Multiplicative functions with sum zero on Beurling prime systems

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Abstract

CMO functions are completely multiplicative functions f for which $\sum_{n=1}^{\infty} f(n) = 0$. Such functions were first defined and studied by Kahane and Saïas [31]. We extend these to multiplicative functions with the aim to investigate the theory of the extended functions and we shall call them MO functions. We give some properties and find examples of MO functions, as well as pointing out the connection between these functions and the Riemann hypothesis at the end of Chapter 2.

Following this, we broaden our scope by generalising both CMO and MO functions to Beurling prime systems with the aim to answer some of the questions which were raised by Kahane and Saïas. We shall call these sets of functions $CMO_{\mathcal{P}}$ and $MO_{\mathcal{P}}$ functions. Such generalisations allow us to look for new examples to extend our knowledge. In particular, we explore how quickly the partial sum of these classes of functions tends to zero with different Beurling generalised prime systems. The findings of this study suggest that for all $CMO_{\mathcal{P}}$ and $MO_{\mathcal{P}}$ functions f over $\mathcal{N}$ with abscissa 1, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} f(n) = \Omega\Big(\frac{1}{\sqrt{x}}\Big).$$

Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

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2020

Certificate

This is to certify that the thesis entitled "**Multiplicative Functions with Sum Zero on Beurling Prime Systems**" has been prepared under my supervision by Ammar Ali Neamah Al-Rammahi for the award of the Degree of Philosophy in Mathematics in the School of Mathematical, Physical and Computational Sciences, the University of Reading.

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List of Symbols

Symbol	Description	Page
χ	A Dirichlet character	8
μ	the Möbius function	8
λ	Liouville's function	8
$M(x)$	the sum of $\mu(n)$ for n not exceeding x	9
$L(x)$	the sum of $\lambda(n)$ for n not exceeding x	9
$*$	Dirichlet convolution	9
$\pi(x)$	the number of primes not exceeding x	10
ζ	the Riemann zeta function	10
$\mathbb{P}$	the set of prime numbers	12
L_χ	Dirichlet L-function	12
$\mu_f(\sigma)$	the infimum of all real numbers A such that $\|f(\sigma+it)\| = O(\|t\|^A)$	18
$\mathcal{P}$	the multi-set of Beurling (generalised) prime systems	35
$\mathcal{N}$	the multi-set of Beurling (generalised) integers	35
$\pi_{\mathcal{P}}(x)$	the number of g-primes not exceeding x	35
$N_{\mathcal{P}}(x)$	the number of g-integers not exceeding x	35
$\zeta_{\mathcal{P}}$	the Beurling (generalised) zeta function	36
$\mu_{\mathcal{P}}$	the generalised Möbius function on $\mathcal{N}$	40
$\Lambda_{\mathcal{P}}$	the generalised Mangoldt function on $\mathcal{N}$	40
$\lambda_{\mathcal{P}}$	the generalized Liouville function on $\mathcal{N}$	40
$\psi_{\mathcal{P}}(x)$	the generalised Chebyshev function on $\mathcal{N}$	41
$\Pi_{\mathcal{P}}(x)$	$\sum_{n=1}^{\infty} \frac{\pi_{\mathcal{P}}(x^{\frac{1}{n}})}{n}$	42
$M_{\mathcal{P}}(x)$	the sum of $\mu_{\mathcal{P}}(n)$ for n not exceeding x	42
$L_{\mathcal{P}}(x)$	the sum of $\lambda_{\mathcal{P}}(n)$ for n not exceeding x	42
$m_{\mathcal{P}}(x)$	the sum of $\frac{\mu_{\mathcal{P}}(n)}{n}$ for n not exceeding x	42
$l_{\mathcal{P}}(x)$	the sum of $\frac{\lambda_{\mathcal{P}}(n)}{n}$ for n not exceeding x	42

Symbol	Description	Page
$U_{\mathcal{P}}(s)$	the Mellin transform of $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n^s}$	46
$Z_{\mathcal{P}}(s)$	the Mellin transform of $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n^s}$	46
$V_{\mathcal{P}}(s)$	the Mellin transform of $\sum_{n\in\mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^s}$	46
$\mathrm{li}(x)$	the Logarithmic integral $\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}$	51

Contents

Introduction

This work aims to study both multiplicative functions with zero sum and Beurling prime systems. This thesis is organised as follows. The first chapter contains an introduction to the theory surrounding both subjects. The purpose of this chapter is to provide some tools which are required and used throughout the thesis to prove the main results. We review some notions which describe the asymptotic behaviour of a function and Riemann-Stieltjes Integral concept. We also introduce some concepts such as arithmetical function, multiplicative and completely multiplicative functions as well as surveying some results which show the connection between the Möbius function and the Prime Number Theorem (PNT). This chapter also reviews a valuable tool known as Dirichlet series which has a significant role in the field of number theory. We also introduce essential theorems which show the link between the Euler product and Dirichlet series of a multiplicative function. Additionally in Chapter 1 we give the Mellin formula which allows one to transform this series into an integral as well as giving Perron's formula. Finally, we review specific relevant results from complex analysis which we use in this work.

Chapter 2 introduces a class of functions which has been defined and studied by Kahane and Saïas [31, 30], called CMO functions. These are completely multiplicative f for which $\sum_{n=1}^{\infty} f(n) = 0$. The main purpose of this chapter is to generalise these functions to multiplicative functions and we shall call them MO functions. More precisely, we define MO functions to be multiplicative functions for which $\sum_{n=1}^{\infty} f(n) = 0$ and $\sum_{k=0}^{\infty} f(p^k) \neq 0$ for all $p \in \mathbb{P}$. The third condition is put in order to avoid trivial examples. We investigate how much of the theory of CMO functions can be generalised. Like CMO functions, MO functions are not so easy to find since the series need to be conditionally convergent, motivating some examples and a closer look at properties of these functions.

Chapter 3 is devoted to reviewing the mathematical background of Beurling prime systems $\mathcal{P}$ which we utilise in the chapters that follow. We introduce this concept, giving some examples. In the following part, we introduce the Beurling analogue of arithmetical functions and Dirichlet convolution of these functions. In the next three sections, we address Abel's Identity, the Mellin transform and its inverse, and Euler products concepts over the Beurling generalised integers $\mathcal{N}$. Recent progress with Beurling's Prime Number Theorem is also considered and surveyed in this chapter. Finally, we quote some known relevant results which are required for this work.

In Chapter 4, we discuss the special functions of Liouville and Möbius over Beurling prime systems (*i.e.* $\lambda_{\mathcal{P}}$, $\mu_{\mathcal{P}}$ respectively). We study the relationship between the partial sums of such functions which play a vital role as examples in chapters 5 and 6. We are also interested in the following question: how small can the partial sum of both Liouville and Möbius functions not exceeding x over $\mathcal{N}$ be made for a system with abscissa equal to 1? In particular, we investigate Beurling prime systems $\mathcal{P}$ for which the counting functions $\psi_{\mathcal{P}}(x)$, $N_{\mathcal{P}}(x)$ and $M_{\mathcal{P}}(x)$ are asymptotically well-behaved, in the sense that $\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon})$, $N_{\mathcal{P}}(x) = \rho x + O(x^{\beta+\varepsilon})$ and

$$M_{\mathcal{P}}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \mu_{\mathcal{P}}(n) = O(x^{\gamma+\varepsilon})$$

hold for all $\varepsilon > 0$, but for no $\varepsilon < 0$, where $\rho > 0$, α, β, $\gamma < 1$ respectively. We show that it is impossible to have both β and γ less than $\frac{1}{2}$ or both α and γ less than $\frac{1}{2}$. We also conclude that out of the three numbers $\{\alpha, \beta, \gamma\}$, the largest two must be equal and at least $\frac{1}{2}$. Lastly, we also examine the behaviour of the sums $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ and $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n}$ of Beurling prime systems under some conditions on $\mathcal{P}$.

In Chapter 5, we investigate CMO functions on Beurling generalised prime systems and find various examples and properties of these functions. In this chapter, we use some concepts of Chapter 3 to generalise CMO functions, whereas earlier work by Kahane and Saïas has depended on the usual primes. This chapter also addresses and considers the questions that have been asked by Kahane and Saïas for CMO functions. For example, we show how quickly partial sums of $CMO_{\mathcal{P}}$ functions tend to zero with different Beurling generalised prime systems.

In Chapter 6 we broaden our scope by considering $MO_{\mathcal{P}}$ functions which is a generalisation of $CMO_{\mathcal{P}}$ functions. We define these functions over multiplicative functions instead of completely multiplicative functions. Such a generalisation allows us to look for new examples to extend our knowledge. In particular, we construct an example which involves the function $f(n) = \frac{a_{\mathcal{P}}(n)}{n^{\alpha}}$ with a g-prime system satisfying

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta})$$

for some $\rho > 0$ and $\beta < 1$, where $a_{\mathcal{P}}(n)$ is $1 - p_0$ if p_0 divides $n \in \mathcal{N}$ and 1 if p_0 does not divide $n \in \mathcal{N}$. We show that $f(n)$ is an $MO_{\mathcal{P}}$ function if and only if $\Re\alpha > \beta$ and $\zeta_{\mathcal{P}}(\alpha) = 0$. For all other examples detailed we find that $\sum_{n \leq x} f(n) = \Omega\big(\frac{1}{x^{\frac{1}{2}+\varepsilon}}\big)$ for all $\varepsilon > 0$. Indeed, this may suggest that for all functions f which are multiplicative over $\mathcal{N}$, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} f(n) = \Omega\Big(\frac{1}{\sqrt{x}}\Big).$$

We notice as a consequence of the above conjecture that if $\mathcal{P}$ is a g-prime system satisfying

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta})$$

for some $\rho > 0$ and $\beta < \frac{1}{2}$, then $\zeta_{\mathcal{P}}(s) \neq 0$ in the strip $\{s \in \mathbb{C} : \beta + \frac{1}{2} < \Re s < 1\}$; *i.e.* a very strong form of Riemann Hypothesis holds.

Chapter 1

Preliminaries

In this chapter, we review some facts and results from real analysis which are necessary of this work. We also give and review some essential mathematical concepts in number theory which are required and used throughout the thesis. Finally, we offer some results from complex analysis which are in order that we may prove some of the main results later on.

1.1 Some facts and results regarding real analysis

In this section, we provide some definitions of special notations which describe the asymptotic behavior of a function like Big oh, Little oh and Omega. Some of these notations are commonly used in the Analytic Number Theory such as the Prime Number Theorem and its equivalences. We also introduce the concept of bounded variation to functions and Riemann-Stieltjes Integral.

1.1.1 Asymptotic Notations

The O (Big oh), o (little oh), Ω (Omega), asymptotic equivalence and order of magnitude estimate notations are the convenient set of notations in asymptotic analysis. We define them as follows: let $f : [A, \infty) \longrightarrow \mathbb{C}$ and $g : [A, \infty) \longrightarrow (0, \infty)$, then

(i) we write $f(x) \ll g(x)$ or $f(x) = O(g(x))$ to mean that there exist constants B and constant y such that $|f(x)| \leq Bg(x)$ for all $x \geq y$.

(ii) we write $f(x) = o(g(x))$ to mean

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

(iii) we write $f(x) \sim g(x)$ to mean

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

(iv) we write $f(x) \asymp g(x)$ to mean that there exist positive constants b, B and y such that $bg(x) \leq f(x) \leq Bg(x)$ holds for $x \geq y$; *i.e.* $f(x) \asymp g(x)$ means that both \"$f(x) \ll g(x)$" and \"$g(x) \ll f(x)$".

(v) we write $f(x) = \Omega(g(x))$ if there exist $x_n \longrightarrow \infty$ and positive constant B such that $|f(x_n)| > Bg(x_n)$ for all n; (*i.e.* $f(x) \neq o(g(x))$)

Remark 1.1. Let $f, h : [A, \infty) \longrightarrow \mathbb{C}$ and $g : [A, \infty) \longrightarrow (0, \infty)$. Then

(i) If we have $h(x) = O(1)$, then this means that $h(x)$ is bounded for sufficiently large x. While if we have $h(x) = o(1)$, then this means that $h(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

(ii) If we have $h(x)=O(B)$, where $B>0$ is a constant, this equivalent to $h(x)=O(1)$.

(iii) An equation of the form $g(x)=f(x)+O(E(x))$ means that $g(x)-f(x)=O(E(x))$, where $O(E(x))$ is an error term and $g(x)$ is a main term. This equation is only meaningful if $f(x)$ is bigger order than $E(x)$; (*i.e.* $E(x) = o(f(x))$.

(iv) If we have $f(x)$ and $g(x)$ are integrable and satisfy $f(x) = O(g(x))$ for $x \geq A$, then $\int_A^x f(x)dx = \int_A^x O(g(x))dx$. Furthermore, $\int_A^x O(g(x))dx = O(\int_A^x g(x)dx)$.

(v) If we have $f(x) = o(g(x))$, then $f(x) = O(g(x))$.

(vi) If we have $f(x) = o(g(x))$, then $\int_A^x f(t)dt = o(\int_A^x g(t)dt)$ if $\int_A^x g(t)dt$ is divergent as $x \longrightarrow \infty$.

1.1.2 Function of Bounded Variation

The function denoted by α will be assumed to be complex valued function.

Definition 1.2. If $[a, b]$ is a compact interval, a set of points $P = \{x_0, x_1, \ldots, x_n\}$, satisfying the inequalities $a = x_0 < x_1 < \ldots, x_{n-1} < x_n = b$, is called a ***partition*** of $[a, b]$. The interval $[x_i, x_{i+1}]$ is called the ith subinterval of P, so that $\sum_{i=0}^{n-1}(x_{i+1} - x_i) = b - a$ [1] page 128.

Definition 1.3. Let α be defined on $[a, b]$ and let P as defined before. If there exists a positive number C such that

$$\sum_{i=0}^{n-1} |\alpha(x_{i+1}) - \alpha(x_i)| \leq C$$

for all the partitions $P = \{x_0, x_1, \ldots, x_n\}$ on $[a, b]$, then α is said to be of ***bounded variation*** on $[a, b]$. As such we can define the ***total variation*** of α defined on an interval $[a, b]$ via

$$V_b^a(\alpha) = \sup \sum_{i=0}^{n-1} |\alpha(x_{i+1}) - \alpha(x_i)|,$$

where the supremum runs over the set of all partitions P of the given interval.

We say that $\alpha : [a, \infty) \longrightarrow \mathbb{C}$ is a function of ***locally bounded variation*** if it is of bounded variation over all compact subintervals of $[a, \infty)$ [1] page 128.

1.1.3 Riemann-Stieltjes Integral

The functions denoted by f and α will be assumed to be (real or complex) valued functions defined and bounded on the compact interval $[a, b]$.

Definition 1.4. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$ and let t_i be a point in a subinterval $[x_i, x_{i+1}]$. A sum of the form

$$S(P, f, \alpha) = \sum_{i=0}^{n-1} f(t_i)(\alpha(x_{i+1}) - \alpha(x_i))$$

is called a ***Riemann-Stieltjes sum*** of f with respect to α. We say f is ***Riemann integrable with respect to*** α on $[a, b]$, and we write ($f \in R(\alpha)$ on $[a, b]$) if there exists $I \in \mathbb{R}$ having the following property: for all $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that for every partition P finer (*i.e.* containing extra points) than

P_ε and every choice of the points t_i in $[x_i, x_{i+1}]$, we have $|S(P, f, \alpha) - I| < \varepsilon$. Then $I = \int_a^b f(x)d\alpha(x)$ is the ***Riemann-Stieltjes integral*** [1] page 141.

Theorem 1.5. *If $f \in R(\alpha)$ on $[a, b]$, then $\alpha \in R(f)$ on $[a, b]$ and we have*

$$\int_a^b f(x)d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)df(x).$$

Proof. See Theorem 7.6 [1] page 144.

□

Theorem 1.6. *Assume $f \in R(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)dx$ exists and we have*

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx.$$

Proof. See Theorem 7.8 [1] page 146.

□

Theorem 1.7. (Abel's summation formula) *Let $a : \mathbb{N} \longrightarrow \mathbb{C}$ be a arithmetic function and let f be a differentiable function such that f' Riemann integrable on $[1, \infty)$ with $A(x) = \sum_{1 \le n \le x} a(n)$. Then*

$$\sum_{1 \le n \le x} a(n)f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt. \tag{1.1}$$

Proof. The formula (1.1) can be deduced by integration by parts for the Riemann-Stieltjes integral (using Theorems 1.5 and 1.6). Indeed, we have

$$\begin{aligned}\sum_{1 \le n \le x} a(n)f(n) &= \int_{1-}^x f(t)dA(t) = \big[A(t)f(t)\big]_{1-}^x - \int_1^x A(t)f'(t)dt \\ &= A(x)f(x) - \int_1^x A(t)f'(t)dt,\end{aligned}$$

where "$1-$" means approaching 1 from below. □

We shall use the next theorem in the proof of Propositions 1.21, 3.26 and 3.27.

Theorem 1.8. *Let $a(n) > 0$ be a sequence. Then product $\prod_{n=1}^\infty \big(1 + |a(n)|\big)$ converges if and only if the series $\sum_{n=1}^\infty |a(n)|$ converges.*

Proof. See Theorem 8.52 of [1] pages 208-209. □

1.2 Some concepts and results in number theory

In this section, we turn our attention to cover materials from number theory. We introduce some concepts such as arithmetical function, multiplicative and completely multiplicative functions with some examples. We also survey some results which show the connection between Möbius and Liouville functions and the Prime Number Theorem (PNT). We then introduce a fundamental number theoretic sum associated with arithmetic functions called Dirichlet series as well as introducing its convergence issues. Finally we review some results and techniques applicable to Dirichlet series such as Euler products and Mellin transforms and the inverse Mellin transforms.

1.2.1 Arithmetical Functions

An arithmetical function $f : \mathbb{N} \longrightarrow \mathbb{C}$ is called ***multiplicative*** if $f(1) = 1$ and it satisfies $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$, where (m, n) is the ***greatest common divisor*** of $m, n \in \mathbb{N}$. Such an f is called ***completely multiplicative*** if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$. Multiplicative functions are determined by their values on prime powers; (*i.e.* once we know the values of $f(p^k)$, we know the values of $f(n)$ for any $n \in \mathbb{N}$). However, completely multiplicative functions are determined by their values on primes; (*i.e.* once we know the values of $f(p)$, we know the values of $f(n)$ for any $n \in \mathbb{N}$).

