in\mathbb{N}} z(t^s) h(t^s) \leq \sum_{n\in\mathbb{N}} z(n) h(n) \leq \left(\sum_{n\in\mathbb{N}} z(n)^2 \sum_{n=1} h(n)^2 \right)^{\frac{1}{2}} = \|z\|_2 \|h\|_2 \ll \epsilon.$$

Therefore, (3.8) is at most of the order

$$\epsilon + \sum_{r\in\mathbb{N}} \sum_{t\in\mathbb{P}} f\left(t\right)^{r} \sum_{s\in\mathbb{N}} z(t^{r+s}) h(t^s) \tag{3.9}$$

Observe that, by the Cauchy-Schwarz inequality, we have

$$\sum_{\substack{t\in\mathbb{P}\\ s\in\mathbb{N}}} f(t)^r z(t^{r+s})h(t^s) \leq \left(\sum_{\substack{t\in\mathbb{P}\\ s\in\mathbb{N}}} \left(f(t)^r z(t^{r+s})\right)^2 \sum_{\substack{t\in\mathbb{P}\\ s\in\mathbb{N}}} h(t^s)^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{t\in\mathbb{P}} f(t)^{2r} \sum_{s\in\mathbb{N}} z(t^{r+s})^2 \sum_{\substack{t\in\mathbb{P}\\ s\in\mathbb{N}}} h(t^s)^2 \right)^{\frac{1}{2}}$$

which is at most

$$\|h\|_2\|z\|_2 \left(\sum_{t\in\mathbb{P}} f(t)^{2r} \right)^{\frac{1}{2}}.$$

As $f \in \mathcal{M}_c^2 \implies f(t)^r \leq c^r < 1$ for all $t \in \mathbb{P}$, we have

$$\sum_{t\in\mathbb{P}} f(t)^{2r} = \sum_{t\in\mathbb{P}} f(t)^2 f(t)^{2r-2} \leq c^{2r-2} \sum_{t\in\mathbb{P}} f(t)^2 \leq c^{2r-2}\|f\|_2^2 \ll c^{2r}.$$

Thus, (3.9) can be estimated as follows,

$$\begin{aligned}
\epsilon + \sum_{r\in\mathbb{N}} \sum_{t\in\mathbb{P}} f(t)^r \sum_{s\in\mathbb{N}} z(t^{r+s})h(t^s) &\leq \epsilon + \|h\|_2\|z\|_2 \sum_{r\in\mathbb{N}} \left(\sum_{t\in\mathbb{P}} f(t)^{2r} \right)^{\frac{1}{2}} \\
&\ll \epsilon + \|h\|_2\|z\|_2 \sum_{r\in\mathbb{N}} c^r \\
&\ll \epsilon + \|h\|_2 \\
&\leq \epsilon + \|h\|_p \\
&\ll \epsilon.
\end{aligned}$$

The same method can be applied to the second term of (3.7) to show that it is also of this order. In brief, for some constant $c > 0$,

$$\begin{aligned}\sum_{t\in\mathbb{P}}\sum_{s=1}^{k} f\left(t^{k-s}\right) h(t^s) \sum_{r=1}^{s} f\left(t^{k-r}\right) z(t^r) &\ll \epsilon + c\sum_{s\in\mathbb{N}}\sum_{t\in\mathbb{P}} f\left(t\right)^s \sum_{r\in\mathbb{N}} h(t^{s+r})z(t^r) \\ &\ll \epsilon + c\|z\|_2\|h\|_2 \sum_{s\in\mathbb{N}}\left(\sum_{t\in\mathbb{P}} f\left(t\right)^{2s}\right)^{\frac{1}{2}} \\ &\ll \epsilon.\end{aligned}$$

Putting these together with the previous statements, this shows that all the terms of (3.6) are of the order ϵ. Therefore, we can conclude

$$\sum_{t\in\mathbb{P}} \eta(t) \ll \epsilon = \|x - z\|_p \to 0,$$

as $x \to z$, as required.

$\square$

Theorem 3.10 shows that for $x = x_1 + x_2$, we have $\|D_f x\|_2 \leq C\|x\|_p$. Attempts to show that D_f is bounded for all $x \in \operatorname{span}(\mathcal{M}_c^p)$ have so far not been successful. Even the case when $x = \lambda_1 x_1 + \lambda_2 x_2$ is unclear. Indeed,

$$\|D_f(\lambda_1 x_2 + \lambda_2 x_2)\|_2 \leq |\lambda_1|\|D_f x_1\|_2 + |\lambda_2|\|D_f x_2\|_2 \leq |\lambda_1|c_1\|x_1\|_2 + |\lambda_2|c_2\|x_2\|_2.$$

However, this is not necessarily smaller than $\|\lambda_1 x_1 + \lambda_2 x_2\|_2$. For example, for $\lambda_1 = 1$, $\lambda_2 = -1$ and $x_1 = x_2 \neq 0$, we have $c_1\|x_1\|_p + c_2\|x_1\|_p \not\lesssim \|x_1 - x_1\|_p = 0$. Although, of course, $\|D_f(x_1 - x_1)\|_2 = 0$.

3.3 Possible counterexample

As establishing $f \in \ell^2$ as a sufficient condition for boundedness of $D_f : \ell^p \to \ell^2$ for $p \in (1, 2)$ has not been possible, we shall investigate the existence of a potential counterexample. That is, given $f \in \ell^2$, does there exist $x \in \ell^p$ for $p \in (1, 2)$, which is not completely multiplicative, such that $D_f x \notin \ell^2$?[21] For simplicity, we choose $f(n) = \frac{1}{n^\alpha}$ with $\alpha > \frac{1}{2}$, and denote D_f by D_α.

[21] Note that this is equivalent to D_f unbounded, see Section 4.2, Example 5 in [49].

Proposition 3.11. *Let $p \in (1,2)$, $q = 2$, and $\alpha > \frac{1}{2}$. Let $x \in \ell^p$. If $x(n) \ll 1/d(n)^{\frac{1}{2-p}}$, then $D_\alpha x \in \ell^2$.*

Proof. First, observe that we can assume x non-negative as D_α is largest when x is positive. Now recall that $f * x = x * f$ and so $y(n) = \sum_{d|n} \frac{d^\alpha x(d)}{n^\alpha} = \sum_{d|n} \frac{x(n/d)}{d^\alpha}$. By the Cauchy-Schwarz inequality, we have

$$|y(n)|^2 \leq \left(\sum_{c|n} \frac{x(n/c)}{c^\alpha} \right)^2 \leq \sum_{c|n} 1 \sum_{c|n} \frac{x(n/c)^2}{c^{2\alpha}} = d(n) \sum_{c|n} \frac{x(n/c)^2}{c^{2\alpha}}.$$