If a function $\chi(n)$ is completely multiplicative, periodic with period $k > 1$, and vanishes when $(n, k) > 1$ then it is called a ***Dirichlet character*** modulo k. A character is called ***principal*** if it has the following properties

$$\chi(n) = \begin{cases} 1 & \text{if } (n,k) = 1, \\ 0 & \text{if } (n,k) > 1. \end{cases}$$

We introduce two significant arithmetic functions which play a crucial role in the thesis. We define ***Möbius function*** to be the function given by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_{i_1} p_{i_2} \cdots p_{i_k} \text{ are distinct primes}, \\ 0 & \text{otherwise}, \end{cases}$$

or equivalently, the multiplicative function defined by $\mu(p) = -1$ and $\mu(p^k) = 0$ if $k > 1$ for all primes p. We also define $\lambda(n)$, also known as ***Liouville's function***, to be the completely multiplicative function which is $\lambda(p) = -1$ for every prime number p.

The following theorem will be of use in later chapters.

Theorem 1.9. *If $f(n)$ is multiplicative, and $f(p^k) \longrightarrow 0$ as $p^k \longrightarrow \infty$, then $f(n) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof. See Theorem 316 of [24]. □

1.2.2 Average order of Arithmetical Functions

In order to study the distribution of values of an arithmetical function, it is useful to consider its average behaviour. To do this we need to know something about the partial sums of the function in order to investigate such behaviour. As such, we can define the partial sum of the arithmetical function f by $F(x) = \sum_{n \le x} f(n)$.

The partial sum of $\mu(n)$ and $\lambda(n)$ functions not exceeding x can be defined, respectively, by

$$M(x) := \sum_{n \le x} \mu(n)$$

and

$$L(x) := \sum_{n \le x} \lambda(n).$$

We use these notations throughout the thesis.

Definition 1.10. Let f and g be two arithmetic functions. Then the ***Dirichlet convolution*** $f * g$ is the following arithmetic function:

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

The next theorem which is called Dirichlet's hyperbola method is a useful elementary technique of finding an asymptotic formula.

Theorem 1.11. *Let f and g be arithmetic functions and write*

$$F(x) = \sum_{n \le x} f(n) \quad \text{and} \quad G(x) = \sum_{n \le x} g(n).$$

Then, for any $ab = x$, where a and b are positive real numbers, we have

$$\sum_{n \le x} (f * g)(n) = \sum_{qd \le x} f(d)g(q) = \sum_{n \le a} f(n)G\Big(\frac{x}{n}\Big) + \sum_{n \le b} g(n)F\Big(\frac{x}{n}\Big) - F(a)G(b).$$

Proof. See Theorem 3.17 of [2].

□

1.2.3 The Prime Number Theorem and Möbius function

The Prime Number Theorem (PNT) that describes the asymptotic distribution of prime numbers was first proved nearly simultaneously by J. Hadamard and C. J. de la Vallée Poussin in 1896 [20] [34]. They proved that

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1$$

by showing that the Riemann zeta function $\zeta(s)$ has no zeros for $\Re s \ge 1$, where $\pi(x)$ is the number of primes not exceeding x. This relation can also be written by using the asymptotic notation,

$$\pi(x) \sim \frac{x}{\log x},$$

or equivalently,

$$p_n \sim n \log n,$$

where p_n is the n^{th} prime number. Several asymptotic formulas have been studied to be equivalent to the PNT by some scholars (see for example [5], [13] , [44]). We review some results which show the relationship between Möbius and Liouville functions and PNT. H. von Mangoldt 1897 [56] proved that knowing the PNT, it is easy to obtain $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$ with same elementary steps. However, E. Landau 1909 showed in [37] the converse of von Mangoldt's result also holds. Another equivalent of the PNT, attributed to E. Landau [35], by $M(x) = o(x)$. He also showed in [37] that the Möbius function can be replaced by the Liouville function in the previous results. In 1912, J. E. Littlewood showed in [41] that the Riemann hypothesis (RH) is equivalent to the following evaluation

$$M(x) = \sum_{n \le x} \mu(n) = O(x^{\frac{1}{2}+\varepsilon}) \quad \text{for all } \varepsilon > 0. \tag{1.2}$$

E. Landau 1924 [38] improved on this result, showing that the RH allows replacement of the error term in (1.2) by

$$O\Big(x^{\frac{1}{2}}\exp\Big(\frac{c\log x\log\log\log x}{\log\log x}\Big)\Big) \quad \text{for some } c>0.$$

E.C. Titchmarsh 1927 [52] later improved this to

$$O\Big(x^{\frac{1}{2}}\exp\Big(\frac{c_1\log x}{\log\log x}\Big)\Big) \quad \text{for some } c_1>0.$$

In 2009, the bound was improved by H. Maier and H. L. Montgomery [42] to

$$O\big(x^{\frac{1}{2}}\exp\big(b(\log x)^{\frac{39}{61}}\big)\big) \quad \text{for some } b>0,$$

and K. Soundararajan 2009 [49] to be

$$O\big(x^{\frac{1}{2}}\exp\big((\log x)^{\frac{1}{2}}(\log\log x)^{14}\big)\big). \tag{1.3}$$

M. Balazard and A. de Roton [3] have slightly improved this bound by using a similar approach as K. Soundararajan. They replaced 14 by $\frac{5}{2}+\varepsilon$ in (1.3). The best possible bound was conjectured by S. M. Gonek (see N. Ng [46]) to be

$$M(x)=O\Big(x^{\frac{1}{2}}(\log\log\log x)^{\frac{5}{4}}\Big).$$

That is, conjecturally, one cannot get $M(x)$ to be $o\big(x^{\frac{1}{2}}(\log\log\log x)^{\frac{5}{4}}\big)$. Following the above conjecture, Theorem 4.7 in Chapter 4 can be used to show that this would imply $L(x)$ is also

$$O\Big(x^{\frac{1}{2}}(\log\log\log x)^{\frac{5}{4}}\Big).$$

It is also well-known that $M(x)$ and $L(x)$ are $\Omega(\sqrt{x})$ since there are zeros of the Riemann zeta function ζ on the line $\Re s=\frac{1}{2}$ (see for example [54]).

1.2.4 Zeta Functions and L-functions; Dirichlet series

It is well-known that L. Euler in the 1737 determined the values of $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ when s is an even integer bigger than 1. A fundamental connection between $\zeta(s)$ and the prime numbers was established by Euler, known as the Euler product representation of the zeta function, given by

$$\zeta(s)=\prod_{p\in\mathbb{P}}\Big(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\Big)=\prod_{p\in\mathbb{P}}\Big(1-\frac{1}{p^s}\Big)^{-1}.$$

This infinite product is convergent for $\Re s>1$. In 1859, B. Riemann [48] defined the ***generalised*** version by viewing $\zeta(s)$ as a function of a complex number s instead of a real number s; *i.e.*

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\quad \text{for } \Re s>1.$$

This was later named the ***Riemann zeta function***. This series is absolutely convergent in the half plane $\Re s>1$ and it has an analytic continuation to the whole complex plane $\mathbb{C}$ except for a simple pole at $s=1$ with residue 1. For this continuation, we have the ***functional equation*** of the zeta function which relates the values of ζ at s and $1-s$ as follows:

$$\zeta(1-s)=2(2\pi)^{-s}cos\Big(\frac{\pi s}{2}\Big)\Gamma(s)\zeta(s),\quad \text{for } \Re s>0,$$

where $\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt$ is the ***Gamma function***. The most interesting mystery of zeta function occurs whenever $0<\Re s<1$. This is not only called the ***critical strip*** but also is central to the famous ***Riemann Hypothesis*** (RH) which states that all zeros in the critical strip are located on the ***critical line*** $\Re s=\frac{1}{2}$.

In 1837, L. Dirichlet [18] introduced another Dirichlet series of the form

$$L_\chi(s)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},$$

which is known as a ***Dirichlet L-function*** in order to prove that there are infinitely many primes in any arithmetic progression $an+d$, where a,d are any two positive coprime integers and n is also a positive integer. These series are absolutely convergent in the half plane $\Re s>1$ and have an analytic continuation to the whole complex plane $\mathbb{C}$ apart from a simple pole at $s=1$ if $\chi(n)$ is a principal character. However, $L_\chi(s)$ is holomorphic everywhere if $\chi(n)$ is a non-principal character. The ***Generalised Riemann Hypothesis*** (GRH) asserts that for every Dirichlet character χ, all the zeros in the critical strip lie along the line $\Re s=\frac{1}{2}$. The Riemann zeta function is a special case of these functions when $\chi(n)=1$ for all $n\in\mathbb{N}$.

The Riemann zeta function and Dirichlet L-functions are examples of the following series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \tag{1.4}$$

which is called an ***Ordinary Dirichlet series*** with coefficients $a(n)$, where $a(n)$ is an arithmetical function. More generally, this series is a type of the ***General Dirichlet series*** which have the form

$$\sum_{n=1}^{\infty} a(n)e^{-\lambda_n s}, \tag{1.5}$$

where s is a complex number and λ_n is a strictly increasing sequence of non-negative real numbers whose limit is infinity. Indeed, if $\lambda_n = \log n$, then (1.5) gives (1.4). However, if $\lambda_n = n$ and the change of variable $e^{-s} = z$ gives the power series $\sum_{n=1}^{\infty} a(n)z^n$. These series not only play a significant role in analytic number theory but also have applications in other areas such that cryptography, physics and applied statistics.

Convergence of Dirichlet Series

In this section, we are concerned with different convergence issues of Dirichlet series. Namely, we introduce some fundamental theorems relating to Dirichlet series, which explain the concept of abscissa of convergence and absolute convergence.

Theorem 1.12. (Abel's theorem on Dirichlet series) *If the Dirichlet series*

$$D(s) := \sum_{n=1}^{\infty} a(n)e^{-\lambda_n s}, \;\; s = \sigma + it, \;\; \lambda_n > 0,$$

converges at the point $s_0 = \sigma_0 + it_0$, then this series is convergent in the half plane $\sigma > \sigma_0$ and uniformly convergent inside any angle $|\arg(s - s_0)| \leq \theta < \frac{\pi}{2}$.

Proof. See Abel's theorem on Dirichlet series given in [23] page 3.

□

Theorem 1.13. *If the series is convergent for $s = s_0$, and has the sum $D(s_0)$, then $D(\sigma + it) \longrightarrow D(\sigma_0 + it_0)$ as $\sigma \longrightarrow \sigma_0^+$ along any path which lies entirely within the region $|\arg(s - s_0)| \leq \theta < \frac{\pi}{2}$.*

Proof. See Abel's theorem on Dirichlet series given in [23] page 6.

□

Theorem 1.14. *The series (1.5) may be convergent everywhere, or divergent everywhere, or there may exist a number σ_c such that the series converges for $\sigma > \sigma_c$ and diverges for $\sigma < \sigma_c$. Indeed this series converges uniformly throughout the region of the half plane $\{s \in \mathbb{C} : \sigma \geq \sigma_c + \delta \text{ for all } \delta > 0\}$.*

Proof. See Theorem 3 and Theorem 4 of [23].

□

Theorem 1.15. *There is a number σ_a such that the series (1.5) converges absolutely if $\sigma > \sigma_a$ but does not converge absolutely if $\sigma < \sigma_a$.*

Proof. See Theorem 8 of [23].

□

Remark 1.16.

(i) The numbers σ_c and σ_a are called the ***abscissa of convergence*** and ***abscissa of absolute convergence*** respectively.

(ii) The regions $H_{\sigma_c} = \{s \in \mathbb{C} : \sigma > \sigma_c\}$ and $H_{\sigma_a} = \{s \in \mathbb{C} : \sigma > \sigma_a\}$ are called the ***half plane of convergence*** and ***half plane of absolute convergence***.

(iii) The lines $\sigma = \sigma_c$ and $\sigma = \sigma_a$ are called the ***line of convergence*** and ***line of absolute convergence*** respectively.

(iv) In general there might be a strip between the lines of convergence and absolute convergence where the series (1.5) is conditionally convergent.

Theorem 1.17. *For the series (1.5), we have*

$$0 \leq \sigma_a - \sigma_c \leq \limsup_{n \longrightarrow \infty} \frac{\log n}{\lambda_n}.$$

Proof. See Theorem 9 of [23].

□

If $\lambda_n = \log n$, then (1.5) gives (1.4) and the maximum possible distance between these two lines is 1.

Theorem 1.18. *Let f, g and h be arithmetic functions, with respective ordinary Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$, $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ and $H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$. Assume that $h = f * g$ the Dirichlet convolution of f and g; i.e.*

$$h(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

Then the series $H(s)$ converges in any domain where both series $F(s)$ and $G(s)$ are absolutely convergent, and in such circumstances we have

$$H(s) = F(s)G(s).$$

Proof. See Theorem 1.2 of Chapter II.1. [51].

□

It is mentioned in the notes of Chapter II.1.[51] that the above theorem can be extended in the following manner: If the series $F(s)$ converges, and if the series $G(s)$ converges absolutely, then the series $H(s)$ converges and we have $H(s) = F(s)G(s)$.

1.2.5 Euler Products

In this section, we introduce the next important theorem which is due to a discovery by Euler in the 1730s.

Theorem 1.19. *Let $f : \mathbb{N} \longrightarrow \mathbb{C}$ be a multiplicative function such that the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent. Then the sum of the series can be expressed as an absolutely convergent infinite product,*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \{1 + f(p) + f(p^2) + \cdots\}$$

extended over all primes. If f is completely multiplicative, the product simplifies and we have

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \frac{1}{1 - f(p)}.$$

Proof. See Theorem 11.6 of [2].

□

Example 1.20. If $f(n) = \frac{1}{n^s}$, then our series is $\zeta(s)$, the Riemann zeta function, and we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}} = \zeta(s) \quad \text{for } \Re(s) > 1.$$

Furthermore, if $f(n) = \frac{\mu(n)}{n^s}, \frac{\lambda(n)}{n^s}$ and $\frac{\chi(n)}{n^s}$, then we can get the following Euler products:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p (1 + \mu(p)p^{-s}) = \frac{1}{\zeta(s)} \quad \text{if } \Re(s) > 1,$$

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p (1 - \lambda(p)p^{-s})^{-1} = \frac{\zeta(2s)}{\zeta(s)} \quad \text{if } \Re(s) > 1,$$

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1} = L_\chi(s) \quad \text{if } \Re(s) > 1.$$

We shall need the following result in Chapter 2.

Propostion 1.21. *Let f be a multiplicative function. Then $\sum_{n=1}^{\infty} |f(n)|$ converges, so that f is absolutely convergent, if and only if $\sum_p \sum_{k=1}^{\infty} |f(p^k)|$ converges.*

Proof. Trivially, the series $\sum_p \sum_{k=1}^{\infty} |f(p^k)|$ converges if $\sum_{n=1}^{\infty} |f(n)|$ converges.

Now suppose $\sum_p \sum_{k=1}^{\infty} |f(p^k)|$ converges. It is follows by Theorem 1.8 that

$$\prod_p \left(1 + \sum_{k=1}^{\infty} |f(p^k)|\right) = \prod_p \left(\sum_{k=0}^{\infty} |f(p^k)|\right) \quad \text{converges.}$$

But the right hand side is at least $\prod_{p \le x} \left\{ \sum_{k=0}^{\infty} |f(p^k)| \right\}$. Therefore, by the proof of Theorem 11.6 of [2] for any x, we have

$$\prod_{p \le x} \left\{ \sum_{k=0}^{\infty} |f(p^k)| \right\} = \sum_{\substack{n \in \mathbb{N} \\ p|n \ \& \ p \le x}} |f(n)| \ge \sum_{n \le x} |f(n)|.$$

Hence $\sum_{n=1}^{\infty} |f(n)|$ converges, so that f is absolutely convergent.

□

1.2.6 Dirichlet Series as Mellin Transform; Perron's formula

In this section, we provide the theorem which allows one to express the general Dirichlet series as integrals. We also introduce Perron's formula which allows one to calculate the partial sum of an arithmetical function $a(n)$ not exceeding x.

Theorem 1.22. *Let $F(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda_n s}$ be a general Dirichlet series with finite abscissa of convergence σ_c, and let $A(x) = \sum_{\lambda_n \leq x} a(n)$. Then we have*

$$F(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda_n s} = s\int_{e^{\lambda_1}}^{\infty} \frac{A(\log y)}{y^{s+1}}\, dy \qquad \text{for } \sigma > \max\{0, \sigma_c\}.$$

Proof. Applying Riemann-Stieltjes integration, we have

$$F(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda_n s} = \int_{\lambda_1 -}^{\infty} e^{-ts} dA(t) = \lim_{T\to\infty} \int_{\lambda_1 -}^{T} e^{-ts} dA(t)$$

$$= \lim_{T\to\infty} \left(\left[A(t)e^{-ts}\right]_{\lambda_1 -}^{T} + s\int_{\lambda_1}^{T} A(t)e^{-ts} dt \right)$$
$$= \lim_{T\to\infty} \left(\frac{A(T)}{e^{Ts}} + s\int_{\lambda_1}^{T} A(t)e^{-ts} dt \right),$$

where "$\lambda_1 -$" means approaching λ_1 from below.

Now if $A(T)$ converges, then $\frac{A(T)}{e^{Ts}} \to 0$ whenever $T \to \infty$ and $\sigma > 0$, while if $A(T)$ diverges, then $A(T) = O(e^{T(\sigma_c+\varepsilon)})$ for all $\varepsilon > 0$ and hence $\frac{A(T)}{e^{Ts}} \to 0$ whenever $T \to \infty$ and $\sigma > \sigma_c$. Therefore, for $\sigma > \max\{0, \sigma_c\}$, we have

$$F(s) = s\int_{e^{\lambda_1}}^{\infty} \frac{A(\log y)}{y^{s+1}}, dy \quad \text{where } A(\log y) = \sum_{e^{\lambda_n} \leq y} a(n).$$

□

Lemma 1.23. *If $c > 0$ we write $\int_{c-i\infty}^{c+i\infty}$ to mean $\lim_{T\to\infty} \int_{c-iT}^{c+iT}$. Then, if x is a real number, we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{xs}}{s}\, ds = \begin{cases} 1 & \text{if } x > 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Moreover, we have

$$\left|\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{xs}}{s}\,ds-\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{e^{xs}}{s}\,ds\right|\leq\begin{cases}\frac{e^{xc}}{\pi T|x|} & \text{if } x\neq 0,\\ \frac{c}{\pi T} & \text{if } x=0.\end{cases}$$

Proof. See Lemma 3 of [23].

□

Theorem 1.24. (Perron's formula) *Let* $F(s)=\sum_{n=1}^{\infty}a(n)e^{-\lambda_n s}$ *be a general Dirichlet series with finite abscissa of convergence* σ_c*, and let* $A(x)=\sum_{\lambda_n\leq x}a(n)$*. Then, for* $x\in\mathbb{R}$*, but* $x\neq\lambda_n$ *for all* n*, we have*

$$A(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(s)}{s}\,e^{xs}\,ds$$

for every $c>\max\{0,\sigma_c\}$*.*

Proof. See Theorem 13 of [23].

□

1.3 Some relevant results from complex analysis

In this section, we quote some useful results from complex analysis which we require for the later chapters.

Definition 1.25. We say that a holomorphic function f on a strip $\{s\in\mathbb{C}:a\leq\Re s\leq b\}$ is of ***finite order*** on the strip if there exists $A>0$ such that

$$f(\sigma+it)=O(|t|^A)\quad\text{as } |t|\longrightarrow\infty \tag{1.6}$$

for all σ in the strip [39] page 367. If for some σ no such A exists, we say f is of ***infinite order*** for this range σ.

A function f defined by Dirichlet series has finite order in the half plane of convergence if (1.6) holds for $\sigma>\sigma_c$. As such, we can define the ***Lindelöf function*** $\mu_f(\sigma)$ to be the infimum of all real numbers A such that $|f(\sigma+it)|=O(|t|^A)$; (*i.e.* $|f(\sigma+it)|=O(|t|^{\mu_f(\sigma)+\varepsilon})$ for all $\varepsilon>0$ but no $\varepsilon<0$). It is also well-known that $\mu_f(\sigma)$ is non-negative, decreasing and convex and for $\sigma>\sigma_a$, $\mu_f(\sigma)=0$ for σ sufficiently large (see for example [53], [54]).

If $f = \zeta$, then $\mu(\sigma) = 0$ for $\sigma \geq 1$ and $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma \leq 0$ by the functional equation. But $\mu(\sigma)$ is not known for any other $0 < \sigma < 1$.

E. L. Lindelöf [40] conjectured (***Lindelöf's Hypothesis***) that one has

$$\mu(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2}, \\ 0 & \text{if } \sigma \geq \frac{1}{2}, \end{cases}$$

or equivalently,

$$|\zeta(\tfrac{1}{2} + it)| = O(|t|^{\varepsilon}) \tag{1.7}$$

for all $\varepsilon > 0$ as $|t| \longrightarrow \infty$. He showed that the upper bound for the left hand side of (1.7) is $O(|t|^{\frac{1}{4}})$ which has been improved by many researchers since then. The current best bound of $|\zeta(\frac{1}{2} + it)|$ which was recently estimated by J. Bourgain [7] is $O(|t|^{\frac{13}{84}+\varepsilon})$ for all $\varepsilon > 0$. It is also known that Lindelöf Hypothesis follows from the Riemann hypothesis which has been mentioned earlier (see for instance [19]).

Theorem 1.26. (Borel-Carathéodory Theorem) *Let $f(z)$ be a holomorphic function on a closed disc of radius R centered at the origin. Then, for $0 < r < R$,*

$$\max_{|z|\leq r}|f(z)| \leq \frac{2r}{R-r} \sup_{|z|\leq R} \Re f(z) + \frac{R+r}{R-r} |f(0)|.$$

Proof. See Theorem 5.5 of [53].

□

Theorem 1.27. (Hadamard Three-Circles Theorem) *Let $f(z)$ be a holomorphic function in the region $r_1 \leq |z| \leq r_3$. Let $r_1 < r_2 < r_3$, and let M_1, M_2, M_3 be the maxima of $|f(z)|$ on the three circles $|z| = r_1, r_2, r_3$ respectively. Then*

$$M_2^{\log(\frac{r_3}{r_1})} \leq M_1^{\log(\frac{r_3}{r_2})} M_3^{\log(\frac{r_2}{r_1})},$$

or equivalently,

$$M_2 \leq M_1^{1-\kappa} M_3^{\kappa}, \quad \textit{where} \quad \kappa = \frac{\log\left(\frac{r_2}{r_1}\right)}{\log\left(\frac{r_3}{r_1}\right)}.$$

Proof. See Theorem 5.3 of [53].

□

Theorem 1.28. *Let f be a holomorphic on an open set U, and let a be any point of U. Then, for every $n \geq 0$, we have*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz,$$

where γ is any oriented circular path with centre at a.

Proof. See Theorem 16.20 of [2]. □

Theorem 1.29. *Let f be continuous on an open set U, and suppose that g is analytic in U and $g' = f$. Let α, β be two points of U, and let γ be a path in U joining α to β. Then*

$$\int_{\gamma} f = g(\beta) - g(\alpha).$$

Proof. See Theorem 2.1 of [39].

□

Chapter 2

Multiplicative functions with Sum Zero

This chapter is organised as follows. In the first section, we review CMO functions which are completely multiplicative functions with sum zero. In the second section, we generalise these to multiplicative functions and shall denote them by MO functions. Furthermore, we give some examples of such functions as well as studying their properties.

2.1 CMO functions

A function $f : \mathbb{N} \longrightarrow \mathbb{C}$ is called a ***CMO*** function if it satisfies the two following conditions:

$$f \quad \text{is a completely multiplicative function} \quad \text{and} \quad \sum_{n=1}^{\infty} f(n) = 0.$$

Such functions were first introduced by J.-P. Kahane and E. Saïas [31]. One motivation for them is to gain further insight into the zeros of Dirichlet series with completely multiplicative coefficients. Namely, the Generalised Riemann Hypothesis as discussed below. One of their aims was to find and give necessary and/or sufficient conditions on $f(p)$ for f being a CMO function. They also gave some properties and examples of such functions. For instance, they discussed various examples of CMO functions including $f(n) = \frac{\lambda(n)}{n}$, where $\lambda(n)$ is the Liouville function and $f(n) = \frac{\chi(n)}{n^{\alpha}}$, where χ is a non-principal Dirichlet character and α is a zero of L_χ with $\Re\alpha > 0$.

This study led them to consider the question of how quickly $\sum_{n\le x} f(n)$ can tend to zero. They suggested that it is always $\Omega(\frac{1}{\sqrt{x}})$ and the Generalised Riemann Hypothesis - Riemann Hypothesis (GRH-RH) would follow if their proposition is true. This is because if GRH-RH is false then there is α which is a zero of L_χ with $\Re\alpha > \frac{1}{2}$ which means $\sum_{n\le x} \frac{\chi(n)}{n^\alpha}$ is not $\Omega(\frac{1}{\sqrt{x}})$ (see [31]). This suggestion is incredibly difficult to prove, but it might be easier to disprove; *i.e.*, to find examples such that

$$\sum_{n\le x} f(n) = O\Big(\frac{1}{x^c}\Big) \text{ for some } c > \frac{1}{2}.$$

To date no such counter examples have been found. One approach to consider this counter example question is to consider examples of generalised CMO functions.

2.2 MO functions

In this section, we introduce new functions which are a natural generalisation of CMO functions. We extend the notion of CMO to multiplicative functions and shall call them MO functions. We would like to see how much the theory of CMO functions can be generalised here. To help motivate our enquiries we consider examples of such functions and properties thereof. For example, let f be a MO function and g a multiplicative function "close" to f. We shall show that g is also an MO function under some extra condition on f. We can also ask a similar question of Kahane and Saïas how quickly the partial sum of MO functions up to and including x; (*i.e.* $\sum_{n\le x} f(n)$) can tend to zero. We define these functions as follows:

Definition 2.1. An arithmetical function $f : \mathbb{N} \longrightarrow \mathbb{C}$ is called an $\boldsymbol{MO}$ function if it is multiplicative and satisfies

$$(i)\ \sum_{n=1}^{\infty} f(n) = 0 \quad \text{and} \quad (ii)\ \sum_{k=0}^{\infty} f(p^k) \neq 0 \text{ for all } p \in \mathbb{P}.$$

The extra condition (ii) says the series converges but not to zero. This is needed to avoid trivial examples. For instance, let $f(1) = 1$, $f(2) = -1$ and $f(n) = 0$ for all $n > 2$. Then $\sum_{n=1}^{\infty} f(n) = 0$ but $\sum_{k=1}^{\infty} f(2^k) = f(1) + f(2) + f(4) + \cdots = 0$, and so does not satisfy the extra condition.

2.2.1 Examples

Like CMO functions which have been studied by Kahane and Saïas [31], MO functions are not so easy to find since these need to be conditionally convergent (as we shall see in Proposition 2.6). To help the readers understanding we give three examples of MO functions. The first is based on the Möbius function, the second on the Dirichlet eta function, which corresponds to the case $k = 2$ in the third example.

Example 2.2. The function $\frac{\mu(n)}{n}$ is an MO function since:

(i) it is clear that $\frac{\mu(n)}{n}$ is a multiplicative function;

(ii) it is well-known that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$ (see for example [2] page 97);

(iii) $\sum_{k=0}^{\infty} \frac{\mu(p^k)}{p^k} = 1 - \frac{1}{p} \neq 0$ for all $p \in \mathbb{P}$.

Example 2.3. Consider $\frac{(-1)^{n-1}}{n^\alpha}$ which is multiplicative. For which values of $\alpha \in \mathbb{C}$ with $\Re\alpha > 0$ is this an MO function?

(i) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha}$ converges for $\Re\alpha > 0$ since $A(x) := \sum_{n\leq x}(-1)^{n-1} = O(1)$. Therefore, $0 \leq A(x) \leq 1$ and using Abel summation (see Theorem 1.7), we have

$$\begin{aligned}\sum_{n\leq x} \frac{(-1)^{n-1}}{n^\alpha} &= \frac{A(x)}{x^\alpha} + \alpha \int_1^x \frac{A(t)}{t^{\alpha+1}} dt \\ &= O\Big(\frac{1}{x^{\Re\alpha}}\Big) + \alpha \int_1^\infty \frac{A(t)}{t^{\alpha+1}} dt - \alpha \int_x^\infty \frac{O(1)}{t^{\alpha+1}} dt = C_\alpha + O\Big(\frac{1}{x^{\Re\alpha}}\Big),\end{aligned}$$

where C_α is a constant, since

$$\Big| \int_x^\infty \frac{O(1)}{t^{\alpha+1}} dt \Big| = O\left(\int_x^\infty \frac{1}{t^{\Re\alpha+1}} dt \right) = O\Big(\frac{1}{x^{\Re\alpha}}\Big).$$

Hence, for $\Re\alpha > 0$,

$$\sum_{n\leq x} \frac{(-1)^{n-1}}{n^\alpha} = C_\alpha + O\Big(\frac{1}{x^{\Re\alpha}}\Big).$$

In particular, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha}$ converges. Now, for $\Re\alpha > 0$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha} = (1 - 2^{1-\alpha})\zeta(\alpha). \tag{2.1}$$

This is zero if and only if $2^\alpha = 2$ or $\zeta(\alpha) = 0$ (for $\alpha = 1$, the sum on the left of (2.1) is not zero).

(ii) It remains to establish for which values of α that $\sum_{k=0}^{\infty} \frac{(-1)^{p^k-1}}{p^{\alpha k}} \neq 0$ for all $p \in \mathbb{P}$.

If $p = 2$, then

$$\sum_{k=0}^{\infty} \frac{(-1)^{2^k-1}}{2^{k\alpha}} = 1 - \sum_{k=1}^{\infty} \frac{1}{2^{\alpha k}} = \frac{2^\alpha - 2}{2^\alpha - 1}.$$

This is non-zero if and only if $2^\alpha \neq 2$; (*i.e.* For $\frac{(-1)^{n-1}}{n^\alpha}$ to be MO we therefore need $2^\alpha \neq 2$). Now if $p \geq 3$, then

$$\sum_{k=0}^{\infty} \frac{(-1)^{p^k-1}}{p^{k\alpha}} = \sum_{k=0}^{\infty} \frac{1}{p^{\alpha k}} = \frac{1}{1 - \frac{1}{p^\alpha}}.$$

This is non-zero for any α with $\Re\alpha > 0$.

We see that $\frac{(-1)^{n-1}}{n^\alpha}$ is not an MO function if $2^\alpha = 2$ since (ii) does not hold. Therefore we conclude that $\frac{(-1)^{n-1}}{n^\alpha}$ is an MO function if and only if $\Re\alpha > 0$ and $\zeta(\alpha) = 0$ since (i) and (ii) hold.

Furthermore, if $\zeta(\alpha) = 0$ with $\Re\alpha > 0$, then

$$\sum_{n \leq x} \frac{(-1)^{n-1}}{n^\alpha} = O\Big(\frac{1}{x^{\Re\alpha}}\Big).$$

This example can be generalised as follows:

Example 2.4. Define $g_k(n)$ as follows:

$$g_k(n) := \begin{cases} 1-k & \text{if } k \text{ divides } n, \\ 1 & \text{if } k \text{ does not divide } n. \end{cases}$$

We ask for which positive integer $k > 1$ and α with $\Re\alpha > 0$ is the function $\frac{g_k(n)}{n^\alpha}$ MO ?

When $k = 2$ we get Example 2.3.

(i) We wish to find all k for which $g_k(n)$ is a multiplicative function as follows: If $m = n = 1$, then $g_k(m)g_k(n) = g_k(mn)$. Now if k divides mn, then we have four cases as follows: Assume $(m, n) = 1$.

(a) If k divides both n and m, then $(m,n) \neq 1$. Hence we cannot have k dividing both m, n since we need $(m,n) = 1$.

(b) If k does not divide n and k divides m, then $g_k(m)g_k(n) = (1-k)(1) = 1-k = g_k(mn)$.

or vice versa

(c) If k does not divide m and k divides n, then $g_k(m)g_k(n) = (1)(1-k) = 1-k = g_k(mn)$.

(d) If k does not divide both n and m, then we have two cases:

i. If k is not a prime power; (*i.e.* $k = p_1^{a_1} \cdot p_2^{a_2} \cdots p_i^{a_i}$, where $i \geq 2$ and $a_i \geq 1$). Then, with $m = p_1^{a_1}$ and $n = p_2^{a_2} \cdots p_i^{a_i}$ such that $(m,n) = 1$, we have $g_k(m)g_k(n) = (1)(1) \neq (1-k) = g_k(mn)$.

ii. If k is a prime power; (*i.e.* $k = p^r$). Then at least one of m or n is not a multiple of p while the other is (*i.e.* p does not divide m, then p^r divides n or p does not divide n, then p^r divides m) and $g_k(m)g_k(n) = (1)(1-k) = (1-k) = g_k(mn)$ or $g_k(m)g_k(n) = (1-k)(1) = (1-k) = g_k(mn)$.

However, if k does not divide mn, then k does not divide both m and n, and $g_k(m)g_k(n) = (1)(1) = 1 = g_k(mn)$.

Thus $g_k(n)$ is multiplicative function if and only if k is a prime power.

(ii) The series $\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha}$ converges for $\Re\alpha > 0$ since

$$\begin{aligned} A(x) := \sum_{n \leq x} g_k(n) &= \sum_{m=1}^{N} \sum_{n=(m-1)k+1}^{mk} g_k(n) + \sum_{n=Nk+1}^{x} g_k(n) = 0 + \sum_{n=Nk+1}^{x} g_k(n) \\ &= g_k(Nk+1) + g_k(Nk+2) + \cdots + g_k(x), \quad \text{where } N = \left\lfloor \frac{x}{k} \right\rfloor \\ &\leq k - 1 = O(1). \end{aligned}$$

Thus $0 \leq A(x) \leq k-1$ and using Abel summation, we have

$$\begin{aligned} \sum_{n \leq x} \frac{g_k(n)}{n^\alpha} &= \frac{A(x)}{x^\alpha} + \alpha \int_1^x \frac{A(t)}{t^{\alpha+1}} dt \\ &= O\Big(\frac{1}{x^{\Re\alpha}}\Big) + \alpha \int_1^\infty \frac{A(t)}{t^{\alpha+1}} dt - \alpha \int_x^\infty \frac{O(1)}{t^{\alpha+1}} dt \\ &= C_\alpha + O\Big(\frac{1}{x^{\Re\alpha}}\Big), \end{aligned}$$

where C_α is a constant, as in Example 2.3. In particular, for $\Re\alpha > 0$, $\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha}$ converges.

Now, for $\Re\alpha > 1$, we have

$$\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha} = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} - \sum_{n=1}^{\infty} \frac{k}{(kn)^\alpha} = (1 - k^{1-\alpha})\zeta(\alpha).$$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha} = C_\alpha = (1 - k^{1-\alpha})\zeta(\alpha)$ for $\Re\alpha > 0$ by analytic continuation.

Also, $\displaystyle\sum_{n=1}^{\infty} \frac{g_k(n)}{n^\alpha} = 0$ if and only if $k^\alpha = k$ or $\zeta(\alpha) = 0$.

(iii) It remains to get all k and α for which $\sum_{m=0}^{\infty} \frac{g_k(p^m)}{p^{m\alpha}} \neq 0$ for all $p \in \mathbb{P}$. Let $k = p_0^r$, p_0 a prime number.

If $p_0 \neq p$, then $g_k(p^m) = 1$ for all $m \geq 0$. Hence

$$\sum_{m=0}^{\infty} \frac{g_k(p^m)}{p^{m\alpha}} = \sum_{m=0}^{\infty} \frac{1}{p^{\alpha m}} = \frac{1}{1 - \frac{1}{p^\alpha}}.$$

This is non-zero for any α with $\Re\alpha > 0$. Now if $p_0 = p$, then

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{g_k(p^m)}{p^{m\alpha}} &= \sum_{m=0}^{r-1} \frac{g_k(p^m)}{p^{\alpha m}} + \sum_{m=r}^{\infty} \frac{g_k(p^m)}{p^{\alpha m}} = \sum_{m=0}^{r-1} \frac{1}{p^{\alpha m}} + (1-k) \sum_{m=r}^{\infty} \frac{1}{p^{\alpha m}} \\
&= 1 + \frac{1}{p^\alpha} + \frac{1}{p^{2\alpha}} + \cdots + \frac{1}{p^{(r-2)\alpha}} + \frac{1}{p^{(r-1)\alpha}} + (1-k) \sum_{m=1}^{\infty} \frac{1}{p^{(m+r-1)\alpha}} \\
&= 1 + \frac{1}{p^\alpha} + \frac{1}{p^{2\alpha}} + \cdots + \frac{1}{p^{(r-2)\alpha}} + \frac{1}{p^{(r-1)\alpha}} + \frac{(1-k)}{p^{(r-1)\alpha}} \cdot \frac{1}{p^\alpha - 1} \\
&= \frac{(p^{(r-1)\alpha} + p^{(r-2)\alpha} + \cdots + p^{2\alpha} + p^\alpha + 1)(p^\alpha - 1) + (1-k)}{p^{(r-1)\alpha}(p^\alpha - 1)} \\
&= \frac{p^{r\alpha} + p^{(r-1)\alpha} + \cdots + p^\alpha - p^{(r-1)\alpha} - p^{(r-2)\alpha} - \cdots - p^\alpha - 1 + 1 - k}{p^{(r-1)\alpha}(p^\alpha - 1)} \\
&= \frac{p^{r\alpha} - k}{p^{(r-1)\alpha}(p^\alpha - 1)} = \frac{k^\alpha - k}{k^\alpha(1 - p^{-\alpha})}.
\end{aligned}$$

This is non-zero if and only if $k^\alpha \neq k$; (*i.e.*, For $\frac{g_k(n)}{n^\alpha}$ to be *MO* we therefore need $k^\alpha \neq k$).