So,

$$\|D_\alpha x\|_2^2 \leq \sum_{n\in\mathbb{N}} d(n) \sum_{c|n} \frac{x(n/c)^2}{c^{2\alpha}} = \sum_{c\in\mathbb{N}} \sum_{m\in\mathbb{N}} d(mc) \frac{x(m)^2}{c^{2\alpha}} \leq \sum_{c\in\mathbb{N}} \frac{d(c)}{c^{2\alpha}} \sum_{m\in\mathbb{N}} d(m) x(m)^2,$$

as $d(mn) \leq d(m)d(n)$. As $\alpha > \frac{1}{2}$, the first series on the RHS is convergent (and given by $\zeta(2\alpha)^2$). Hence,

$$\|D_\alpha x\|_2^2 \leq C \sum_{m\in\mathbb{N}} d(m) x(m)^2,$$

for some constant C. This summation is convergent if $x(m)^2 d(m) \ll x(m)^p$ as $x \in \ell^p$. By rearranging, this is equivalent to $x(m) \ll 1/d(m)^{\frac{1}{2-p}}$ as required. $\square$

From Proposition 3.11, we can conclude that any counterexample, say x, must satisfy $|x(n)| > 1/d(n)^{\frac{1}{2-p}}$ for infinitely many $n \in \mathbb{N}$. As such, given $x \in \ell^p$, we define

$$\mathcal{S} = \left\{ n \in \mathbb{N} : |x(n)| > 1/d(n)^{\frac{1}{2-p}} \right\},$$

and we may assume that the support of x is contained within the set $\mathcal{S}$, i.e. $x(n) = 0$ if $n \notin \mathcal{S}$. However, some care must be taken in choosing $\mathcal{S}$ (if an example is possible) as

$$\sum_{n\in\mathcal{S}} \frac{1}{d(n)^{\frac{p}{2-p}}} \leq \sum_{n\in\mathcal{S}} |x(n)|^p < \infty, \tag{3.10}$$

must be satisfied since $x \in \ell^p$. First, $\mathcal{S}$ must be a "sparse" set; consider the function which counts the number of $n \in \mathcal{S}$ below a given c, $\mathcal{S}(c) = \sum_{\substack{n \le c \\ n \in \mathcal{S}}} 1$. Then

$$\mathcal{S}(c) = \sum_{\substack{n \le c \\ n \in \mathcal{S}}} \left| \frac{x(n)}{x(n)} \right|^p \ll c^\epsilon \sum_{\substack{n \le c \\ n \in \mathcal{S}}} |x(n)|^p \ll c^\epsilon \text{ for all } \epsilon > 0,$$

as $1/|x(n)|^p \le d(n)^{\frac{p}{2-p}} \ll n^\epsilon \le c^\epsilon$ for all $\epsilon > 0$, see Proposition 1.42. For example, choosing $\mathcal{S} = \mathbb{N}$ fails. Secondly, $\mathcal{S}$ must contain n with large numbers of divisors, otherwise $1/d(n)^{\frac{p}{2-p}} \not\to 0$ as $n \to \infty$ and so (3.10) will not be satisfied ($\mathcal{S}$ cannot be a subset of $\mathbb{P}$, for example). However, the following example indicates the difficulty of choosing $\mathcal{S}$ to yield D_α unbounded: suppose $\mathcal{S} = \left\{2^k : k \in \mathbb{N}\right\}$. We see that (3.10) is satisfied since

$$\sum_{n \in \mathcal{S}} \frac{1}{d(n)^{\frac{p}{2-p}}} = \sum_{k \in \mathbb{N}} \frac{1}{(k+1)^{\frac{p}{2-p}}} < \infty \quad \text{as} \quad \frac{p}{2-p} > 1 \quad \text{for} \quad p \in (1,2).$$

Now,

$$y_n = \sum_{2^k | n} \frac{2^{k\alpha} x(2^k)}{n^\alpha}.$$

Write $n = 2^l m$ where m is odd. Then, for some $\delta > 0$,

$$\begin{aligned} |y_{2^l m}|^2 &= \left| \sum_{k=0}^{l} \frac{x(2^k)}{(2^{l-k} m)^\alpha} \right|^2 = \frac{1}{m^{2\alpha}} \left| \sum_{k=0}^{l} \frac{x(2^{l-k})}{2^{k\alpha}} \right|^2 = \frac{1}{m^{2\alpha}} \left| \sum_{k=0}^{l} \frac{x(2^{l-k})}{2^{k(\alpha-\delta)}} \frac{1}{2^{k\delta}} \right|^2 \\ &\le \frac{1}{m^{2\alpha}} \sum_{k=0}^{l} \left| \frac{x(2^{l-k})}{2^{k(\alpha-\delta)}} \right|^2 \sum_{k=0}^{l} \frac{1}{2^{2k\delta}} \ll \frac{1}{m^{2\alpha}} \sum_{k=0}^{l} \left| \frac{x(2^{l-k})}{2^{k(\alpha-\delta)}} \right|^2 . \end{aligned}$$

We now sum over all l and m,

$$\begin{aligned} \sum_{l \in \mathbb{N}_0} \sum_{\substack{m \in \mathbb{N} \\ m \text{ odd}}} |y_{2^l m}|^2 &\ll \sum_{l \in \mathbb{N}_0} \sum_{\substack{m \in \mathbb{N} \\ m \text{ odd}}} \frac{1}{m^{2\alpha}} \sum_{k=0}^{l} \left| \frac{x(2^{l-k})}{2^{k(\alpha-\delta)}} \right|^2 \ll \sum_{l \in \mathbb{N}_0} \sum_{k=0}^{l} \left| \frac{x(2^{l-k})}{2^{k(\alpha-\delta)}} \right|^2 \\ &\ll \sum_{k \in \mathbb{N}_0} \sum_{l \in \mathbb{N}_0} \left| \frac{x(2^l)}{2^{k(\alpha-\delta)}} \right|^2 = \sum_{k \in \mathbb{N}_0} \frac{1}{2^{2k(\alpha-\delta)}} \sum_{l \in \mathbb{N}_0} \left| x(2^l) \right|^2 \ll \|x\|_2^2, \end{aligned}$$

which is finite as $\|x\|_2^2 \le \|x\|_p^2 < \infty$. The following proposition suggests some further structure of $\mathcal{S}$.

Proposition 3.12. *Let $\alpha > \frac{1}{2}$ and $\beta = \frac{p}{(2-p)(2\alpha-1)}$. Let $x \in \ell^p$ be supported on $\mathcal{S} = \left\{n \in \mathbb{N} : |x(n)| > 1/d(n)^{\frac{1}{2-p}}\right\}$, where $p \in (1,2)$. And, we write $D_\alpha x = \gamma + \mu$, where $\gamma = (\gamma_n)$ and $\mu = (\mu_n)$ are given by*

$$\gamma(n) = \sum_{\substack{d|n \\ d\in\mathcal{S} \\ d\le \frac{n}{d(n)^\beta}}} \frac{d^\alpha x(d)}{n^\alpha} \quad \text{and } \mu(n) = \sum_{\substack{d|n \\ d\in\mathcal{S} \\ d> \frac{n}{d(n)^\beta}}} \frac{d^\alpha x(d)}{n^\alpha}.$$

Then, $\gamma \in \ell^2$.

Proof. First, observe that we can assume x non-negative as D_α is largest when x is positive. For ease of notation, we shall use $\mathcal{S}_-$ to denote the set $\left\{c \in \mathcal{S} : c < \frac{n}{d(n)^\beta}\right\}$. By the Cauchy-Schwarz inequality,

$$\gamma(n)^2 \le \sum_{\substack{c|n \\ c\in\mathcal{S}_-}} x(c)^2 \sum_{\substack{c|n \\ c\in\mathcal{S}_-}} (c/n)^{2\alpha} \le \|x\|_2^2 \sum_{\substack{c|n \\ c\in\mathcal{S}_-}} (c/n)^{2\alpha} \ll \sum_{\substack{c|n \\ c\in\mathcal{S}_-}} (c/n)^{2\alpha},$$

as $\|x\|_2 \le \|x\|_p < \infty$. Since $c \mid n \implies cm = n$ for some $m \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{n\in\mathbb{N}} \gamma(n)^2 &\ll \sum_{n\in\mathbb{N}} \sum_{\substack{c|n \\ c\in\mathcal{S}_-}} (c/n)^{2\alpha} \le \sum_{c\in\mathcal{S}} \sum_{\substack{m\in\mathbb{N} \\ d(cm)^\beta \le m}} \frac{1}{m^{2\alpha}} \le \sum_{c\in\mathcal{S}} \sum_{d(c)^\beta \le m} \frac{1}{m^{2\alpha}} \\ &\ll \sum_{c\in\mathcal{S}} \frac{1}{d(c)^{\beta(2\alpha-1)}},\end{aligned}$$

as, for $s > 1$,

$$\sum_{n\ge m} \frac{1}{n^s} \ll m^{1-s},$$

(see [2], page 55). As $x \in \ell^p$, we have

$$\sum_{c\in\mathcal{S}} \frac{1}{d(c)^{\beta(2\alpha-1)}} = \sum_{c\in\mathcal{S}} \frac{1}{d(c)^{\frac{p}{2-p}}} \le \sum_{c\in\mathcal{S}} x(c)^p < \infty,$$

as required. □

From Proposition 3.12, we can see that any counterexample must yield $\mu \notin \ell^2$. Note that by writing $d \mapsto \frac{n}{d}$, we have

$$\mu(n) = \sum_{\substack{d|n \\ \frac{n}{d} \in \mathcal{S} \\ d < d(n)^\beta}} \frac{x(n/d)}{d^\alpha}.$$

So, $\mathcal{S}$ must contain $n \in \mathbb{N}$ such that n has a large number of small divisors so that $d < d(n)^\beta$ is satisfied often and, in turn, ensuring that many divisors contribute to the summation. Finding a suitable support set $\mathcal{S}$ which yields $\mu \notin \ell^2$ has not been possible, and the investigation gives little indication of a successful counterexample.

The results presented demonstrate the challenging nature of the open question posed at the beginning of the chapter. Theorem 3.1 gives a partial criterion which provides a sufficient condition for boundedness. From the investigation into whether this is also necessary for the simpler case D_f, another connection with multiplicative number theory and these operators emerges. The connection, which is echoed by the literature discussed in Section 2.4, leads us to consider the boundedness of D_f on $\mathcal{M}_c^p \to \ell^2$. A new sufficient condition is given in Theorem 3.8. It is unclear which, if either, of these sufficient conditions is also a necessary condition, at least in the case when $p \in (1, 2)$ and $q = 2$. We conclude this chapter by summarising the open problems that have arisen:

- Is $f \in \mathcal{M}_c^r$ a necessary condition for $D_f : \ell^p \to \ell^q$ to be bounded for any p and q?
- Can we generalise Theorem 3.8 from multiplicative subsets to $D_f : \ell^p \to \ell^2$ for $p = (1, 2)$? Equivalently, is $\|D_f x\|_p \leq C\|x\|_2$ for all $x \in \text{span}(\mathcal{M}_c^p)$ and $p \in (1, 2)$?
- Or can we find a counterexample to this?
- Does D_f attain its supremum on $\mathcal{M}_c$ for general p and q?

Chapter 4

Spectral properties on Besicovitch spaces

In this chapter, we investigate the second set of open questions as discussed in Chapter 2; what is the spectrum of M_F and what can this tell us about the classical setting? The main results are presented within Section 4.1 where we characterise the invertibility of the operator D_F and, moreover, we establish the spectrum of D_F. As a result, we also find the spectrum of the multiplication operator on $\mathcal{W}(\mathbb{T}^\infty)$.

In the latter sections, we find an expression for the semi-commutator, $M_{FG} - M_F M_G$, and give an example of when the semi-commutator is not compact. Finally, some further observations on Coburn's lemma and Wiener's factorisation are made.

4.1 Spectrum of D_F

Our investigation into the spectrum of M_F starts with the simpler case of D_F. We begin with the observation that the point spectrum of D_F is easily found.

Proposition 4.1. *Let $F \in \mathcal{W}_{\mathbb{N}}$. Then D_F has an eigenvalue if and only if F is constant, in which case the constant is the eigenvalue. Equivalently, let $f \in \ell^1$. Then $D_f : \ell^2 \to \ell^2$ has an eigenvalue if and only if $f = \lambda e_1$ where $\lambda \in \mathbb{C}$, in which case λ is the eigenvalue.*

Proof. Suppose $\lambda \in \mathbb{C}$ and that $D_f x = \lambda x$ for $x \in \ell^2$ where x is non-zero. Then $f * x = \lambda e_1 * x$, so $f = \lambda e_1$. $\square$

Before we present the main results of this section, we require some further definitions and results. Let $\mathbb{C}_+ = \{s \in \mathbb{C} : \Re s > 0\}$, and let $F \in \mathcal{W}_{\mathbb{N}}$ be given by $\sum_{n\in\mathbb{N}} f(n)n^{it}$. We use $\tilde{F}$ to denote the series given by

$$\tilde{F}(s) = \sum_{n\in\mathbb{N}} f(n)n^{-s} \text{ for } s \in \mathbb{C}_+.$$

In other words, $\tilde{F}$ denotes the Dirichlet series formed from the Dirichlet Fourier coefficients of F. A generalisation of the celebrated Wiener's Lemma, Theorem 1.16 to Dirichlet Fourier series is stated below.

Theorem 4.2.