We see that $\frac{g_k(n)}{n^\alpha}$ is not an *MO* function if $k^\alpha = k$ since (iii) fails. Therefore, we conclude that $\frac{g_k(n)}{n^\alpha}$ is an MO function if and only if k is a prime power, $\Re\alpha > 0$ and $\zeta(\alpha) = 0$ since (i), (ii) and (iii) hold.

Furthermore, if $\zeta(\alpha) = 0$ with $\Re\alpha > 0$, then

$$\sum_{n\leq x} \frac{g_k(n)}{n^\alpha} = O\Big(\frac{1}{x^{\Re\alpha}}\Big).$$

2.2.2 Some properties of MO functions

In this section, we establish some preliminary properties of MO functions.

Propostion 2.5. *If f is a CMO function, then f is an MO function; (i.e $CMO \subset MO$).*

Proof. It is clear that f is multiplicative and $\sum_{n=1}^{\infty} f(n) = 0$. It remains to show that $\sum_{k=0}^{\infty} f(p^k) \neq 0$ for all $p \in \mathbb{P}$. Now since f is completely multiplicative, then $f(p^k) = f(p)^k$. Therefore

$$\sum_{k=0}^{\infty} f(p^k) = \sum_{k=0}^{\infty} f(p)^k = \frac{1}{1 - f(p)} \neq 0.$$

This series converges since $|f(p)| < 1$.

Hence, by Definition 2.1, f is an MO function.

□

Propostion 2.6. *Let f be an MO function. Then $\sum_{n=1}^{\infty} |f(n)|$ diverges. Indeed $\sum_p \sum_{k=1}^{\infty} |f(p^k)|$ diverges.*

Proof. Let us assume that the statement is false, so that

$$\sum_{n=1}^{\infty} |f(n)| \text{ converges.}$$

Then, by multiplicative property,

$$\sum_{n=1}^{\infty} f(n) = \prod_p \sum_{k=0}^{\infty} f(p^k) \neq 0 \text{ since } \sum_{k=0}^{\infty} f(p^k) \neq 0.$$

Yielding a contradiction since f is an MO function and hence

$$\sum_{n=1}^{\infty} |f(n)| \text{ diverges.}$$

Furthermore, Proposition 1.21 gives $\sum_p \sum_{k=1}^{\infty} |f(p^k)|$ diverges, as required.

□

2.2.3 Partial sums of MO functions

We know that the partial sum of an MO function not exceeding x tends to zero when x tends to infinity. A question raised by Kahane and Saïas [31] regarding CMO functions is: can one show, given $g(x)$, that there exist a CMO function f with

$$\sum_{n \le x} f(n) = \Omega(g(x))?$$

We are not considering this question, but we are interested in a related question which is: how small can we make $g(x)$, so that the above is true for all MO functions f? This question motivates the following propositions:

Propostion 2.7. *If f is an MO function, then*

$$\sum_{n \le x} f(n) = \Omega\Big(\frac{1}{x \log x}\Big).$$

Proof. Let us assume that the statement is false, so that

$$\sum_{n \le x} f(n) = O\Big(\frac{1}{x \log x}\Big).$$

We know that for $n \in \mathbb{N}$,

$$f(n) = \sum_{m \le n} f(m) - \sum_{m < n} f(m) = O\Big(\frac{1}{n \log n}\Big).$$

Hence

$$f(p^k) = O\Big(\frac{1}{p^k \log p^k}\Big).$$

Now it follows that $\sum_p \sum_{k=1}^{\infty} |f(p^k)|$ converges since

$$\begin{aligned}
\sum_p \sum_{k=1}^{\infty} \frac{1}{p^k \log p^k} &\leq \sum_p \sum_{k=1}^{\infty} \frac{1}{p^k \log p} \quad (\text{since } \log p^k \geq \log p) \\
&= \sum_p \frac{1}{\log p} \sum_{k=1}^{\infty} \frac{1}{p^k} \\
&= \sum_p \frac{1}{(p-1)\log p} \quad \text{converges (since } p_n \log p_n \sim n(\log n)^2).
\end{aligned}$$

Thus

$$\sum_p \sum_{k=1}^{\infty} \frac{1}{p^k \log p^k} \text{ converges.}$$

Hence, by Proposition 1.21, $\sum_{n=1}^{\infty} |f(n)|$ converges. However, by Proposition 2.6, we have a contradiction, and so it follows that

$$\sum_{n \leq x} f(n) = \Omega\Big(\frac{1}{x \log x}\Big).$$

□

Remark 2.8. Similarly, if f is an MO function, then

$$\sum_{n \leq x} f(n) = \Omega\big(\frac{1}{x(\log x)^{\varepsilon}}\big) \quad \text{for all } \varepsilon > 0.$$

We can improve Proposition 2.7 using the fact that $\sum_p \frac{1}{p(\log \log p)^2}$ converges.

Propostion 2.9. *If f is an MO function, then*

$$\sum_{n \leq x} f(n) = \Omega\Big(\frac{1}{x(\log \log x)^2}\Big).$$

Proof. Let us assume that the statement is false, so that

$$\sum_{n \leq x} f(n) = O\Big(\frac{1}{x(\log \log x)^2}\Big).$$

We know that for $n \in \mathbb{N}$,

$$f(n) = \sum_{m \le n} f(m) - \sum_{m < n} f(m) = O\Big(\frac{1}{n(\log\log n)^2}\Big).$$

Hence

$$f(p^k) = O\Big(\frac{1}{p^k(\log\log p^k)^2}\Big).$$

Now it follows that $\sum_p \sum_{k=1}^\infty |f(p^k)|$ converges since

$$\begin{aligned}\sum_{p\ge 3}\sum_{k=1}^\infty \frac{1}{p^k(\log\log p^k)^2} &\le \sum_{p\ge 3}\sum_{k=1}^\infty \frac{1}{p^k(\log\log p)^2} \quad (\text{since } (\log\log p^k)^2 \ge (\log\log p)^2)\\ &= \sum_{p\ge 3} \frac{1}{(\log\log p)^2} \sum_{k=1}^\infty \frac{1}{p^k}\\ &= \sum_{p\ge 3} \frac{1}{(p-1)(\log\log p)^2} \text{ converges (since } (\log\log p_n)^2 \sim (\log\log n)^2).\end{aligned}$$

For $p = 2$,

$$\sum_{k=1}^\infty \frac{1}{2^k(\log\log 2^k)^2} \le \frac{1}{2(\log\log 2)^2} + \frac{1}{(\log\log 4)^2}\sum_{k\ge 2}\frac{1}{2^k} \quad \text{converges.}$$

Thus

$$\sum_p \sum_{k=1}^\infty \frac{1}{p^k(\log\log p^k)^2} \text{ converges.}$$

Hence, by Proposition 1.21, $\sum_{n=1}^\infty |f(n)|$ converges. However, by Proposition 2.6, we have a contradiction, and so it follows that

$$\sum_{n\le x} f(n) = \Omega\Big(\frac{1}{x(\log\log x)^2}\Big).$$

□

Remark 2.10. Similarly, if f is an MO function, then

$$\sum_{n\le x} f(n) = \Omega\big(\frac{1}{x(\log\log x)^{1+\varepsilon}}\big) \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

Kahane and Saïas [31] have shown that if f is a CMO function, then

$$\sum_{n\leq x} f(n) = \Omega\Big(\frac{1}{x}\Big)$$

by using a deep result of D. Koukoulopoulos in [33]. We attempted to improve (2.2) to $\Omega(\frac{1}{x})$ as with the work of Kahane and Saïas [31], but the question is still open.

2.2.4 Closeness relation between two multiplicative functions

Let $\mathcal{M} := \{f : \mathbb{N} \longrightarrow \mathbb{C} \text{ multiplicative}\}$, and let us define an ***(extended) metric*** on $\mathcal{M}$ to be the distance function

$$D(f,g) := \sum_{p}\sum_{k=0}^{\infty} |g(p^k) - f(p^k)|.$$

Then $\mathcal{M}$ is an ***extended metric space*** since $D(f,g)$ can attain the value ∞. It is straightforward to check for all $f, g, h \in \mathcal{M}$

(i) $D(f,g) = 0$ if and only if $f = g$,

(ii) $D(f,g) = D(g,f)$,

(iii) $D(f,h) \leq D(f,g) + D(g,h)$,

hold. We aim to extend Theorem 3 of Kahane and Saïas in [31] by showing that if f is an MO function and g is a multiplicative function "close" to f, (*i.e.* g has finite distance from f), then g is also an MO function. We can do this under an extra condition on f, as the following theorem shows.

Theorem 2.11. *Let f be an MO function for which*

$$\Big|\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}\Big| \geq a \quad \textit{for some } a > 0,\ \textit{for all } p \textit{ and all } \Re s \geq 0, \tag{2.3}$$

and let g be a multiplicative function such that $D(f,g)$ is finite and

$$\sum_{k=0}^{\infty} g(p^k) \neq 0 \quad \textit{for all } p. \tag{2.4}$$

Then g is an MO function.

Proof. Let $F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ and $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$. Then the series for $F(s)$ is absolutely convergent for $\Re s > 1$ and it is convergent for $\Re s > 0$ and $s = 0$ since $\sum_{n=1}^{\infty} f(n) = 0$. We note that the assumption $D(f, g)$ is finite and the fact that f is an MO function imply $|g(p^k)| \longrightarrow 0$ as $p^k \longrightarrow \infty$. Then, by Theorem 1.9, $g(n) \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore the series for $G(s)$ converges for $\Re s > 1$ since g is bounded. Therefore $F(s)$ and $G(s)$ can be written as follows:

$$F(s) = \prod_p \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \quad \text{and} \quad G(s) = \prod_p \sum_{k=0}^{\infty} \frac{g(p^k)}{p^{ks}} \quad \Re s > 1.$$

Now

$$H(s) := \prod_p \left(\frac{\sum_{k=0}^{\infty} \frac{g(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}} \right) = \prod_p \left(1 + \frac{\sum_{k=0}^{\infty} \frac{g(p^k)-f(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}} \right)$$

converges absolutely for $\Re s \geq 0$ if and only if

$$\sum_p \frac{\left| \sum_{k=0}^{\infty} \frac{g(p^k)-f(p^k)}{p^{ks}} \right|}{\left| \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right|} \tag{2.5}$$

converges for $\Re s \geq 0$. But

$$\sum_p \frac{\left| \sum_{k=0}^{\infty} \frac{g(p^k)-f(p^k)}{p^{ks}} \right|}{\left| \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right|} \leq \frac{1}{a} \sum_p \sum_{k=0}^{\infty} |g(p^k) - f(p^k)|$$

by (2.3) so, since $D(f, g)$ is finite, (2.5) converges for $\Re s \geq 0$ and $H(s)$ converges absolutely to holomorphic function for $\Re s > 0$. However, $H(s) = (G/F)(s)$ for $\Re s > 1$ then $G(s) = F(s)H(s)$, where the series for $F(s)$ converges for $\Re s > 0$ and $s = 0$ since f is an MO function, and $H(s)$ converges absolutely for $\Re s \geq 0$. Therefore $G(s)$ converges for $\Re s > 0$ and $s = 0$ using the extension of Theorem 1.18. Thus we have $G(0) = F(0)H(0) = 0$. Hence, by assumption (2.4) and $G(0) = 0$, g is an MO function.

□

The proof of Theorem 2.11 also gives the following result.

Corollary 2.12. *Let f and g both be multiplicative functions such that $D(f, g)$ is* *finite and satisfies*

$$\Big|\sum_{k=0}^{\infty}\frac{f(p^k)}{p^{ks}}\Big| \geq a \quad \textit{for some } a>0 \textit{ and all } \Re s \geq 0,$$

$$\Big|\sum_{k=0}^{\infty}\frac{g(p^k)}{p^{ks}}\Big| \geq b \quad \textit{for some } b>0 \textit{ and all } \Re s \geq 0.$$

Then the following two assertions are equivalent:

$$\sum_{n=1}^{\infty} f(n) = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} g(n) = 0.$$

2.2.5 Open problems

(i) Let f be an MO function. Can we show that

$$\sum_{n\leq x} f(n) = \Omega\Big(\frac{1}{x}\Big) \ ?$$

(ii) As pointed out earlier Kahane and Saïas suggested that for all CMO functions, one has $\sum_{n\leq x} f(n) = \Omega\big(\frac{1}{\sqrt{x}}\big)$. As also mentioned, GRH-RH (Generalised Riemann Hypothesis-Riemann Hypothesis) would follow if their suggestion is correct.

In Example 2.2, it is known that $\sum_{n\leq x}\mu(n) = \Omega(\sqrt{x})$ since there are zeros of the Riemann zeta function ζ on the line $\Re s = \frac{1}{2}$ (see [54]). Thus, by Abel summation,

$$\sum_{n\leq x}\frac{\mu(n)}{n} = \Omega\Big(\frac{1}{\sqrt{x}}\Big).$$

However, for $\sum_{n\leq x}\frac{(-1)^{n-1}}{n^\alpha}$ and $\sum_{n\leq x}\frac{g_k(n)}{n^\alpha}$ to converge to zero in Examples 2.3 and 2.4, it is necessary that α be a zero of $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$ and $\sum_{n=1}^{\infty}\frac{g_k(n)}{n^s}$ with $\Re\alpha > 0$; (*i.e.* $\zeta(\alpha)=0$). Suppose this is the case. We then have

$$\sum_{n\leq x}\frac{(-1)^{n-1}}{n^\alpha} = O\Big(\frac{1}{x^{\Re\alpha}}\Big) \text{ and } \sum_{n\leq x}\frac{g_k(n)}{n^\alpha} = O\Big(\frac{1}{x^{\Re\alpha}}\Big),$$

and

$$\sum_{n\le x}\frac{(-1)^{n-1}}{n^\alpha}=\Omega\Big(\frac{1}{x^{\Re\alpha}}\Big) \text{ and } \sum_{n\le x}\frac{g_k(n)}{n^\alpha}=\Omega\Big(\frac{1}{x^{\Re\alpha}}\Big).$$

In our results, we have not found any examples with $\sum_{n\le x} f(n)=O\big(\frac{1}{x^c}\big)$ for $c>\frac{1}{2}$. This may suggest the following conjecture.

Conjecture 2.13. For all multiplicative function f (MO functions), we have

$$\sum_{n\le x} f(n)=\Omega\Big(\frac{1}{\sqrt{x}}\Big).$$

Furthermore, the RH would follow if Conjecture 2.13 were true since if RH is false then there is α which is a zero of ζ with $\Re\alpha>\frac{1}{2}$ which means $\sum_{n\le x}\frac{(-1)^{n-1}}{n^\alpha}$ and $\sum_{n\le x}\frac{g_k(n)}{n^\alpha}$ is not $\Omega(\frac{1}{\sqrt{x}})$.

Chapter 3

Beurling generalised prime systems

In this chapter, we turn our attention to introduce Beurling generalised prime systems, along with Beurling's Prime Number Theorem in relation to these systems. We also give some relevant known results about these systems which we use in this work.

3.1 g-prime systems

The concept of ***generalised primes*** and ***generalised integers*** was introduced by A. Beurling in the 1930s and has been studied by many researchers since then (see for instance [4], [15], [21], [25] , [58]). The structure of this system is defined to be a sequence of real positive numbers $\mathcal{P} = \{p_1, p_2, p_3, \ldots\}$ which need not be actual primes (or even integers) satisfying

$$1 < p_1 \leq p_2 \leq \cdots \leq p_i \leq \cdots$$

and for which $p_i \longrightarrow \infty$ as $i \longrightarrow \infty$. With this sequence we can form a new increasing sequence

$$1 < n_1 \leq n_2 \leq \cdots \leq n_i \leq \cdots$$

of real numbers which represent all possible products $\prod_{i=1}^{k} p_i^{a_i}$, where $k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. These new elements are called ***generalised integers*** associated to $\mathcal{P}$ and denoted by $\mathcal{N}$; (*i.e.* $\mathcal{N} = \{n_i\}_{i\geq 1}$). Attached to these systems we have the usual counting functions $\pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$ which are the sum over all the g-primes and g-integers not exceeding the positive real number x, counting multiplicities, respectively; that is

$$\pi_{\mathcal{P}}(x) = \sum_{\substack{p_i \leq x \\ i \in \mathbb{N}}} 1 \quad \text{and} \quad N_{\mathcal{P}}(x) = \sum_{\substack{n_i \leq x \\ i \in \mathbb{N}}} 1,$$

which can be written equivalently in the more standard notation in many books and papers:

$$\pi_{\mathcal{P}}(x) = \sum_{\substack{p \leq x \\ p \in \mathcal{P}}} 1 \quad \text{and} \quad N_{\mathcal{P}}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} 1.$$

These type of systems are discrete systems, where π and N are step functions with integer jumps. There is also a concept of continuous g-prime systems [12] [26], but they shall not concern us here. The ***generalised zeta function***, the associated zeta function, is formally defined by

$$\begin{aligned} \zeta_{\mathcal{P}}(s) &= \prod_{i=1}^{\infty} \frac{1}{1 - p_i^{-s}} = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}} \\ &= \sum_{i=0}^{\infty} \frac{1}{n_i^s} = \sum_{n \in \mathcal{N}} \frac{1}{n^s}. \end{aligned} \tag{3.1}$$

We say that a g-prime system $\mathcal{P}$ has an ***abscissa of convergence*** σ_c if and only if (3.1) converges for $\Re s > \sigma_c$ and diverges for $\Re s < \sigma_c$. The product is called the Euler product of the Beurling zeta function. The sum of (3.1) represents the generalised Dirichlet series which will be generated by multiplying out this product in the same way as the standard Euler product, defined in Chapter 1.