1. *Let $F \in \mathcal{W}_{\mathbb{Q}^+}$. Then*

$$F^{-1} \in \mathcal{W}_{\mathbb{Q}^+} \iff 0 \notin \overline{\operatorname{ran}(F)}.$$

2. *Let $F \in \mathcal{W}_{\mathbb{N}}$. Then*

$$F^{-1} \in \mathcal{W}_{\mathbb{N}} \iff 0 \notin \overline{\operatorname{ran}(\tilde{F})}.$$

Proof. See Theorems[22] 1 and 2 in [29]. □

It is useful to note how these statements differ. For $F \in \mathcal{W}_{\mathbb{Q}^+}$ to be invertible, we require that F must be bounded away from 0. In contrast, for $F \in \mathcal{W}_{\mathbb{N}}$ to be invertible in $\mathcal{W}_{\mathbb{N}}$, we require $\tilde{F}$ to be bounded away from 0 for $s \in \mathbb{C}_+$. We make the observation that similar behaviour is seen in Wiener's Lemma in the classical setting, see Theorem 1.16. For $\Phi \in \mathcal{W}$, Φ must be non-zero on the boundary of the unit disc. Whereas, by the properties of holomorphic functions, for $\Phi \in \mathcal{W}$ whose coefficients are supported on $\mathbb{N}$, Φ must be non-zero for all points in the disc.

Finally, recall that $\mathcal{D}^2$ denotes the space of Dirichlet series for which $\sum_{n\in\mathbb{N}} |f(n)|^2 < \infty$. We are now in a position to present the first of the main findings within this chapter; a result which characterises the invertibility of the operator D_F.

[22] Note here that the statement of Theorem 4.2 corresponds with the original statement of the theorem as given in [29], since $\tilde{F}$ is continuous. Indeed, for $\lambda \in \mathbb{C}_+$, $0 \notin \overline{\operatorname{ran}(\tilde{F})}$ if and only if there exists $\epsilon > 0$ such that $\epsilon \leq |\tilde{F}(s)|$ for all $s \in \mathbb{C}_+$. Moreover, if $|\tilde{F}(s)| > \epsilon$ for all $s \in \mathbb{C}_+$ then it follows that $|\tilde{F}(s)| > \epsilon$ for all $\Re s \geq 0$.

Theorem 4.3. *Let $F \in \mathcal{W}_{\mathbb{N}}$. Then*

$$D_F \text{ is invertible } \iff F^{-1} \in \mathcal{W}_{\mathbb{N}}, \text{ in which case } D_F^{-1} = D_{F^{-1}}.$$

Proof. We shall begin by assuming that $F^{-1} \in \mathcal{W}_{\mathbb{N}}$. Then, for all $X \in \mathcal{B}^2_{\mathbb{N}}$,

$$D_F D_{F^{-1}} X = F F^{-1} X = X \quad \text{and} \quad D_{F^{-1}} D_F X = F^{-1} F X = X.$$

Hence, D_F is invertible and is given by $D_{F^{-1}}$.

We will now prove the other direction. Suppose that D_F is invertible then there exists an operator $L \in B(\mathcal{B}^2_{\mathbb{N}})$ such that

$$LD_F = D_F L = I,$$

where I is the identity operator. That is, for all $X \in \mathcal{B}^2_{\mathbb{N}}$,

$$L(FX)(s) = F(s)L(X)(s) = X(s). \tag{4.1}$$

From the isometric isomorphism given in (2.6), we can define the operator $\tilde{L} \in B(\mathcal{D}^2)$ by $\tilde{L} = \tau L \tau^{-1}$. Then, (4.1) becomes

$$\tilde{L}(\tilde{F}\tilde{X})(s) = \tilde{F}(s)\tilde{L}(\tilde{X})(s) = \tilde{X}(s), \tag{4.2}$$

for all $\tilde{X} \in \mathcal{D}^2$. Choose $\tilde{X}(s) = 1$ for all $s \in \mathbb{R}$. From above, we have

$$\tilde{F}(s)\tilde{L}(1)(s) = 1 \text{ for } \Re s > \frac{1}{2}.$$

Therefore,

$$\tilde{F}(s) \neq 0 \text{ for } \Re s > \frac{1}{2}.$$

As such, $\tilde{F}^{-1}(s)$ exists for $\Re s > 1/2$. Now, multiplying (4.2) by $\tilde{F}^{-1}$, we have

$$\tilde{F}^{-1}(s)\tilde{F}(s)\tilde{L}(\tilde{X})(s) = \tilde{L}(\tilde{X})(s) = \tilde{F}^{-1}(s)\tilde{X}(s).$$

As $\tilde{L} \in B(\mathcal{D}^2)$, it follows that

$$\tilde{F}^{-1}(s)\tilde{X}(s) \in \mathcal{D}^2 \text{ for all } \tilde{X} \in \mathcal{D}^2,$$

and by choosing $\tilde{X}(s) = 1$, we have

$$\tilde{F}^{-1} \in \mathcal{D}^2.$$

In particular, $\tilde{F}^{-1}(s)$ is analytic in the half plane $\Re s > 1/2$. Now, by Theorem 2.12, it follows that $\tilde{F}^{-1}(s)$ extends to a bounded analytic Dirichlet series on $\mathbb{C}_+$. That is, there exists $c \in \mathbb{R}$ such that

$$\left|\tilde{F}^{-1}(s)\right| \leq c \text{ for all } s \in \mathbb{C}_+.$$

Therefore,

$$0 < 1/c \leq \left|\tilde{F}(s)\right| \text{ for all } s \in \mathbb{C}_+.$$

Therefore, $0 \notin \overline{\operatorname{ran}(\tilde{F})}$. Now by Theorem 4.2, F is invertible in $\mathcal{W}_{\mathbb{N}}$ as required. $\square$

Following on from Theorem 4.3, we now establish the spectrum of D_F.

Theorem 4.4. *Let $F \in \mathcal{W}_{\mathbb{N}}$. Then $\sigma(D_F) = \overline{\operatorname{ran}(\tilde{F})}$.*

Proof. Suppose that $\lambda \notin \overline{\operatorname{ran}(\tilde{F})}$. By linearity, it follows that $0 \notin \overline{\operatorname{ran}(\tilde{F} - \lambda)}$. From Theorem 4.2, we can conclude that $(F - \lambda)^{-1}$ exists and lies in $\mathcal{W}_{\mathbb{N}}$. Hence, by Theorem 4.3, $D_{F-\lambda}$ is invertible, i.e. $\lambda \notin \sigma(D_F)$.

Let $\lambda \notin \sigma(D_F)$. Therefore, $D_F - \lambda I = D_{F-\lambda}$ is invertible. By Theorem 4.3, $(F - \lambda)^{-1}$ exists in $\mathcal{W}_{\mathbb{N}}$. It follows from Theorem 4.2, that $0 \notin \overline{\operatorname{ran}(\tilde{F} - \lambda)}$. That is, $\lambda \notin \overline{\operatorname{ran}(\tilde{F})}$ as required. $\square$

Example 4.5. Let $f(n) = n^{-\alpha}$ for some $\alpha > 1$, so $F(t)$ is given by $\sum_{n \in \mathbb{N}} f(n) n^{it} = \zeta(\alpha - it)$. We denote the operator D_F by D_ζ. Observe that

$$\tilde{F}(s) = \sum_{n \in \mathbb{N}} n^{-(\alpha+s)} = \zeta(\alpha + s).$$

Therefore, by Theorem 4.4,

$$\sigma\left(D_{\zeta}\right) = \overline{\{\zeta(s) \text{ for } \Re s \geq \alpha\}}.$$

Example 4.6. Suppose F has Dirichlet Fourier coefficients supported on positive powers of 2. In this case, the operator D_F is given by C_Φ, and Theorem 4.4 yields

$$\sigma\left(C_{\Phi}\right) = \overline{\Phi(\mathbb{D})}$$

since $\tilde{F}(s) = \sum_{k\in\mathbb{N}} f(2^k)2^{k-s} = \sum_{k\in\mathbb{N}} \phi(k)t^k = \Phi(t)$ where $\phi(k) = f(2^k)$ and $t = 2^{-s}$. This corresponds with Theorem 2.22.

Theorem 4.4 also yields the spectrum of the multiplication operator acting upon infinite dimensions.

Corollary 4.7. *Let $\Phi \in \mathcal{W}(\mathbb{T}^\infty)$. Then $\sigma\left(C_{\Phi}\right) = \overline{ran(\tilde{\Phi})}$*

Proof. The result follows immediately from Theorem 4.4, by using the Bohr lift to write D_F as C_Φ acting on the infinite polydisc. □

Essential spectrum of D_F

We conclude this section with a brief comment on the essential spectrum. As $\sigma_{\mathrm{e}}\left(D_F\right) \subset \sigma\left(D_F\right)$ follows immediately, we can investigate whether $\sigma\left(D_F\right) \subset \sigma_{\mathrm{e}}\left(D_F\right)$. Assume that $\lambda \notin \sigma_{\mathrm{e}}\left(D_F\right)$. We wish to show that $\lambda \notin \sigma\left(D_F\right)$ i.e. $D_F - \lambda I$ is invertible. Recall from Section 1.3, it suffices to show that $\ker(D_F - \lambda I) = \{0\}$ and $\operatorname{im}(D_F - \lambda I) = \mathcal{B}^2_{\mathbb{N}}$. Since $\lambda \notin \sigma_{\mathrm{e}}\left(D_F\right)$, by definition $\dim\ker(D_F - \lambda I) < \infty$.

Corollary 4.8. *Let $F \in \mathcal{W}_{\mathbb{N}}$. Then either* $\ker(D_F) = \{0\}$ *or* $\ker(D_F) = \mathcal{B}^2_{\mathbb{N}}$.

Proof. Let $F \in \mathcal{W}_{\mathbb{N}}$ and let $X \in \mathcal{B}^2_{\mathbb{N}}$ such that $X \neq 0$. If $X \in \ker(D_F)$, then $FX = 0 = 0X$. That is, 0 is an eigenvalue. Now by Proposition 4.1, $F = 0$. Therefore, $\ker(D_F) = \mathcal{B}^2_{\mathbb{N}}$. □

Applying Corollary 4.8 yields $\ker(D_F - \lambda I) = \{0\}$. Therefore, it remains to prove that $\operatorname{im}(D_F - \lambda I) = \mathcal{B}^2_{\mathbb{N}}$ which is an open problem. Recall that given an operator L acting on a Hilbert space $\mathcal{H}$, we have the following identity

$$\operatorname{im}(L) \oplus \ker(L^*) = \mathcal{H}.$$

Therefore, if it were possible to establish that $\ker((D_f - \lambda I)^*) = \{0\}$, then it would follow that $\operatorname{im}(D_F - \lambda I) = \mathcal{B}^2_{\mathbb{N}}$. On the other hand, we observe here that the failure of a Coburn's Lemma type result would show that there exists a symbol that is analytic for which $\ker(D_F - \lambda I) \neq \{0\}$. This, by the above identity, would yield that $\operatorname{im}(D_F - \lambda I) \neq \mathcal{B}^2_{\mathbb{N}}$. As previously discussed in Section 2.4.2, Coburn's Lemma has been shown to fail in the two dimensional case (see [17]), however the symbol considered is not analytic and, therefore, this result can not be utilised.

4.2 Spectral properties of M_F

In the rest of this chapter we present findings regarding the spectral behaviour of M_F for $F \in \mathcal{W}_{\mathbb{Q}^+} \setminus \mathcal{W}_{\mathbb{N}}$. Recall, from Section 1.3.1, that the spectrum of compact operators can be easily characterised and, therefore, we begin by considering compactness. The following theorem shows that, like in the classical setting, M_F is never compact, except in the trivial case when $F = 0$. A more general result is given for $M_f : \ell^p \to \ell^q$ for $1 < p \leq q \leq \infty$. By taking $p = q = 2$, the non-compactness of M_F is obtained.

Theorem 4.9. *Let $1 < p \leq q < \infty$. Let $f \in \mathcal{A}$ such that $M_f : \ell^p \to \ell^q$ is bounded. The operator $M_f : \ell^p \to \ell^q$ is only compact when $f = 0$.*

Proof. Recall the sequence $(e_n)_{n\in\mathbb{N}}$ forms an orthonormal basis in ℓ^p and, therefore, converges weakly to 0 (see Example 4.8-6 in [37]). Now, suppose $M_f : \ell^p \to \ell^q$ is compact. We know, from Theorem 8.1-7 in [37], that a compact operator maps a weakly convergent sequence, namely $(x_j) \rightharpoonup x$, to a strongly convergent sequence, $M_f x_j$ whose limit is $M_f x$. So, for every $\epsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$\|M_f e_j - M_f 0\|_q^q \leq \epsilon^q.$$

Therefore,

$$\sum_{n\in\mathbb{N}} \left| \sum_{d\in\mathbb{N}} f\left(\frac{n}{d}\right) e_j(d) \right|^q \leq \epsilon^q$$

Since $e_j(d) = 0$ when $d \neq j$, we have

$$\sum_{n\in\mathbb{N}} \left| f\left(\frac{n}{j}\right)\right|^q \leq \epsilon^q$$

In particular, $|f(n/j)|^q \leq \epsilon^q$ for all $n \in \mathbb{N}$ and $j \geq j_0$. We note that given any $s \in \mathbb{Q}^+$, we can choose n, j such that $s = n/j$. Let $s = u/v \in \mathbb{Q}^+$, where $(u, v) = 1$. Choose $n = uj_0$ and $j = vj_0$. Of course, $n \in \mathbb{N}$ and $j \geq j_0$. Moreover, $n/j = u/v$. Therefore,

$$|f(s)| < \epsilon \text{ for all } s \in \mathbb{Q}^+.$$

As this is true for all $\epsilon > 0$, we must have $f(s) = 0$ for $s \in \mathbb{Q}^+$. □

4.2.1 Semi-commutator of M_F

As previously mentioned, the challenge with establishing the spectrum of T_Φ is that the multiplication of two Toeplitz operators is not, in general, a Toeplitz operator itself, and in particular the semi-commutator $T_{\Phi\Gamma} - T_\Phi T_\Gamma \neq 0$. In this section, we give an explicit formula for the semi-commutator of M_F. For $F \in \mathcal{W}_{\mathbb{Q}^+}$ and $m \in \mathbb{N}$, define

$$F_m(t) := \frac{F(t)}{m^{it}} = \left(\frac{1}{m}\right)^{it} \sum_{q\in\mathbb{Q}^+} f(q)q^{it} = \sum_{q\in\mathbb{Q}^+} f(qm)q^{it}.$$

Theorem 4.10. *Let $F, G \in \mathcal{W}_{\mathbb{Q}^+}$. The semi-commutator, $M_{FG} - M_F M_G$, is given by*

$$\sum_{m\geq 2} \sum_{d|m} \mu(d) M_{F_{d/m}} M_{G_{m/d}},$$

where $\mu(n)$ is the Möbius function.