Remark 3.1.

(i) Beurling prime systems generalise the concept of the primes and natural numbers which are obtained from the original prime numbers by taking all possible products of these.

(ii) Note that $\mathcal{P}$ and $\mathcal{N}$ are in general multi-sets and $\mathcal{N}$ is the semi-group generated by $\mathcal{P}$ under multiplication.

(iii) We write g-primes, g-integers and g-zeta (*i.e.* $\mathcal{P}$, $\mathcal{N}$ and $\zeta_{\mathcal{P}}$) to mean ***Beurling (or generalised) prime systems***, ***integers*** and ***zeta function*** respectively.

(iv) The abscissa of convergence of the series $\sum_{n \in \mathcal{N}} \frac{1}{n^s}$ and $\sum_{p \in \mathcal{P}} \frac{1}{p^s}$ is the same.

(v) If abscissa of $\mathcal{P}$ is σ $(0 < \sigma < \infty)$, then $\mathcal{P}' := \mathcal{P}^\sigma = \{p^\sigma : p \in \mathcal{P}\}$ has abscissa 1 and as such $\mathcal{N}' = \mathcal{N}^\sigma = \{n^\sigma : n \in \mathcal{N}\}$. At the outset we are only interested in those systems for which the abscissa of convergence of the Dirichlet series for $\zeta_{\mathcal{P}}$ is 1.

3.1.1 Examples

We provide some examples that describe systems $\mathcal{P}$ which have different abscissa of convergence as follows:

Example 3.2. If $\mathcal{P}_1 = \{3, 5, 7, 11, \cdots\} = \mathbb{P} \setminus \{2\}$; *i.e.* $p_n = n^{th}$ odd prime. Then $\mathcal{N}_1 = \{1, 3, 5, 7, 9, \cdots\}$ which is the set of odd numbers. The counting functions $\pi_{\mathcal{P}_1}(x)$ and $N_{\mathcal{P}_1}(x)$ are:

$$\begin{aligned}\pi_{\mathcal{P}_1}(x) = \sum_{\substack{p \leq x \\ p \in \mathcal{P}_1}} 1 &= \pi(x) - 1 \quad \text{for } x \geq 2 \\ &= \frac{x}{\log x}(1 + o(1)) \quad \text{by PNT,}\end{aligned}$$

and

$$N_{\mathcal{P}_1}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}_1}} 1 = \sum_{\substack{2k-1 \leq x \\ k \in \mathbb{N}}} 1 = \frac{x}{2} + O(1).$$

The abscissa of convergence of the Dirichlet series of $\mathcal{P}_1$ is 1 since

$$\sum_{n \in \mathcal{N}} \frac{1}{n^s} = \sum_{\substack{n = 2k-1 \\ k \in \mathbb{N}}} \frac{1}{n^s} \quad \text{converges } \Re s > 1 \text{ and diverges } \Re s \leq 1.$$

Example 3.3. If $\mathcal{P}_2 = \{2, 2, 3, 3, 5, 5, 7, 7, \cdots\}$ (every prime appears twice), then $\mathcal{N}_2 = \{1, 2, 2, 3, 3, 4 = 2{\cdot}2, 4 = 2{\cdot}2, 4 = 2{\cdot}2, 5, 5, 6 = 2{\cdot}3, 6 = 2{\cdot}3, 6 = 2{\cdot}3, 6 = 2{\cdot}3, \cdots\}$ with multiplicity $d(n)$ (every integer appears $d(n)$ times) since

$$\begin{aligned}\zeta_{\mathcal{P}}(s) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1} &= \left(\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}\right)\left(\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}\right) = \zeta^2(s) \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)^2 = \sum_{n=1}^{\infty} \frac{(1 * 1)(n)}{n^s} = \sum_{n=1}^{\infty} \frac{d(n)}{n^s},\end{aligned}$$

where $d(n)$ is the divisor counting function. The counting functions $\pi_{\mathcal{P}_2}(x)$ and $N_{\mathcal{P}_2}(x)$

are:

$$\pi_{\mathcal{P}_2}(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P}_2}} 1 = 2\sum_{p\le x} 1 = 2\pi(x) = \frac{2x}{\log x}(1+o(1)) \quad \text{by PNT,}$$

and

$$N_{\mathcal{P}_2}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}_2}} 1 = \sum_{\substack{n \le x \\ n \in \mathbb{N}}} d(n) \sim x \log x \quad \text{as } x \longrightarrow \infty,$$

by the proof of Theorem 3.3 of [2]. This system has abscissa of convergence 1 since

$$\sum_{n\in\mathcal{N}} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad \text{converges } \Re s > 1 \text{ and diverges } \Re s \le 1,$$

by the Theorem 291 of [24].

Example 3.4. If $\mathcal{P}_3 = \{4, 9, 16, 25, \cdots\}$, where $p_n = (n+1)^2$ for $n \in \mathbb{N}$. Then we find $\mathcal{N}_3 = \{1, 4, 9, 16, 16, 25, 36, 36, \cdots\}$ with all elements $(n+1)^2$. The counting function $\pi_{\mathcal{P}_3}(x)$ is:

$$\pi_{\mathcal{P}_3}(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P}_3}} 1 = \sum_{(n+1)^2\le x} 1 = \sum_{n\le\sqrt{x}-1} 1 = \sqrt{x} + O(1).$$

But, the asymptotic behaviour of $N_{\mathcal{P}_3}(x)$ may not clear. The abscissa of convergence of the Dirichlet series of $\mathcal{P}_3$ is $\frac{1}{2}$ since

$$\sum_{p\in\mathcal{P}} \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^{2s}} \quad \text{converges } \Re s > \tfrac{1}{2} \text{ and diverges } \Re s \le \tfrac{1}{2}.$$

Example 3.5. Let $\mathcal{P}_4 = \{2, 4, 8, 16, \cdots, 2^n, \cdots\}$, where $p_n = 2^n$ for $n \in \mathbb{N}$. Then we find $\mathcal{N}_4 = \{1, 2, 4, 4, 8, 8, 8, 16, 16, 16, 16, 16, \cdots\}$ which made from 2^n with multiplicity $p(n)$, where $p(n)$ is the number of partitions of n. The counting functions $\pi_{\mathcal{P}_4}(x)$ and $N_{\mathcal{P}_4}(x)$ are:

$$\pi_{\mathcal{P}_4}(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P}_4}} 1 = \sum_{2^n\le x} 1 = \sum_{n\le\frac{\log x}{\log 2}} 1 = \frac{\log x}{\log 2} + O(1),$$

and

$$N_{\mathcal{P}_4}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}_4}} 1 = \sum_{\substack{2^n \le x \\ n \ge 0}} p(n) = \sum_{0\le n\le\frac{\log x}{\log 2}} p(n) \;= e^{(c+o(1))\sqrt{\frac{\log x}{\log 2}}}$$

since $p(n) = e^{(c+o(1))\sqrt{n}}$, where $c = \pi\sqrt{\frac{2}{3}}$. A more detailed asymptotic formula was independently discovered by G. H. Hardy and S. Ramanujan [22] and Y. V. Uspensky [55]. This system has abscissa of convergence 0 since

$$\sum_{p\in\mathcal{P}} \frac{1}{p^s} = \left(\frac{1}{2^s}\right)^n \quad \text{converges } \Re s > 0 \text{ and diverges } \Re s \leq 0.$$

Example 3.6. If $\mathcal{P}_5 = \{2, 2+\log 2, 2+\log 3, 2+\log 4, \cdots\}$, where $p_n = 2+\log n$ and $n \in \mathbb{N}$, then $\mathcal{N}_5 = \{1, 2, 2+\log 2, \cdots, 2+\log 100, 4, 2+\log 101, \cdots\}$. The counting function $\pi_{\mathcal{P}_5}(x)$ can be found as follows:

$$\pi_{\mathcal{P}_5}(x) = \sum_{\substack{p \,\leq\, x \\ n \,\in\, \mathcal{P}_5}} 1 = \sum_{(2+\log n)\leq x} 1 = \sum_{n\leq e^{x-2}} 1 = e^{x-2} + O(1).$$

However, the asymptotic behaviour of $N_{\mathcal{P}_5}(x)$ is less clear. The abscissa of convergence of the Dirichlet series of $\mathcal{P}_5$ is ∞ since

$$\sum_{p\in\mathcal{P}} \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{1}{(2+\log n)^s} \quad \text{diverges for all } s.$$

3.2 Arithmetical functions over $\mathcal{N}$

This section introduces the concept of divisibility that will be necessary for this thesis to define the greatest common divisor of two integers over Beurling generalised integers. It also provides a definition of functions that are defined over $\mathcal{P}$ and $\mathcal{N}$ such as arithmetic, multiplicative and completely multiplicative with some examples like Liouville and Möbius functions. This section also introduces Dirichlet convolution of arithmetical functions over $\mathcal{N}$.

Definition 3.7. Let $m, n \in \mathcal{N}$, say $m = p_1^{a_1} \cdots p_k^{a_k}$ and $n = p_1^{b_1} \cdots p_k^{b_k}$, where $p_i \in \mathcal{P}$ are distinct, $k \in \mathbb{N}$ and $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{N}_0$. As such, we say m ***divides*** n if $a_i \leq b_i$ for all i, or equivalently, $n = rm$ for some $r \in \mathcal{N}$. In all other cases, we say m does not divide n.

E. M. Horadam [28] only defined this in the case where the n_i are distinct; (*i.e.* the multiplicities all 1). We use the same but now the n_i do not have to be distinct. We could here have $m = n \in \mathcal{N}$ numerically the same but m does not divide n; (*i.e.* if they are made from different g-primes).

Definition 3.8. With m and n as in Definition 3.7, the ***greatest common divisor*** (m, n) of any $m, n \in \mathcal{N}$, is defined as the largest g-integer that divides both m and n; ($i.e.$ with m and n as above, $(m, n) = p_1^{c_1} \cdots p_k^{c_k} \in \mathcal{N}$, where each $c_i = \min\{a_i, b_i\}$ and $i = 1, \ldots, k$) (see [28]).

Definition 3.9. An ***arithmetical function*** with domain $\mathcal{N}$ is a function $f : \mathcal{N} \to \mathbb{C}$ which is defined on the multi-set of Beurling integers $\mathcal{N}$.

Remark 3.10. Note that we are abusing the notion of function in case of multiplicities. This is done for clarity of notation. In much of our work, we are not interested in the arithmetic function $f : \mathcal{N} \longrightarrow \mathbb{C}$ itself, but in the partial sum of the function $f(n)$ up to and including x; $i.e.$

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} f(n),$$

which is a function because it counts all the possible elements of $\mathcal{N}$ up to x.

3.2.1 Some special functions

In this section, we define the functions $\mu_{\mathcal{P}}(n)$, $\Lambda_{\mathcal{P}}(n)$, and $\lambda_{\mathcal{P}}(n)$ which generalise the standard functions $\mu(n)$, $\Lambda(n)$, and $\lambda(n)$.

Example 3.11. (Möbius's function) We define ***generalised Möbius function*** on $\mathcal{N}$ to be $\mu_{\mathcal{P}}(1) = 1$, $\mu_{\mathcal{P}}(p_{i_1} \cdots p_{i_k}) = (-1)^k$ for distinct g-primes; (i.e. $i_1, \ldots, i_k$ are distinct) and zero otherwise.

Example 3.12. (Mangoldt's function) We define ***generalised Mangoldt function*** for $n \in \mathcal{N}$ as follows:

$$\Lambda_{\mathcal{P}}(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some } g\text{-prime } p \in \mathcal{P} \text{ and integer } k \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.13. (Liouville's function) We define ***generalised Liouville function*** on $\mathcal{N}$ to be $\lambda_{\mathcal{P}}(1) = 1$ and $\lambda_{\mathcal{P}}(n) = (-1)^{a_1 + \cdots + a_k}$ for $n = p_1^{a_1} \cdots p_k^{a_k} \in \mathcal{N}$, where $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{N}_0$.

Remark 3.14. As mentioned these examples may not necessarily be functions if a g-integer can be made from different g-primes. We could have two g-integers n_1 and $n_2 \in \mathcal{N}$ that are numerically the same but n_1 does not divide n_2. From Example 3.3, for $\mathcal{N}_2$, we have here 4=4 but $4 = 2 \cdot 2$ does not divide $4 = 2 \cdot 2$; ($i.e.$ $2 \cdot 2 \nmid 2 \cdot 2$).

Moreover, we notice that $\mu_{\mathcal{P}}(2 \cdot 2) = (-1)^2 = 1$ and $\mu_{\mathcal{P}}(2 \cdot 2) = \mu_{\mathcal{P}}(2 \cdot 2) = 0$ while $\lambda_{\mathcal{P}}(2 \cdot 2) = \lambda_{\mathcal{P}}(2 \cdot 2) = \lambda_{\mathcal{P}}(2 \cdot 2) = (-1)^2 = 1$.

We also note that

$$\sum_{\substack{d|(n\,=\,2\,\cdot\,2) \\ d\,\in\,\mathcal{N}_2}} \lambda_{\mathcal{P}}(d) = \lambda_{\mathcal{P}}(1) + \lambda_{\mathcal{P}}(2) + \lambda_{\mathcal{P}}(2 \cdot 2) = 1 - 1 + 1 = 1,$$

whereas $$\sum_{\substack{d|(n\,=\,2\,\cdot\,2) \\ d\,\in\,\mathcal{N}_2}} \lambda_{\mathcal{P}}(d) = \lambda_{\mathcal{P}}(1) + \lambda_{\mathcal{P}}(2) + \lambda_{\mathcal{P}}(2) + \lambda_{\mathcal{P}}(2 \cdot 2) = 1 - 1 - 1 + 1 = 0.$$

3.2.2 Multiplicative functions on $\mathcal{N}$

In this section, we present the definition of multiplicative and completely multiplicative functions. We also introduce the function $\psi_{\mathcal{P}}(n)$ which generalises the standard function $\psi(n)$.

Definition 3.15. A function $f : \mathcal{N} \longrightarrow \mathbb{C}$ is said to be ***multiplicative*** on $\mathcal{N}$ if $f(1) = 1$ and it satisfies

$$f(mn) = f(m)f(n) \quad \text{whenever } (m, n) = 1.$$

Such an f is said to be ***completely multiplicative*** [57] if we also have

$$f(mn) = f(m)f(n) \quad \text{for all values of } m, n \in \mathcal{N}.$$

The functions $\mu_{\mathcal{P}}(n)$ and $\lambda_{\mathcal{P}}(n)$ are examples of multiplicative and completely multiplicative functions.

As for classical multiplicative functions, if f and g are multiplicative functions and $f(p^k) = g(p^k)$ for all g-primes $p \in \mathcal{P}$ and $k \in \mathbb{N}_0$, then $f = g$.

Definition 3.16. (Chebyshev's ψ-function) We define the ***generalised Chebyshev function*** over $\mathcal{N}$ with the sum extending over all g-prime numbers $p \in \mathcal{P}$ that are less than or equal to x as follows:

$$\psi_{\mathcal{P}}(x) = \sum_{\substack{p^k\,\leq\,x \\ p\,\in\,\mathcal{P} \\ k\,\in\,\mathbb{N}}} \log p = \sum_{\substack{n\,\leq\,x \\ n\,\in\,\mathcal{N}}} \Lambda_{\mathcal{P}}(n).$$

As in classical prime number theory, we introduce the g-prime counting function

$$\Pi_{\mathcal{P}}(x) = \sum_{n=1}^{\infty} \frac{\pi_{\mathcal{P}}(x^{\frac{1}{n}})}{n}.$$

It is related to $\psi_{\mathcal{P}}(x)$ via

$$\psi_{\mathcal{P}}(x) = \int_{p_1}^{x} \log t \, d\, \Pi_{\mathcal{P}}(t).$$

We can also define the functions $M_{\mathcal{P}}$, $L_{\mathcal{P}}$, $m_{\mathcal{P}}$ and $l_{\mathcal{P}}$ which represent the following partial sums:

$$M_{\mathcal{P}}(x) := \sum_{\substack{n_i \le x \\ i \in \mathbb{N}}} \mu_{\mathcal{P}}(n_i) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \mu_{\mathcal{P}}(n),$$

$$L_{\mathcal{P}}(x) := \sum_{\substack{n_i \le x \\ i \in \mathbb{N}}} \lambda_{\mathcal{P}}(n_i) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n),$$

$$m_{\mathcal{P}}(x) := \sum_{\substack{n_i \le x \\ i \in \mathbb{N}}} \frac{\mu_{\mathcal{P}}(n_i)}{n_i} = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(n)}{n},$$

and

$$l_{\mathcal{P}}(x) := \sum_{\substack{n_i \le x \\ i \in \mathbb{N}}} \frac{\lambda_{\mathcal{P}}(n_i)}{n_i} = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{\lambda_{\mathcal{P}}(n)}{n}.$$

We will use these throughout the rest of this thesis.

3.2.3 Dirichlet convolution of arithmetical functions over $\mathcal{N}$

In this section, we provide the definition of the ***Dirichlet convolution*** of arithmetical functions $f * g$ which is multiplicative if f and g are multiplicative over Beurling integers $\mathcal{N}$.

Definition 3.17. The convolution of arithmetical functions f and g over $\mathcal{N}$ was defined by E. M. Horadam [29] as follows:

$$(f * g)(n) = \sum_{\substack{d|n \\ d \in \mathcal{N}}} f(d)\, g\Big(\frac{n}{d}\Big).$$

E. M. Horadam only defined this in the case where the n_i are distinct as previously mentioned. We use the same but now the n_i do not have to be distinct. Again this need not be a function in the sense that we described earlier (a g-integer can be made from different g-primes).