Proof. We proceed by considering the matrix entries induced by the semi-commutator. Let $F, G \in \mathcal{W}_{\mathbb{Q}^+}$ and let A_F and A_G denote the matrix representations of M_F and M_G respectively[23]. By matrix multiplication, it follows that

$$(A_F A_G)_{i,j} = \sum_{n\in\mathbb{N}} (A_F)_{i,n} (A_G)_{n,j} = \sum_{n\in\mathbb{N}} f\left(\frac{i}{n}\right) g\left(\frac{n}{j}\right).$$

[23] recall that the entries of A_F are given by the Dirichlet Fourier coefficients $f(i/j)$ for $i, j \in \mathbb{N}$

Furthermore, observe that

$$FG = \sum_{q\in\mathbb{Q}^+} f(q)\chi_q \sum_{r\in\mathbb{Q}+} g(r)\chi_r = \sum_{s\in\mathbb{Q}^+}\sum_{qr=s} f(q)g(r)\chi_s = \sum_{s\in\mathbb{Q}^+} h(s)\chi_s$$

where $h(s) = \sum_{qr=s} f(q)g(r)$. Hence,

$$(A_{FG})_{i,j} = h(i/j) = \sum_{\substack{q,r\in\mathbb{Q}^+\\ qr=\frac{i}{j}}} f(q)g(r) = \sum_{w\in\mathbb{Q}^+} f\left(\frac{i}{w}\right) g\left(\frac{w}{j}\right),$$

where $w = jr$. By writing $w = n/m$ where $(n,m) = 1$, the above is equal to

$$\sum_{\substack{n,m\in\mathbb{N}\\ (n,m)=1}} f\left(\frac{im}{n}\right) g\left(\frac{n}{mj}\right).$$

Therefore, the matrix entries of the semi-commutator are given by

$$(A_{FG} - A_F A_G)_{i,j} = \sum_{m\geq 2} \sum_{\substack{n\in\mathbb{N}\\ (n,m)=1}} f\left(\frac{im}{n}\right) g\left(\frac{n}{mj}\right) \tag{4.3}$$

From Theorem 1.43, the above is equal to

$$= \sum_{m\geq 2}\sum_{n\in\mathbb{N}} \left(\sum_{d|(n,m)} \mu(d)\right) f\left(\frac{im}{n}\right) g\left(\frac{n}{mj}\right).$$

As $d \mid (n,m) \implies d \mid n$ and $d \mid m$, we can replace n by nd, which yields

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \left(\sum_{d|(n,m)} \mu(d)\right) f\left(\frac{im}{n}\right) g\left(\frac{n}{mj}\right) &= \sum_{d|m} \mu(d) \sum_{n\in\mathbb{N}} f\left(\frac{im}{dn}\right) g\left(\frac{dn}{mj}\right) \\
&= \sum_{d|m} \mu(d) \sum_{n\in\mathbb{N}} \left(A_{F_{\frac{m}{d}}}\right)_{i,n} \left(A_{G_{\frac{d}{m}}}\right)_{n,j} \\
&= \sum_{d|m} \mu(d) \left(A_{F_{\frac{m}{d}}} A_{G_{\frac{d}{m}}}\right)_{i,j}.
\end{aligned}$$

□

We recall from Section 2.4 that the semi-commutator in the two dimensional case is already known to be non-compact in general. The following gives an example which fails within the infinite dimensional case.

Example 4.11. Recall from Corollary 2.31 that for $F \in \mathcal{W}_{\overline{\mathbb{N}}}$ (with $G \in \mathcal{W}_{\mathbb{Q}^+}$) or $G \in \mathcal{W}_{\mathbb{N}}$ (with $F \in \mathcal{W}_{\mathbb{Q}^+}$), the semi-commutator is zero. This can also be computed from Theorem 4.10. Suppose that $F \in \mathcal{W}_{\overline{\mathbb{N}}}$. Observe that f never contributes to the summation in (4.3), since we must have

$$\frac{im}{n} = \frac{1}{k} \text{ for } k \in \mathbb{N}.$$

This means that $n/m = ki \in \mathbb{N}$. However, there does not exist n, m where this occurs since $(n, m) = 1$ with $m \geq 2$. The same argument applies for $G \in \mathcal{W}_{\mathbb{N}}$ with $F \in \mathcal{W}_{\mathbb{Q}^+}$.

Further, recall from Section 2.4 attempts have been made to classify if there are any classes of Toeplitz operators on two dimensions for which the semi-commutator is compact. The following example examines this in the infinite case. We firstly simplify expression (4.3) to the case when $F \in \mathcal{W}_{\mathbb{N}}$ and $G \in \mathcal{W}_{\mathbb{Q}^+}$.

Corollary 4.12. *Let $F \in \mathcal{W}_{\mathbb{N}}$ and $G \in \mathcal{W}_{\mathbb{Q}^+}$. Then*

$$(A_{FG} - A_F A_G)_{i,j} = \sum_{\substack{n \geq 1 \\ n \nmid i}} f(n) g\left(\frac{i}{nj}\right).$$

Proof. From the proof of Theorem 4.10,

$$(A_{FG})_{i,j} = h(i/j) = \sum_{\substack{n \in \mathbb{N}, q \in \mathbb{Q}^+ \\ nq = \frac{i}{j}}} f(n) g(q).$$

Fix $i, j \in \mathbb{N}$. Given any $n \in \mathbb{N}$, we can choose $q \in \mathbb{Q}^+$ such that $nq = \frac{i}{j}$. Hence, the above is equal to

$$\sum_{n \in \mathbb{N}} f(n) g\left(\frac{i}{nj}\right).$$

Furthermore, from the proof of Theorem 4.10,

$$(A_F A_G)_{i,j} = \sum_{n\in\mathbb{N}} f\left(\frac{i}{n}\right) g\left(\frac{n}{j}\right) = \sum_{m|i} f(m) g\left(\frac{i}{mj}\right)$$

as $n \mid i$ implies $i = mn$ for some $m \in \mathbb{N}$. Therefore,

$$(A_{FG} - A_F A_G)_{i,j} = \sum_{\substack{n\in\mathbb{N} \\ n\nmid i}} f(n) g\left(\frac{i}{nj}\right).$$