Theorem 3.18. *Let $f, g : \mathcal{N} \longrightarrow \mathbb{C}$ both be multiplicative functions. Then their Dirichlet convolution $f * g$ is also multiplicative.*

Proof. Let $h = f * g$ and let $(m, n) = 1$, where $m, n \in \mathcal{N}$. Then

$$h(mn) = \sum_{\substack{d \mid mn \\ d \in \mathcal{N}}} f(d)\, g\Big(\frac{mn}{d}\Big).$$

Now every divisor d of mn can be written uniquely as $d = xy$, where $x \mid m$ and $y \mid n$. In addition, $(x, y) = 1$ and $(\frac{m}{x}, \frac{n}{y}) = 1$. Hence

$$\begin{aligned} h(mn) &= \sum_{\substack{x \mid m \\ y \mid n}} f(xy)\, g\Big(\frac{mn}{xy}\Big) = \sum_{\substack{x \mid m \\ y \mid n}} f(x) f(y)\, g\Big(\frac{m}{x}\Big) g\Big(\frac{n}{y}\Big) \\ &= \sum_{\substack{x \mid m \\ x \in \mathcal{N}}} f(x)\, g\Big(\frac{m}{x}\Big) \sum_{\substack{y \mid n \\ y \in \mathcal{N}}} f(y)\, g\Big(\frac{n}{y}\Big) = h(m) h(n) \end{aligned}$$

□

3.3 Abel's Identity over $\mathcal{N}$

Abel summation (see Theorem 1.7) can be extended to series over $\mathcal{N}$. The basic theorem allows one to express the partial sum of the form

$$\sum_{\substack{n_i \le x \\ i \in \mathbb{N}}} a(n_i) f(n_i) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} a(n) f(n),$$

where $a : \mathcal{N} \longrightarrow \mathbb{C}$ is an arithmetic function over $\mathcal{N}$ in terms of

$$A(x) = \sum_{\substack{n_i \le x \\ i \in \mathbb{N}}} a(n_i) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} a(n),$$

where $f(n)$ is a continuous differentiable function.

Theorem 3.19. *Let $a : \mathcal{N} \longrightarrow \mathbb{C}$ be a function, and let the function f be a continuously differentiable function on $[1, \infty)$ with $A(x)$ as above. Then*

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} a(n) f(n) = A(x) f(x) - \int_1^x A(t) f'(t) dt.$$

Proof. The formula can be deduced by integration by parts for the Riemann-Stieltjes integral (using Theorems 1.5 and 1.6). Indeed, we have

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} a(n) f(n) = \int_{1-}^x f(t) dA(t) = \big[A(t) f(t)\big]_{1-}^x - \int_1^x A(t) f'(t) dt$$

$$= A(x) f(x) - \int_1^x A(t) f'(t) dt.$$

□

For example, letting $B(x) = \sum_{n \le x} \frac{a(n)}{n}$, we have the relations

$$B(x) = \frac{A(x)}{x} + \int_1^x \frac{A(t)}{t^2} dt \tag{3.2}$$

and

$$A(x) = B(x)\, x - \int_1^x B(t) dt. \tag{3.3}$$

We are interested in estimating lower bound of the partial sums of $\mu_{\mathcal{P}}(n)$, $\lambda_{\mathcal{P}}(n)$, $\frac{\mu_{\mathcal{P}}(n)}{n}$ and $\frac{\lambda_{\mathcal{P}}(n)}{n}$ not exceeding x in Chapters 5 and 6. To estimate such bound we need the following:

Propostion 3.20. *Let $A(x)$ and $B(x)$ as defined above, if*

(i) $A(x) = o\big(\sqrt{x}\big)$*, then* $B(x) = C + o\big(\frac{1}{\sqrt{x}}\big)$ *for some constant* C*.*

(ii) $B(x) = o\big(\frac{1}{\sqrt{x}}\big)$*, then* $A(x) = o\big(\sqrt{x}\,\big)$*.*

Proof. (i) By (3.2), we have

$$
\begin{aligned}
B(x) &= \frac{A(x)}{x} + \int_1^x \frac{A(t)}{t^2}dt = \frac{o\big(\sqrt{x}\big)}{x} + \int_1^x \frac{A(t)}{t^2}dt \\
&= o\bigg(\frac{1}{\sqrt{x}}\bigg) + \int_1^\infty \frac{A(t)}{t^2}dt - \int_x^\infty \frac{A(t)}{t^2}dt = o\bigg(\frac{1}{\sqrt{x}}\bigg) + \int_1^\infty \frac{A(t)}{t^2}dt - \int_x^\infty \frac{o\big(\sqrt{t}\big)}{t^2}dt \\
&= o\bigg(\frac{1}{\sqrt{x}}\bigg) + C + \int_x^\infty o\big(t^{-\frac{3}{2}}\big)dt = C + o\bigg(\frac{1}{\sqrt{x}}\bigg), \quad \text{where } C \text{ is constant.}
\end{aligned}
$$

(ii) By (3.3), we have

$$
\begin{aligned}
A(x) &= B(x)\,x - \int_1^x B(t)dt \\
&= o\bigg(\frac{1}{\sqrt{x}}\bigg)x + \int_1^x o\bigg(\frac{1}{\sqrt{t}}\bigg)dt = o\bigg(\frac{x}{\sqrt{x}}\bigg) + o\bigg(\int_1^x t^{-\frac{1}{2}}dt\bigg) \\
&= o(\sqrt{x}\,) + o\Big(\big[2t^{\frac{1}{2}}\big]_1^x\Big) = o(\sqrt{x}\,) + o\Big(2\big[\sqrt{x}\,\big] - 2\big[\sqrt{1}\,\big]\Big) = o(\sqrt{x}\,).
\end{aligned}
$$

□

As a result, we also have (i) and (ii) below, which we shall use in later chapters.

(i) if $B(x) = C + \Omega\big(\frac{1}{\sqrt{x}}\big)$ for some constant C, then $A(x) = \Omega\big(\sqrt{x}\big)$.

(ii) if $A(x) = \Omega\big(\sqrt{x}\,\big)$, then $B(x) = \Omega\big(\frac{1}{\sqrt{x}}\big)$.

3.4 The Mellin transform and its inverse over $\mathcal{N}$

In this section, we apply Theorems 1.22 and 1.24 to give the Mellin transform and Perron's formula for Beurling's numbers $\mathcal{N}$ as follows: If $\lambda_i = \log(n_i)$, then, for $\Re s > \sigma_c$, we have

(i) Theorem 1.22 gives

$$
F(s) = \sum_{n\in\mathcal{N}} \frac{a(n)}{n^s} = s\int_{n_1}^\infty \frac{A(y)}{y^{s+1}}\,dy, \quad \text{where } A(y) = \sum_{\substack{n \le y \\ n \in \mathcal{N}}} a(n).
$$

(ii) Theorem 1.24 (Perron's formula) gives for $y \notin \mathcal{N}$,

$$A(y) = \sum_{\substack{n \le y \\ n \in \mathcal{N}}} a(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{F(s)}{s} y^s \, ds, \quad \text{where } F(s) = \sum_{n \in \mathcal{N}} \frac{a(n)}{n^s}.$$

If $y \in \mathcal{N}$, an extra term appears as in the usual case.

Example 3.21. Taking $a(n_i) = 1$, $\mu(n_i)$, $\lambda(n_i)$, $\Lambda(n_i)$, respectively for the Dirichlet series, we get the usual Mellin transforms:

$$\begin{aligned}
\zeta_{\mathcal{P}}(s) &= \sum_{n \in \mathcal{N}} \frac{1}{n^s} = s \int_1^\infty \frac{N_{\mathcal{P}}(x)}{x^{s+1}} \, dx, \\
U_{\mathcal{P}}(s) := \frac{1}{\zeta_{\mathcal{P}}(s)} &= \sum_{n \in \mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n^s} = s \int_1^\infty \frac{M_{\mathcal{P}}(x)}{x^{s+1}} \, dx, \\
Z_{\mathcal{P}}(s) := \frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)} &= \sum_{n \in \mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n^s} = s \int_1^\infty \frac{L_{\mathcal{P}}(x)}{x^{s+1}} \, dx, \\
V_{\mathcal{P}}(s) := -\frac{\zeta_{\mathcal{P}}'(s)}{\zeta_{\mathcal{P}}(s)} &= \sum_{n \in \mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^s} = s \int_1^\infty \frac{\psi_{\mathcal{P}}(x)}{x^{s+1}} \, dx.
\end{aligned}$$

If the abscissa is 1, then all these hold for at least $\Re s > 1$.

We can also get the usual inverse Mellin transforms (Perron's formula) of the above Dirichlet series for $x > 0$, but $x \notin \mathcal{N}$:

$$\begin{aligned}
N_{\mathcal{P}}(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta_{\mathcal{P}}(s)}{s(s+1)} x^{s+1} \, ds, \\
M_{\mathcal{P}}(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{U_{\mathcal{P}}(s)}{s} x^s \, ds, \\
L_{\mathcal{P}}(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{Z_{\mathcal{P}}(s)}{s} x^s \, ds, \\
\psi_{\mathcal{P}}(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{V_{\mathcal{P}}(s)}{s} x^s \, ds.
\end{aligned}$$

These all hold with $c > \max\{0, \sigma_c\}$, where σ_c is the abscissa of convergence of the Dirichlet series for $\zeta_{\mathcal{P}}$ as mentioned earlier.

3.5 Euler products over $\mathcal{N}$

The significance of multiplicativity underpins the next theorem which is an extension of Theorem 1.19 over $\mathcal{N}$. This theorem in its original form is sometimes referred to as the analytic version of unique-prime-factorization.

Theorem 3.22. *Let $f : \mathcal{N} \longrightarrow \mathbb{C}$ be a multiplicative function, and let $\sum_{n\in\mathcal{N}} f(n)$ be an absolutely convergent series. Then*

$$\sum_{n\in\mathcal{N}} f(n) = \prod_{p\in\mathcal{P}} \{1 + f(p) + f(p^2) + \cdots\}, \tag{3.4}$$

where the infinite product ranges over all g-primes. If f is completely multiplicative, then the right hand side of (3.4) simplifies to be

$$\sum_{n\in\mathcal{N}} f(n) = \prod_{p\in\mathcal{P}} \frac{1}{1 - f(p)}.$$

Proof. For every x, let

$$W(x) := \prod_{\substack{p \leq x \\ p \in \mathcal{P}}} \{1 + f(p) + f(p^2) + \cdots\}.$$

Every $n \in \mathcal{N}$ can be uniquely written in the form $p_1^{a_1} \cdots p_k^{a_k}$, where the $p_i \in \mathcal{P}$ and $a_i > 0$ and by multiplicativity of f we have $f(p_1^{a_1}) \cdots f(p_k^{a_k}) = f(p_1^{a_1} \cdots p_k^{a_k}) = f(n)$. Hence the result of multiplying out the series is precisely the following

$$W(x) = \sum_{n\in A} f(n),$$

where A is the set of all those g-integers whose g-prime factors are at most x. Hence

$$\Big| \sum_{n\in\mathcal{N}} f(n) - W(x) \Big| = |\sum_{n\notin A} f(n)| \leq \sum_{\substack{n > x \\ n \in \mathcal{N}}} |f(n)|.$$

Since $n \notin A$ implies at least one g-prime factor of n is $> x$, so $n > x$. As $x \longrightarrow \infty$, the sum on the right tends to 0 since $\sum_{n\in\mathcal{N}} f(n)$ is absolutely convergent and the result follows.

Note that by the same argument with $f(p^k)$ replaced by $|f(p^k)|$, the product can be seen to be absolutely convergent. Finally, we have $f(p^k) = f(p)^k$ when f is completely multiplicative and

$$\sum_{n\in\mathcal{N}} f(n) = \prod_{p\in\mathcal{P}}\{1 + f(p) + f(p)^2 + \cdots\} = \prod_{p\in\mathcal{P}} \frac{1}{1-f(p)}$$

as we will see later.

□

Corollary 3.23. *Assume* $\sum_{n\in\mathcal{N}} f(n)n^{-s}$ *converges absolutely for* $\sigma > \sigma_a$. *If* $f : \mathcal{N} \longrightarrow \mathbb{C}$ *is multiplicative we have*

$$\sum_{n\in\mathcal{N}} \frac{f(n)}{n^s} = \prod_{p\in\mathcal{P}} \left\{1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right\} \quad \text{if } \sigma > \sigma_a, \tag{3.5}$$

and if f *is completely multiplicative we have*

$$\sum_{n\in\mathcal{N}} \frac{f(n)}{n^s} = \prod_{p\in\mathcal{P}} \frac{1}{1-\frac{f(p)}{p^s}} \quad \text{if } \sigma > \sigma_a. \tag{3.6}$$

Proof. Equations (3.5) and (3.6) can be obtained by applying Theorem 3.22 to an absolutely convergent Dirichlet series.

□

Example 3.24. Suppose $\mathcal{P}$ has abscissa 1 and let $f(n) = 1$, $\mu_{\mathcal{P}}(n)$, $\lambda_{\mathcal{P}}(n)$, respectively for the Dirichlet series, then we get the following Euler products over $\mathcal{P}$:

$$\begin{aligned}
\sum_{n\in\mathcal{N}} \frac{1}{n^s} &= \prod_{p\in\mathcal{P}} \frac{1}{1-\frac{1}{p^s}} = \zeta_{\mathcal{P}}(s),\\
U_{\mathcal{P}}(s) = \sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n^s} &= \prod_{p\in\mathcal{P}} \left(1 + \frac{\mu_{\mathcal{P}}(p)}{p^s}\right) = \prod_{p\in\mathcal{P}} (1 - \frac{1}{p^s}) = \frac{1}{\zeta_{\mathcal{P}}(s)},\\
Z_{\mathcal{P}}(s) = \sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n^s} &= \prod_{p\in\mathcal{P}} \frac{1}{1-\frac{\lambda_{\mathcal{P}}(p)}{p^s}} = \prod_{p\in\mathcal{P}} \frac{1}{1+\frac{1}{p^s}} = \prod_{p\in\mathcal{P}} \frac{1-\frac{1}{p^s}}{1-\frac{1}{p^{2s}}} = \frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}.
\end{aligned}$$

These all hold for at least $\Re s > 1$. They may converge for $\Re s = 1$. For example, if $\sum_{n\in\mathcal{N}} \frac{1}{n}$ converges.

The following propositions concerns multiplicative functions over $\mathcal{P}$ and $\mathcal{N}$.

Propostion 3.25. *Let* f *be a completely multiplicative function over* $\mathcal{N}$ *and assume* $\sum_{n\in\mathcal{N}} f(n)$ *converges. Then* $\sup_{p\in\mathcal{P}} |f(p)| < 1$ *for all* $p \in \mathcal{P}$.

Proof. Since $\sum_{n\in\mathcal{N}} f(n)$ converges, then $f(n) \longrightarrow 0$ as $n \longrightarrow \infty$. This also means that $f(p) \longrightarrow 0$ as $p \longrightarrow \infty$, so $|f(p)| < \frac{1}{2}$ for some $p > p_0 \in \mathcal{P}$.

Now if $|f(p)| \geq 1$, then $|f(p^k)| = |f(p)|^k \geq 1$ for all k since f is completely multiplicative function and $f(p^k)$ does not tend to zero as $k \longrightarrow \infty$. Thus give a contradiction. Thus $|f(p)| < 1$ for all $p \in \mathcal{P}$. Since $|f(p)| < \frac{1}{2}$ for some $p > p_0 \in \mathcal{P}$, it follows that $|f(p)| \leq c < 1$ for all $p \in \mathcal{P}$.

□

Propostion 3.26. *Let f be a completely multiplicative function over $\mathcal{N}$. Then $\sum_{n\in\mathcal{N}} |f(n)|$ converges if and only if $\sum_{p\in\mathcal{P}} |f(p)|$ converges and $\sup_{p\in\mathcal{P}} |f(p)| < 1$ for all $p \in \mathcal{P}$.*

Proof. Using Proposition 3.25, the series $\sum_{p\in\mathcal{P}} |f(p)|$ converges and $\sup_{p\in\mathcal{P}} |f(p)| < 1$ for all $p \in \mathcal{P}$ if $\sum_{n\in\mathcal{N}} |f(n)|$ converges.

Now suppose $\sum_{p\in\mathcal{P}} |f(p)|$ converges and $|f(p)| < 1$ for all $p \in \mathcal{P}$. We want to prove $\sum_{n\in\mathcal{N}} |f(n)|$ converges.

From above, $c = \sup_{p\in\mathcal{P}} |f(p)| < 1$. Therefore $\prod_{p\in\mathcal{P}} \frac{1}{1-|f(p)|}$ converges since

$$1 \leq \prod_{p\in\mathcal{P}} \frac{1}{1-|f(p)|} = \prod_{p\in\mathcal{P}} \Big(1 + \frac{|f(p)|}{1-|f(p)|}\Big) \leq \prod_{p\in\mathcal{P}} \Big(1 + \frac{1}{1-c}\,|f(p)|\Big),$$

where $|f(p)| < c$ where we have used Theorem 1.8. Now

$$\begin{aligned} \prod_{p\leq x} \frac{1}{1-|f(p)|} &= \sum_{\substack{n\,\in\,\mathcal{N} \\ p|n\ \&\ p\,\leq\,x}} |f(n)| \quad \text{converges} \quad \text{(by Theorem 1.8)} \\ &\geq \sum_{n\leq x} |f(n)|. \end{aligned}$$

Hence $\sum_{n\in\mathcal{N}} |f(n)|$ converges.

□

Propostion 3.27. *Let f be a multiplicative function over $\mathcal{N}$. Then $\sum_{n\in\mathcal{N}} |f(n)|$ converges if and only if $\sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{N}} |f(p^k)|$ converges.*

Proof. Trivially, the series $\sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{N}} |f(p^k)|$ converges if $\sum_{n\in\mathcal{N}} |f(n)|$ converges.

Now suppose $\sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{N}} |f(p^k)|$ converges. We wish to show that $\sum_{n\in\mathcal{N}} |f(n)|$

converges. Since

$$\sum_{p\in\mathcal{P}}\sum_{k\in\mathbb{N}}|f(p^k)| \text{ converges, it is follows by Theorem 1.8 that}$$

$$\prod_{p\in\mathcal{P}}\Big(1+\sum_{k=1}^{\infty}|f(p^k)|\Big)=\prod_{p\in\mathcal{P}}\Big(\sum_{k=0}^{\infty}|f(p^k)|\Big) \text{ converges.}$$

The right hand side is at least $\prod_{p\leq x}\Big\{\sum_{k=0}^{\infty}|f(p^k)|\Big\}$. But, by the proof of Theorem 3.22, we have

$$\begin{aligned}\prod_{p\leq x}\Big\{\sum_{k=0}^{\infty}|f(p^k)|\Big\} &= \sum_{\substack{n\,\in\,\mathcal{N}\\ p|n\ \&\ p\,\leq\,x}}|f(n)|\\ &\geq \sum_{n\leq x}|f(n)|.\end{aligned}$$

Hence $\sum_{n\in\mathcal{N}}|f(n)|$ converges.

□

3.6 Beurling's Prime Number Theorem

A. Beurling in 1937 [6] found a condition on $N_{\mathcal{P}}$ to ensure the validity of the Prime Number Theorem (PNT) as follows: if

$$N_{\mathcal{P}}(x)=\rho x+O\Big(\frac{x}{(\log x)^{\gamma}}\Big), \tag{3.7}$$

where ρ is a positive constant and $\gamma>\frac{3}{2}$, then

$$\pi_{\mathcal{P}}(x)\sim\frac{x}{\log x}\,. \tag{3.8}$$

Beurling, and later Diamond, showed that Beurling's condition is sharp by providing different examples of generalised prime systems where PNT fails when $\gamma=\frac{3}{2}$ (see [6] [12]). Many researchers have refined this result with error terms; (*i.e* if we assume something more in (3.7), can we say something more in (3.8)?)

E. Landau 1903 [36] showed that if

$$N_{\mathcal{P}}(x)=\rho x+O(x^{\beta}) \tag{3.9}$$

for some $\rho > 0$ and $\beta < 1$, then

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O\Big(\frac{x}{e^{k\sqrt{\log x}}}\Big) \tag{3.10}$$

for some $k > 0$, where $\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}$ is the ***Logarithmic integral function***. H. G. Diamond, H. L. Montgomery and U. M. A. Vorhauer 2006 [14] showed that the above result is best possible by establishing a g-prime system for which (3.9) holds for some $\rho > 0$ and $\beta < 1$, but the error term of (3.10) is $\Omega\big(\frac{x}{e^{k_1\sqrt{\log x}}}\big)$ for some $k_1 > 0$.

In 1949, B. Nyman [47] treated the converse of PNT by proving that if

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O\Big(\frac{x}{(\log x)^{\delta}}\Big)$$

holds for all $\delta > 0$, then

$$N_{\mathcal{P}}(x) = \rho x + O\Big(\frac{x}{(\log x)^{\delta}}\Big) \tag{3.11}$$

holds for some $\rho > 0$ and for all $\delta > 0$. In 1961, P. Malliavin [43] verified that if

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O\Big(\frac{x}{e^{c_2(\log x)^b}}\Big) \tag{3.12}$$

for some $0 < b < 1$ and $c_2 > 0$, then (3.11) holds with the error term $O\big(\frac{x}{e^{c_1(\log x)^a}}\big)$ for some $c_1, \rho > 0$ and $a = \frac{b}{b+2}$. In 1970, H. G. Diamond [11] improved Malliavin's result by proving that if (3.12) holds for some $0 < b < 1$ and $c_2 = 1$, then (3.11) holds with error term $O\big(\frac{x}{e^{(\log x \log\log x)^a}}\big)$ for some $\rho > 0$ and $a = \frac{b}{b+1}$. In 2006, T. W. Hilberdink and M. L. Lapidus [27] extended Diamond's result by showing that if (3.12) holds for $b = 1$, (*i.e* $\psi_{\mathcal{P}}(x) = x + O(x^{\alpha})$ for some $0 < \alpha < 1$), then (3.11) holds with the error term $O\big(\frac{x}{e^{C\sqrt{\log x \log\log x}}}\big)$ for some $\rho, C > 0$. The open question here is: is this result best possible?

Several asymptotic formulas that are "equivalent" to Beurling's PNT have been recently investigated by Diamond and Zhang [15, 16]. For instance, this includes

$$\psi_{\mathcal{P}}(x) \sim x,$$
$$M_{\mathcal{P}}(x) = o(x),$$
$$m_{\mathcal{P}}(x) = o(1),$$

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{\Lambda_{\mathcal{P}}(n)}{n} = \log x - c + o(1),$$

where c is not Euler's constant in general. They researched conditions under which these relations do or do not hold, and they found some of the implications between such formulas without additional assumptions, while the others can fail unconditionally. Many of their results have been recently improved by means of different approaches which are based on recent complex tauberian theorems for Laplace transforms with pseudo function boundary behavior [9]. In 2018, G. Debruyne, H. G. Diamond and J. Vindas [8] gave conditions that imply the Beurling version of PNT equivalence related to the partial sum of Möbius's function not exceeding x. They have also shown that such sum estimates fail by giving some examples which violate their necessary condition for $M_{\mathcal{P}}(x) = o(x)$.

3.7 Known results about Beurling numbers where $\psi_{\mathcal{P}}$ or $N_{\mathcal{P}}$ is well-behaved

In this section, we outline some relevant ideas and results about g-primes and g-integers, in order to prove the main results in later chapters, where we are interested in g-prime systems for which both $N_{\mathcal{P}}(x)$ and $\psi_{\mathcal{P}}(x)$ are simultaneously "well-behaved". These systems were investigated by T. W. Hilberdink in 2005 [25] and have the following properties:

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta+\varepsilon}) \quad \text{for some } \rho > 0 \tag{3.13}$$

and

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon}) \tag{3.14}$$

hold for all $\varepsilon > 0$, but for no $\varepsilon < 0$ and $0 \le \alpha, \beta < 1$. For the usual primes, (3.13) holds with $\beta = 0$ and if the RH is true, then (3.14) would hold for $\alpha = \frac{1}{2}$. Such systems exist as was shown by Zhang [58]. Indeed, $\mathcal{P}_Z$ (his system) satisfies these with $\alpha = \beta = \frac{1}{2}$. We also study such systems where either of (3.13) or (3.14) holds with $\alpha, \beta < \frac{1}{2}$.

Theorem 3.28. *Suppose that for some $0 \le \alpha < 1$, we have*

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon}) \quad \text{for all } \varepsilon > 0.$$

Then $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to the half plane $\{s \in \mathbb{C} : \Re s > \alpha\}$ except for a simple (non-removable) pole at $s = 1$ and $\zeta_{\mathcal{P}}(s)$ has no zeros in this region.

Conversely, suppose that for some $0 \le \alpha < 1$, $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to the half plane $\{s \in \mathbb{C} : \Re s > \alpha\}$, except for a simple (non-removable) pole at $s = 1$, and that $\zeta_{\mathcal{P}}(s) \neq 0$ in this region. Further assume that $|V_{\mathcal{P}}(\sigma + it)| = O(|t|^{\varepsilon})$ holds for all $\varepsilon > 0$, uniformly for $\sigma \ge \alpha + \delta$ with any $\delta > 0$. Then

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon}) \quad \text{for all } \varepsilon > 0.$$

Proof. See Theorem 2.1 of [27].

□

Theorem 3.29. *Suppose that $\psi_{\mathcal{P}}(x) = x + O(x^{\alpha})$ for some $0 < \alpha < 1$. Then there exist positive constants ρ and c such that*

$$N_{\mathcal{P}}(x) = \rho x + O(xe^{-c\sqrt{\log x \log\log x}}).$$

Proof. See Theorem 2.2 of [27].

□

Corollary 3.30.

(i) If $\psi_{\mathcal{P}}(x) = x + O(x^{\alpha})$ for some constant $\alpha < \frac{1}{2}$ (which implies that $N_{\mathcal{P}}(x) \sim \rho x$ for some $\rho > 0$), then for every $\alpha < \eta < \frac{1}{2}$, $N_{\mathcal{P}}(x) - \rho x = \Omega(x^{\eta})$ and $\zeta_{\mathcal{P}}(s)$ does not have finite order throughout the strip $\{s \in \mathbb{C} : \eta < \Re s < 1\}$.

(ii) If $N_{\mathcal{P}}(x) = \rho x + O(x^{\beta})$ for some constants $\rho > 0$ and $\beta < \frac{1}{2}$, then for every $\beta < \eta' < \frac{1}{2}$, $\psi(x) - x = \Omega(x^{\eta'})$ and $\zeta_{\mathcal{P}}(s)$ has infinitely many zeros in the strip $\{s \in \mathbb{C} : \eta' < \Re s < 1\}$.

Proof. See Corollary 2 of [25].

□

Theorem 3.31. *Let $\mathcal{P}$ a g-prime system satisfying (3.13) and (3.14) for some $0 \le \alpha, \beta < 1$. Then $\max\{\alpha, \beta\} \ge \frac{1}{2}$.*

Proof. See Theorem 1 of [25].

□

Theorem 3.32. *Let $\mathcal{P}$ a g-prime system satisfying (3.13) and (3.14) for some $0 \leq \alpha, \beta < 1$. Then for $\sigma > \Theta = \max\{\alpha, \beta\}$, and uniformly for $\sigma \geq \Theta + \delta$ (any $\delta > 0$),*

$$V_{\mathcal{P}}(\sigma + it) = O\Big((\log |t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}\Big) \quad \text{and} \quad \zeta_{\mathcal{P}}(\sigma + it) = O\Big(\exp\big\{(\log |t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}\big\}\Big),$$

for all $\varepsilon > 0$. In particular, $\zeta_{\mathcal{P}}(\sigma + it) = O(|t|^{\varepsilon})$ for all $\varepsilon > 0$.

Proof. See Theorem 2.3 of [27].

□

Remark 3.33.

(i) If $\alpha < \beta$ and we already know that $\zeta_{\mathcal{P}}(s)$ is of finite order for $\sigma > \eta$ for some $\alpha < \eta < \beta$, then $\zeta_{\mathcal{P}}(s)$ and $V_{\mathcal{P}}(s)$ have zero order in this range.

(ii) If $\beta < \alpha$ and we we already know that $V_{\mathcal{P}}(s)$ has only finitely many poles for $\sigma > \eta'$ for some $\alpha < \eta' < \beta$ (equivalently, $\zeta_{\mathcal{P}}(s)$ has finitely many zeros here), then $\zeta_{\mathcal{P}}(s)$ and $V_{\mathcal{P}}(s)$ have zero order in this range.

The following result shows the existence of a system which satisfies (3.13) and (3.14) unconditionally.

Theorem 3.34. *There is a system of Beurling primes $\mathcal{P}_Z$ such that*

(i) the associated Beurling integer counting function satisfies

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\frac{1}{2}} e^{c(\log x)^{\frac{2}{3}}}) \quad \text{with } \rho > 0;$$

(ii) the associated zeta function $\zeta_{\mathcal{P}}(s)$ is analytic for $\sigma > \frac{1}{2}$ except a simple pole at $s = 1$ with residue ρ;

(iii) the function $\zeta_{\mathcal{P}}(s)$ has no zeros on the half plane $\sigma > \frac{1}{2}$;

(iv) the prime counting function satisfies

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(x^{\frac{1}{2}}).$$

Proof. See Theorem 1 of [58] and Theorem 17.11 of [16].

□

We shall use the next theorem in the proof of Theorem 4.15 (see also the preprint [45]). An eealier version was proved in [25].

Theorem 3.35. *Let* $f(s) = \sum_{n \in \mathcal{N}} \frac{a(n)}{n^s}$ *be a Dirichlet series with abscissa of convergence* $\sigma_c \leq 1$.

(i) Suppose that for some $0 \leq \vartheta < 1$ *and* $\rho \in \mathbb{C}$*, we have*

$$A(x) := \sum_{\substack{n \le x \\ n \in \mathcal{N}}} a(n) = \rho x + O(x^{\vartheta+\varepsilon}) \quad \text{for all } \varepsilon > 0. \tag{3.15}$$

Then $f(s)$ *has an analytic continuation to the half plane* $\{s \in \mathbb{C} : \Re s > \vartheta\}$ *with a simple (removable if* $\rho = 0$*) pole at* $s = 1$ *with residue* ρ *and* $f(s)$ *has finite order; indeed* $\mu_f(\sigma) \leq 1$ *for* $\sigma > \vartheta$.

(ii) Conversely, suppose that for some $0 \leq \vartheta < 1$, $f(s)$ *has an analytic continuation to the half plane* $\{s \in \mathbb{C} : \Re s > \vartheta\}$ *except for a simple pole at* $s = 1$ *with residue* ρ. *Further assume that* $\mu_f(\sigma) = 0$ *for* $\sigma > \vartheta$ *(see Definition 1.25) and either*

(a) $a(n) \geq 0$ *or*

(b)

$$\sum_{\substack{x-1 < n \le x \\ n \in \mathcal{N}}} |a(n)| = O(x^{\vartheta+\varepsilon}) \quad \text{for all } \varepsilon > 0. \tag{3.16}$$

*Then (*3.15*) holds.*

Proof.

(i) The proof of the first part follows on writing

$$f(s) = s\int_1^\infty \frac{A(x)}{x^{s+1}}\,dx = \frac{\rho s}{s-1} + s\int_1^\infty \frac{A(x)-\rho x}{x^{s+1}}\,dx. \tag{3.17}$$

The integral on the side of (3.17) converges to a holomorphic function for $\Re s > \vartheta$ since

$$\left|\frac{A(x)-\rho x}{x^{s+1}}\right| = \frac{|A(x)-\rho x|}{|x^{s+1}|} = \frac{|O(x^{\vartheta+\varepsilon})|}{x^{\sigma+1}} \leq \frac{Bx^{\vartheta+\varepsilon}}{x^{\sigma+1}} = \frac{B}{x^{1+\sigma-\vartheta-\varepsilon}}.$$

Furthermore,

$$
\begin{aligned}
|f(s)| &= \left|\frac{\rho s}{s-1} + s\int_1^\infty \frac{A(x)-\rho x}{x^{s+1}}\,dx\right| \\
&\le |\rho|\left|\frac{\sigma+it}{\sigma-1+it}\right| + |\sigma+it|\int_1^\infty \frac{|A(x)-\rho x|}{x^{\sigma+1}}\,dx \\
&\le C_\rho\left|1+\frac{1}{\sigma-1+it}\right| + C_\sigma|t|\left|1+\frac{\sigma}{it}\right| \ \ (\text{since} \int_1^\infty \frac{A(x)-\rho x}{x^{\sigma+1}}\,dx \text{ converges for } \sigma>\vartheta) \\
&= O(|t|), \quad \text{as } |t| \longrightarrow \infty.
\end{aligned}
$$

(ii) For the converse, let $c > 1$, $x, T > 0$ such that $x \notin \mathcal{N}$, then, for $n \in \mathcal{N}$, we have

$$
\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^s \frac{ds}{s} = O\Big(\frac{(\frac{x}{n})^c}{T|\log\frac{x}{n}|}\Big) + \begin{cases} 1 & \text{if } n < x \\ 0 & \text{if } n > x \end{cases}.
$$

Multiply by $a(n)$ and sum over all $n \in \mathcal{N}$. Therefore, for $x \notin \mathcal{N}$, we have

$$
\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{f(s)x^s}{s}\,ds = O\Big(\frac{x^c}{T}\sum_{n\in\mathcal{N}} \frac{|a(n)|}{n^c|\log\frac{x}{n}|}\Big) + \sum_{\substack{n \le x \\ n \in \mathcal{N}}} a(n).
$$

The range is split into ($n \ge 2x$ & $n \le \frac{x}{2}$) and ($\frac{x}{2} < n < 2x$) in order to use the bound $|\log\frac{x}{n}| \ge \log 2$ for the first range. This gives

$$
A(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{f(s)x^s}{s}\,ds + O\left(\frac{x^c}{T}\sum_{\substack{n \ge 2x \,\&\, n \le \frac{x}{2} \\ n \in \mathcal{N}}} \frac{|a(n)|}{n^c|\log\frac{x}{n}|}\right) + O\left(\frac{x^c}{T}\sum_{\substack{\frac{x}{2} < n < 2x \\ n \in \mathcal{N}}} \frac{|a(n)|}{n^c|\log\frac{x}{n}|}\right).
$$

Using $\left|\log\frac{x}{n}\right| = \left|\log\left(1+\frac{n-x}{x}\right)\right| \asymp \frac{|n-x|}{x}$ for the second range, we obtain

$$
A(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{f(s)x^s}{s}ds + O\left(\frac{x^c}{T}\sum_{\substack{n \ge 2x \,\&\, n \le \frac{x}{2} \\ n \in \mathcal{N}}} \frac{|a(n)|}{n^c}\right) + O\left(\frac{x}{T}\sum_{\substack{\frac{x}{2} < n < 2x \\ n \in \mathcal{N}}} \frac{|a(n)|}{|n-x|}\right).
$$

Therefore

$$A(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{f(s)x^s}{s}ds + O\left(\frac{x^c}{T(c-1)}\right) + O\left(\frac{x}{T}\sum_{\substack{\frac{x}{2} < n < 2x \\ n \in \mathcal{N}}}\frac{|a(n)|}{|n-x|}\right)$$

since $f(c) = O(\frac{1}{c-1})$.

Now consider the right integral of the above equation. We move the contour past the line $s = 1$ to the line $\Re s = \sigma$ for any $\sigma > \vartheta$ (see Figure 3.1). The residue at 1 is ρx since $f(s)$ is holomorphic in this region.