$\square$

Example 4.13. For a fixed $k \in \mathbb{N}$, let $s_1, \ldots, s_k \in \mathbb{N}$ be pairwise co-prime i.e $(s_n, s_m) = 1$ for every $1 \leq n, m \leq k$ when $n \neq m$. We put

$$\mathcal{S}_k = \left\{ \frac{1}{s_1}, \ldots, \frac{1}{s_k} \right\}.$$

Furthermore, let $\mathcal{W}_{\mathbb{N}\cup\mathcal{S}_k}$ be the subset of functions in $\mathcal{W}_{\mathbb{Q}^+}$, such that the coefficients are supported on $\mathbb{N} \cup \mathcal{S}_k$. Suppose $F \in \mathcal{W}_{\mathbb{N}}$ and $G \in \mathcal{W}_{\mathbb{N}\cup\mathcal{S}_k}$ such that $f(s_n) \neq 0$, and $g(1/s_n) \neq 0$, for some $n \in [1, k]$. Without loss of generality, we can assume that $f(s_k) \neq 0$ and $g(1/s_k) \neq 0$. Let $(i_m)_{m\in\mathbb{N}}$ be the sequence of positive integers such that $i_m \equiv s_1 \ldots s_{k-1} (\bmod s_1 \ldots s_k)$. That is,

$$i_m = (s_1 \ldots s_k)m + s_1 \ldots s_{k-1} = (s_1 \ldots s_{k-1})(s_k m + 1).$$

Clearly, $s_1, \ldots, s_{k-1} \mid i_m$. Moreover, as $s_1, \ldots, s_k$ are co-prime, it follows that $s_k \nmid i_m$ for all $m \in \mathbb{N}$. From Corollary 4.12,

$$(A_{FG} - A_F A_G)_{i_m,i_m} = \sum_{\substack{n\geq 1 \\ n\nmid i_m}} f(n) g\left(\frac{i_m}{n i_m}\right) = \sum_{\substack{n=1 \\ s_n\nmid i_m}}^{k} f(s_n) g\left(\frac{1}{s_n}\right) = f(s_k) g\left(\frac{1}{s_k}\right) \neq 0,$$

for all i_m. Since this does not converge to 0 as $i \to \infty$, the semi-commutator cannot be compact.

The above example shows that there exists a large class of symbols for which the semi-commutator is not compact, unless of course it is the trivial symbol. These symbols are those with coefficients supported on a finite number of points in $\mathbb{Q}^+$, We conclude this chapter with comments on the failure of further analogous results. Specifically, we consider the Wiener factorisation and Coburn's Lemma.

4.2.2 Wiener factorisation.

Recall that the Wiener-Hopf factorisation, given in Theorem 2.23, facilitated the description of the spectrum of T_Φ. It states that symbols in the Wiener algebra which are never zero are factorisable. Further, recall from Section 2.4.2, that $\mathcal{FW}_{\mathbb{Q}^+}$ is the set of $F \in \mathcal{W}_{\mathbb{Q}^+}$ that are factorisable. In other words, $F = F_-\chi_q F_+$ where $F_- \in \mathcal{W}_{\overline{\mathbb{N}}}$, $F_+ \in \mathcal{W}_{\mathbb{N}}$ are invertible and $\chi_q(t) = q^{it}$ with $t \in \mathbb{R}$ and $q \in \mathbb{Q}^+$. As stated in Theorem 2.30, it is known that if $F \in \mathcal{FW}_{\mathbb{Q}^+}$, then

$$M_F = M_{F_-} M_{\chi_q} M_{F_+}.$$

From this, given that $F \in \mathcal{FW}_{\mathbb{Q}^+}$, a necessary and sufficient condition for M_F to be invertible is given in Theorem 2.32. We ask therefore which symbols in F are factorisable.

Proposition 4.14. *Let $F \in \mathcal{W}_{\mathbb{Q}^+}$. Then,*

$$F \in \mathcal{FW}_{\mathbb{Q}^+} \iff F = \chi_q e^{G_- + G_+},$$

where $G_- \in \mathcal{W}_{\overline{\mathbb{N}}}$, and $G_+ \in \mathcal{W}_{\mathbb{N}}$.

Proof. Assume $F \in \mathcal{FW}_{\mathbb{Q}^+}$. Then, there exists $F_- \in \mathcal{W}_{\overline{\mathbb{N}}}$ and $F_+ \in \mathcal{W}_{\mathbb{N}}$ which are invertible such that $F = F_-\chi_q F_+$. From [30], (c) on page 95, it follows that

$$F_- = e^{G_-} \text{ and } F_+ = e^{G_+},$$

for some $G_- \in \mathcal{W}_{\overline{\mathbb{N}}}$ and $G_+ \in \mathcal{W}_{\mathbb{N}}$. Therefore,

$$F = e^{G_-}\chi_q e^{G_+} = \chi_q e^{G_- + G_+}.$$

Conversely, assume $F = \chi_q e^{G_- + G_+}$. Let $F_- = e^{G_-}$ and $F_+ = e^{G_+}$. Now, by Proposition 4.2, F_- and F_+ are invertible, and therefore, $F \in \mathcal{FW}_{\mathbb{Q}^+}$. $\square$

However, unlike the Wiener-Hopf factorisation, there exists many symbols that are

are bounded away from 0 (i.e. invertible), but not factorisable, and in which case, the invertibility of M_F is unknown. We show this through the following example.

Example 4.15. Let $f(q) = \frac{1}{n!}$ if $q = r^n$, and 0 otherwise, where $n \in \mathbb{N}_0$. Fix $r \in \mathbb{Q}^+$, such that $r, \frac{1}{r} \notin \mathbb{N}$. Then,

$$F(t) = \sum_{n\in\mathbb{N}_0} \frac{1}{n!}(r^n)^{it} = \sum_{n\in\mathbb{N}_0} \frac{(r^{it})^n}{n!} = e^{r^{it}}$$

Clearly $0 \notin \overline{\operatorname{ran}(F)}$, and by Theorem 4.2, F is invertible. Assume that $F \in \mathcal{FW}_{\mathbb{Q}^+}$. Then from Proposition 4.14,

$$F = \chi_q e^{G_- + G_+}$$

where $G_- \in \mathcal{W}_{\overline{\mathbb{N}}}$, and $G_+ \in \mathcal{W}_{\mathbb{N}}$ and $q \in \mathbb{Q}^+$. On the other hand, $\log F(t) = r^{it}$. Therefore,

$$r^{it} = it \log q + G_-(t) + G_+(t) \text{ for all } t \in \mathbb{R}.$$

Since r^{it} is bounded for all $t \in \mathbb{R}$, it must follow that $q = 1$. This gives

$$r^{it} = G_-(t) + G_+(t) \text{ for all } t \in \mathbb{R}.$$

By multiplying through by r^{-it}, integrating and taking limits, this becomes

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (r/r)^{it} dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G_-(t) r^{-it} dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G_+(t) r^{-it} dt$$

We note here that given $\lambda > 0$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \lambda^{it} dt = \begin{cases} 1 \text{ if } \lambda = 1, \\ 0 \text{ otherwise} \end{cases}$$

Therefore, we must have

$$1 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G_-(t) r^{-it} dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G_+(t) r^{-it} dt$$

Now since, G_- is absolutely convergent we can re-arrange as follows,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}G_-(t)r^{-it}dt=\sum_{n\in\mathbb{N}}g_-(n)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(rn)^{-it}dt=0,$$

since $nr\neq 1$ as $1/r\notin\mathbb{N}$. Similarly, for G_+, we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}G_+(t)r^{-it}dt=\sum_{n\in\mathbb{N}}g_+(n)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(n/r)^{it}dt=0$$

since $n/r\neq 1$ as $r\notin\mathbb{N}$. Therefore, we have $1=0$, which is contradiction. Thus, $F\notin\mathcal{F}\mathcal{W}_{\mathbb{Q}^+}$.

Example 4.15 highlights how the "size" of $\mathcal{W}_{\mathbb{Q}^+}$, in comparison with the analogous Wiener space in the classical setting, creates a significant difference in the behaviour of the operator. Namely, it is not true that functions in $\mathcal{W}_{\mathbb{Q}^+}$ can be written as the sum of two functions in $\mathcal{W}_{\mathbb{N}}$ and $\mathcal{W}_{\overline{\mathbb{N}}}$, which in turn leads to the failure of the Wiener-Hopf factorisation. It is, however, true in the classical setting; that is $\mathcal{L}^2=\mathcal{H}^2\oplus\overline{\mathcal{H}^2}$ (see Section 1.3 in [10]). Therefore, as long as the symbol of T_Φ has no zeros, the operator itself can be factorised.

4.2.3 Coburn's Lemma

First, we must compute the adjoint of M_F.

Proposition 4.16. *Let $F\in\mathcal{W}_{\mathbb{Q}^+}$. Then $M_F^*=M_{\overline{F}}$.*

Proof. We start by writing[24] $M_F=PF=PFP$. Now,

$$(PFP)^*=P^*(PF)^*=P^*F^*P^*=PF^*.$$

Therefore, for $X,Y\in\mathcal{W}_{\mathbb{Q}^+}$, we compute [25]

$$\langle FX,Y\rangle=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}F(t)X(t)\overline{Y(t)}dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t)\overline{\overline{F(t)}Y(t)}dt=\langle X,\overline{F}Y\rangle.$$

□

[24] By this, we mean $M_FX=P(FX)$. For convenience, we briefly use F to denote the multiplication by F, rather than a function in $\mathcal{W}_{\mathbb{Q}^+}$.

[25] We revert back to $F\in\mathcal{W}_{\mathbb{Q}^+}$

Proposition 4.17. *There exists F in $\mathcal{W}_{\mathbb{Q}^+}$, not identically zero, such that* $\ker M_F \neq \{0\}$, *and* $\ker M_F^* \neq \{0\}$.

Proof. Fix $m \in \mathbb{N}$. We choose $F \in \mathcal{W}_{\mathbb{Q}^+}$ such that $f(n/m) = f(m/n) = 0$ for $n \in \mathbb{N}$. Consider $\chi_m(t) = m^{it}$. Therefore,

$$M_F\chi_m = P\left(\sum_{q\in\mathbb{Q}^+} f(q)\chi_{qm}\right) = P\left(\sum_{q\in\mathbb{Q}^+} f(q/m)\chi_q\right) = \sum_{n\in\mathbb{N}} f(n/m)\chi_n = 0.$$

Hence, $\chi_m \in \ker M_F$. Furthermore,

$$M_F^*\chi_m = P\left(\sum_{q\in\mathbb{Q}^+} \overline{f(q)}\chi_{m/q}\right) = P\left(\sum_{q\in\mathbb{Q}^+} \overline{f(m/q)}\chi_q\right) = \sum_{n\in\mathbb{N}} \overline{f(m/n)}\chi_n = 0.$$

So $\chi_m \in \ker(M_F)$ □

The failure of Coburn's Lemma in this setting stems from the structure of the Besicovitch spaces over which multiplicative Toeplitz operators are acting. We give an example to illustrate this below.

Example 4.18. Let $m = 1$ so, $X = 1$, and $f(q) = 0$ if $q \in \mathbb{N}$ or $q \in \overline{\mathbb{N}}$. This leads to

$$M_F X = \sum_{n\in\mathbb{N}} f(n)\chi_n = 0 \text{ and } M_F^* X = \sum_{n\in\mathbb{N}} \overline{f(1/n)}\chi_n = 0.$$

Observe that the coefficients of the symbol which contribute to M_F and M_F^* are only those defined on $\mathbb{N}$ and $\overline{\mathbb{N}}$ respectively. In other words, there may exist many non-zero coefficients on $\mathbb{Q}^+ \setminus \mathbb{N} \cup \overline{\mathbb{N}}$ that vanish within the mapping. Therefore, F can be defined to be not identically zero, but be zero at all coefficients which contribute to both operators. In turn, this allows non-zero elements to be simultaneously in the kernel of M_F and M_F^*.

As stated in Section 2.4, Coburn's Lemma is already known to fail for two dimensional Toeplitz operators. Proposition 4.17 gives another proof of this by choosing f supported on $\left\{2^k 3^l : k, l \in \mathbb{N}\right\}$ such that $f(q) = 0$ if $q \in \mathbb{N}$ or $\overline{\mathbb{N}}$.

It is a challenging problem to determine the kernel of M_F and M_F^*, and as such establishing the spectrum, or the essential spectrum, is very difficult. Furthermore, it is clear from the literature and the findings presented within this chapter that establishing the spectrum of M_F is a much more delicate problem than that of T_Φ, and will require further mathematical tools than used within the classical setting.

We conclude this chapter with a summary of the open problems which have arisen within:

- What is the spectrum of $M_F : \mathcal{B}^2_{\mathbb{N}} \to \mathcal{B}^2_{\mathbb{N}}$ for $F \in \mathcal{W}_{\mathbb{Q}^+}$? Equivalently, can you find the spectrum of $T_\Phi : \mathcal{H}^2(\mathbb{T}^\infty) \to \mathcal{H}^2(\mathbb{T}^\infty)$ for $\Phi \in \mathcal{W}(\mathbb{T}^\infty)$?
- What is the essential spectrum of D_F for $F \in \mathcal{W}_{\mathbb{N}}$?
- Can we find symbols $F \in \mathcal{W}_{\mathbb{N}}$ and $G \in \mathcal{W}_{\mathbb{Q}^+}$ for which the semi-commutator?
- Given $f \notin \mathcal{F}\mathcal{W}_{\mathbb{Q}^+}$, when is M_F invertible?

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