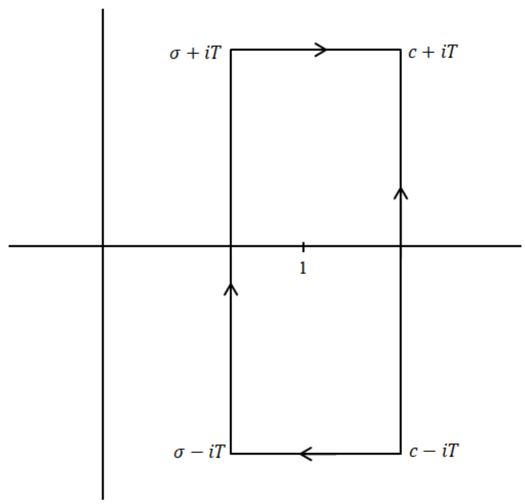

Figure 3.1: rectangular contour

Hence

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{f(s)x^s}{s}ds = \rho x + \frac{1}{2\pi i}\left(\int_{c-iT}^{\sigma-iT} + \int_{\sigma-iT}^{\sigma+iT} + \int_{\sigma+iT}^{c+iT}\right)\frac{f(s)x^s}{s}ds.$$

The integrals will be estimated by using the bound $|f(s)| = O(|t|^{\varepsilon})$ for all $\varepsilon > 0$. The integral over the horizontal path $[\sigma + iT, c + iT]$ is

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\sigma+iT}^{c+iT}\frac{f(s)}{s}x^s ds\right| &= \left|\frac{1}{2\pi i}\int_{\sigma}^{c}\frac{f(y+iT)}{y+iT}x^{y+iT}dy\right| \\ &\leq \frac{x^c}{2\pi T}\int_{\sigma}^{c}|f(y+iT)|dy \\ &= O\left(\frac{x^c}{T}T^{\varepsilon}\right) = O\left(\frac{x^c}{T^{1-\varepsilon}}\right) \text{ for all } \varepsilon > 0,\end{aligned}$$

by the above bound. Similarly for the integral over $[c-iT,\sigma-iT]$. On the line $\Re s=\sigma$, we will have

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{f(s)}{s}x^s ds\right| &= \left|\frac{1}{2\pi i}\int_{-T}^{T}\frac{f(\sigma+it)}{\sigma+it}x^{\sigma+it}dt\right| \\ &\le \frac{x^\sigma}{2\pi}\int_{-T}^{T}\frac{|f(\sigma+it)|}{|\sigma+it|}dt = \frac{x^\sigma}{\pi}\int_0^T\frac{|f(\sigma+it)|}{|\sigma+it|}dt \\ &= \frac{x^\sigma}{\pi}\int_0^1\frac{|f(\sigma+it)|}{|\sigma+it|}dt + \frac{x^\sigma}{\pi}\int_1^T\frac{|f(\sigma+it)|}{t}dt \\ &= O(x^\sigma)+O(x^\sigma T^\varepsilon) = O(x^\sigma T^\varepsilon) \text{ for all } \varepsilon>0,\end{aligned}$$

by the above bound. Hence

$$A(x)=\rho x+O\Big(\frac{x^c}{T^{1-\varepsilon}}\Big)+O(x^\sigma T^\varepsilon)+O\Big(\frac{x^c}{T(c-1)}\Big)+O\Bigg(\frac{x}{T}\sum_{\substack{\frac{x}{2}<n<2x\\ n\in\mathcal{N}}}\frac{|a(n)|}{|n-x|}\Bigg).$$

Choosing $c=1+\frac{1}{\log x}$ gives

$$A(x)=\rho x+O\Big(\frac{x}{T^{1-\varepsilon}}\Big)+O(x^\sigma T^\varepsilon)+O\Big(\frac{x\log x}{T}\Big)+O\Bigg(\frac{x}{T}\sum_{\substack{\frac{x}{2}<n<2x\\ n\in\mathcal{N}}}\frac{|a(n)|}{|n-x|}\Bigg) \tag{3.18}$$

for $x\notin\mathcal{N}$ and for all $\varepsilon>0$. We need to bound the term on the right hand side, which is difficult for general x when n is an integer close to x, as then $|n-x|^{-1}$ could be very large. To take into account this eventuality we choose x here such that $|n-x|<\frac{1}{x^2}$. This ensures that it stays away from these integer n; *i.e.*

$$\big(x-\frac{1}{x^2},\,x+\frac{1}{x^2}\big)\cap\mathcal{N}=\phi. \tag{3.19}$$

Then, for such x,

$$\sum_{\substack{\frac{x}{2}<n<2x\\ n\in\mathcal{N}}}\frac{|a(n)|}{|n-x|}\le x^2\sum_{\frac{x}{2}<n<2x}|a(n)|.$$

In case (a) of Theorem 3.35 (ii), the term on the right hand side of the above inequality is $O(x^{3+\varepsilon})$ since the abscissa of convergence $\sigma_c\le 1$, while in case (b), it is $O(x^{3+\vartheta+\varepsilon})$ by (3.16).

Taking $T = x^4$, (3.18) gives $A(x) = \rho x + O(x^{\sigma+\varepsilon})$ for all $\varepsilon > 0$. This holds for all $\sigma > \vartheta$, so (3.15) holds whenever $x \to \infty$ satisfying (3.19). Now we follow the method used in the proof of Theorem 2.2, originally given in [27]. We show for all x sufficiently large for which

$$\Big(x - \frac{1}{x^2}, x + \frac{1}{x^2}\Big) \cap \mathcal{N} \neq \phi,$$

there exist $x_1 \in (x-1, x)$ and $x_2 \in (x, x+1)$ such that

$$\Big(x_1 - \frac{1}{x_1^2}, x_1 + \frac{1}{x_1^2}\Big) \cap \mathcal{N} = \phi \quad \text{and} \quad \Big(x_2 - \frac{1}{x_2^2}, x + \frac{1}{x_2^2}\Big) \cap \mathcal{N} = \phi.$$

For case (a), positivity of $a(n)$ gives

$$A(x) \leq A(x_2) = \rho x_2 + O(x_2^{\vartheta+\varepsilon}) = \rho x + O(x^{\vartheta+\varepsilon})$$

and

$$A(x) \geq A(x_1) = \rho x_1 + O(x_1^{\vartheta+\varepsilon}) = \rho x + O(x^{\vartheta+\varepsilon}).$$

Hence (3.15) follows for x. While for case (b), we have

$$|A(x) - A(x_1)| \leq \Big| \sum_{\substack{x_1 - 1 < n \leq x \\ n \in \mathcal{N}}} a(n) \Big| \leq \sum_{\substack{x - 1 < n \leq x \\ n \in \mathcal{N}}} |a(n)| \ll x^{\vartheta+\varepsilon}$$

for all $\varepsilon > 0$ by (3.16). Hence (3.15) follows. It remains to prove (3.19).

Assume x is sufficiently large, so that $N_{\mathcal{P}}(x) < L$, where $L = [\frac{(x-1)^2}{2}]$. Divide $(x-1, x)$ into L intervals of equal length. Then one of them contains no elements of $x \notin \mathcal{N}$. Let its midpoint be x_1. Then $\big(x_1 - \frac{1}{2L}, x_1 + \frac{1}{2L}\big) \cap \mathcal{N} = \phi$. Thus the equation (3.19) holds with such x_1 when $\frac{1}{x_1^2} \leq \frac{1}{2L}$; (*i.e.* $L \leq \frac{1}{2}\, x_1^2$).

Similarly $(x, x+2)$ contains suitable x_2.

□

Remark 3.36. The following conjecture has been suggested by T. Hilberdink [personal communication].

Conjecture 3.37. Let $\mathcal{P}$ be a g-prime system for which the integer counting function satisfies (3.13) for some $\beta < \frac{1}{2}$. Then $\zeta_{\mathcal{P}}(s)$ has at least one zero on the line $\Re s = \frac{1}{2}$ or to the right of this line.

Chapter 4

The special functions $\lambda_{\mathcal{P}}$ and $\mu_{\mathcal{P}}$

In this chapter, we firstly study the relationship between the partial sum of $\lambda_{\mathcal{P}}$ and $\mu_{\mathcal{P}}$ which play a significant role as examples in chapters 5 and 6. We then investigate generalised prime systems for which the counting functions $\psi_{\mathcal{P}}(x)$, $N_{\mathcal{P}}(x)$ and $M_{\mathcal{P}}(x)$ are asymptotically well-behaved, in the sense that $\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon})$, $N_{\mathcal{P}}(x) = \rho x + O(x^{\beta+\varepsilon})$ and $M_{\mathcal{P}}(x) = O(x^{\gamma+\varepsilon})$ for some $\rho > 0$ and $\alpha, \beta, \gamma < 1$ respectively. We shall explore which values of α, β, γ are feasible. We also study the behaviour of the sums $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ and $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n}$ under some conditions on g-prime systems $\mathcal{P}$. Finally, we study the behaviour of the sums $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ and $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n}$ under some conditions on g-prime systems $\mathcal{P}$.

4.1 Relationship between $\lambda_{\mathcal{P}}$ and $\mu_{\mathcal{P}}$

In this section, we derive results which establish relationships between the $\lambda_{\mathcal{P}}$ and $\mu_{\mathcal{P}}$ functions as in the classical case. Of course, we shall always be aware that these are not necessarily functions if they are made from different g-primes.

Theorem 4.1. *For every $n \in \mathcal{N}$, we have*

$$\lambda_{\mathcal{P}}(n) = \sum_{\substack{d^2|n \\ d\in\mathcal{N}}} \mu_{\mathcal{P}}\Big(\frac{n}{d^2}\Big).$$

Proof. Let

$$F_{\mathcal{P}}(n) := \sum_{\substack{d^2|n \\ d \in \mathcal{N}}} \mu_{\mathcal{P}}\left(\frac{n}{d^2}\right) = \sum_{\substack{c|n \\ c \in \mathcal{N}}} \mu_{\mathcal{P}}\left(\frac{n}{c}\right) \cdot S(c) = (\mu_{\mathcal{P}} * S)(n),$$

where

$$S(c) = \begin{cases} 1 & \text{if } c = m^2 \text{ with } m \in \mathcal{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We would like to show that $F_{\mathcal{P}}(n) = \lambda_{\mathcal{P}}(n)$. It is easy to see that $S(n)$ is multiplicative and since $\mu_{\mathcal{P}}(n)$ is multiplicative, then, by Theorem 3.18, the function $(\mu_{\mathcal{P}} * S)(n)$ in the Dirichlet convolution algebra of arithmetic functions, is also multiplicative. We know that $\lambda_{\mathcal{P}}(n)$ is a multiplicative function. It therefore suffices to show that $F_{\mathcal{P}}(p^k) = \lambda_{\mathcal{P}}(p^k)$ for all g-primes $p \in \mathcal{P}$ and all $k \in \mathbb{N}$. Now, for every p^k, where $p \in \mathcal{P}$, we have

$$\begin{aligned} F_{\mathcal{P}}(p^{2m}) &= \sum_{\substack{d^2|p^{2m} \\ d \in \mathcal{N}}} \mu_{\mathcal{P}}\left(\frac{p^{2m}}{d^2}\right) = \sum_{\substack{r \geq 0 \ s.t. \\ r \leq m}} \mu_{\mathcal{P}}(p^{2(m-r)}) \\ &= \sum_{\substack{r \geq 0 \ s.t. \\ r < m}} \mu_{\mathcal{P}}(p^{2(m-r)}) + \mu_{\mathcal{P}}(p^{2(m-m)}) \\ &= 0 + \mu_{\mathcal{P}}(1) = 1 \end{aligned}$$

and

$$\begin{aligned} F_{\mathcal{P}}(p^{2m+1}) &= \sum_{\substack{d^2|p^{2m+1} \\ d \in \mathcal{N}}} \mu_{\mathcal{P}}\left(\frac{p^{2m+1}}{d^2}\right) = \sum_{\substack{r \geq 0 \ s.t. \\ r \leq m}} \mu_{\mathcal{P}}(p^{2(m-r)+1}) \\ &= \sum_{\substack{r \geq 0 \ s.t. \\ r < m}} \mu_{\mathcal{P}}(p^{2(m-r)+1}) + \mu_{\mathcal{P}}(p^{2(m-m)+1}) \\ &= 0 + \mu_{\mathcal{P}}(p) = -1. \end{aligned}$$

Thus

$$F_{\mathcal{P}}(p^k) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases} = \lambda_{\mathcal{P}}(p^k), \quad \text{as required.}$$

□

Theorem 4.2. *For every $n \in \mathcal{N}$, we have*

$$\mu_{\mathcal{P}}(n) = \sum_{\substack{d^2 | n \\ d \in \mathcal{N}}} \lambda_{\mathcal{P}}\Big(\frac{n}{d^2}\Big)\mu_{\mathcal{P}}(d).$$

Proof. Let

$$G_{\mathcal{P}}(n) := \sum_{\substack{d^2 | n \\ d \in \mathcal{N}}} \lambda_{\mathcal{P}}\Big(\frac{n}{d^2}\Big)\mu_{\mathcal{P}}(d) = \sum_{\substack{c | n \\ c \in \mathcal{N}}} \lambda_{\mathcal{P}}\Big(\frac{n}{c}\Big) \cdot T(c) = (\lambda_{\mathcal{P}} * T)(n),$$

where

$$T(c) = \begin{cases} \mu_{\mathcal{P}}(m) & \text{if } c = m^2 \text{ with } m \in \mathcal{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We would like to show that $G_{\mathcal{P}}(n) = \mu_{\mathcal{P}}(n)$. It is easy to see that $T(n)$ is multiplicative and since $\lambda_{\mathcal{P}}(n)$ is multiplicative, then, by Theorem 3.18, the function $(\lambda_{\mathcal{P}} * T)(n)$ in the Dirichlet convolution algebra of arithmetic functions, is also multiplicative. We know that $\mu_{\mathcal{P}}(n)$ is a multiplicative function. It therefore suffices to show that $G_{\mathcal{P}}(p^k) = \mu_{\mathcal{P}}(p^k)$ for all g-primes $p \in \mathcal{P}$ and all $k \in \mathbb{N}$. Now, for every p^k, where $p \in \mathcal{P}$, we have

$$\begin{aligned} G_{\mathcal{P}}(p^k) = \sum_{\substack{d^2 | p^k \\ d \in \mathcal{N}}} \lambda_{\mathcal{P}}\Big(\frac{p^k}{d^2}\Big)\mu_{\mathcal{P}}(d) &= \sum_{\substack{r \geq 0 \ s.t. \\ 2r \leq k}} \lambda_{\mathcal{P}}(p^{k-2r})\mu_{\mathcal{P}}(p^r) \\ &= \lambda_{\mathcal{P}}(p^k)\mu_{\mathcal{P}}(1) + \lambda_{\mathcal{P}}(p^{k-2})\mu_{\mathcal{P}}(p). \end{aligned}$$

Note: $\lambda_{\mathcal{P}}(p^{k-2})\mu_{\mathcal{P}}(p)$ exists only if $k \geq 2$. Thus

$$G_{\mathcal{P}}(p^k) = \begin{cases} 1 & \text{if } k = 0 \\ -1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases} = \mu_{\mathcal{P}}(p^k), \quad \text{as required.}$$

□

For the following, we recall that

$$m_{\mathcal{P}}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(n)}{n} \quad \text{and} \quad l_{\mathcal{P}}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \frac{\lambda_{\mathcal{P}}(n)}{n}.$$

As consequences of Theorems 4.1 and 4.2 we have

$$
\begin{aligned}
l_{\mathcal{P}}(x) &= \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \sum_{\substack{d^2 | n \\ d \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(\frac{n}{d^2})}{n} = \sum_{\substack{d^2 \le x \\ d \in \mathcal{N}}} \sum_{\substack{n \le x \ s.t. \ d^2 | n \\ n \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(\frac{n}{d^2})}{n} \\
&= \sum_{\substack{d^2 \le x \\ d \in \mathcal{N}}} \sum_{\substack{m \le \frac{x}{d^2} \\ m \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(m)}{md^2} = \sum_{\substack{d^2 \le x \\ d \in \mathcal{N}}} \frac{m_{\mathcal{P}}(\frac{x}{d^2})}{d^2} \qquad (4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
m_{\mathcal{P}}(x) &= \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \sum_{\substack{d^2 | n \\ d \in \mathcal{N}}} \frac{\lambda_{\mathcal{P}}(\frac{n}{d^2})}{n} \mu_{\mathcal{P}}(d) \\
&= \sum_{\substack{d^2 \le x \\ d \in \mathcal{N}}} \sum_{\substack{n \le x \ s.t. \ d^2 | n \\ n \in \mathcal{N}}} \frac{\lambda_{\mathcal{P}}(\frac{n}{d^2})}{n} \mu_{\mathcal{P}}(d) \\
&= \sum_{\substack{d^2 \le x \\ d \in \mathcal{N}}} \sum_{\substack{m \le \frac{x}{d^2} \\ m \in \mathcal{N}}} \frac{\lambda_{\mathcal{P}}(m)}{md^2} \mu_{\mathcal{P}}(d) = \sum_{\substack{d^2 \le x \\ d \in \mathcal{N}}} \frac{l_{\mathcal{P}}(\frac{x}{d^2})}{d^2} \mu_{\mathcal{P}}(d). \qquad (4.2)
\end{aligned}
$$

Lemma 4.3. *Let $\mathcal{P}$ be a g-prime system for which $\sum_{n \in \mathcal{N}} \frac{1}{n^2}$ converges. Then $N_{\mathcal{P}}(x) = o(x^2)$.*

Proof. Since $\sum_{n \in \mathcal{N}} \frac{1}{n^2}$ converges and put

$$
A(x) := \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{1}{n^2} = C + o(1),
$$

then, by Abel summation,

$$
\begin{aligned}
N_{\mathcal{P}}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} 1 &= A(x) \cdot x^2 - 2 \int_1^x A(t) t dt \\
&= (C + o(1))x^2 - 2 \int_1^x (C + o(1)) t dt \\
&= Cx^2 + o(x^2) - 2C \int_1^x t dt - 2 \int_1^x o(t) dt \\
&= Cx^2 + o(x^2) - C[x^2 - 1] + o(x^2) = o(x^2).
\end{aligned}
$$

□

We now establish a relationship between $l_{\mathcal{P}}(x)$ and $m_{\mathcal{P}}(x)$ in terms of these sum functions tending to zero with increasing terms.

Theorem 4.4. *Let $\mathcal{P}$ be a g-prime system for which $\sum_{n \in \mathcal{N}} \frac{1}{n^2}$ converges. Then $l_{\mathcal{P}}(x) = o(1)$ if and only if $m_{\mathcal{P}}(x) = o(1)$.*

Proof. Suppose $m_{\mathcal{P}}(x) = o(1)$. We want to show that $l_{\mathcal{P}}(x) = o(1)$. Let $\varepsilon > 0$. Then $|m_{\mathcal{P}}(x)| < \varepsilon$ for $x \geq x_0$, some x_0. Thus $|m_{\mathcal{P}}(\frac{x}{d^2})| < \varepsilon$ if $\frac{x}{d^2} \geq x_0$. Hence (4.1) gives

$$\begin{aligned} |l_{\mathcal{P}}(x)| = \sum_{\substack{d^2 \leq x \\ d \in \mathcal{N}}} \frac{|m_{\mathcal{P}}(\frac{x}{d^2})|}{d^2} &\leq \varepsilon \sum_{\substack{d^2 \leq \frac{x}{x_0} \\ d \in \mathcal{N}}} \frac{1}{d^2} + \sum_{\substack{\frac{x}{x_0} < d^2 \leq x \\ d \in \mathcal{N}}} \frac{|m_{\mathcal{P}}(\frac{x}{d^2})|}{d^2} \\ &\leq \varepsilon \sum_{\substack{d^2 \leq \frac{x}{x_0} \\ d \in \mathcal{N}}} \frac{1}{d^2} + A\,\frac{x_0}{x} \sum_{\substack{\frac{x}{x_0} < d^2 \leq x \\ d \in \mathcal{N}}} 1 \quad (\text{since } |m_{\mathcal{P}}(\tfrac{x}{d^2})| \leq A) \\ &\leq \varepsilon\, \zeta_{\mathcal{P}}(2) + A x_0\, \frac{N_{\mathcal{P}}(\sqrt{x})}{x}. \end{aligned}$$

Letting $x \longrightarrow \infty$ and using $N_{\mathcal{P}}(\sqrt{x}) = o(x)$ (by Lemma 4.3), we find

$$\limsup_{x \longrightarrow \infty} |l_{\mathcal{P}}(x)| \leq \varepsilon\, \zeta_{\mathcal{P}}(2).$$

This is true for all $\varepsilon > 0$. Since ε is arbitrary, then $l_{\mathcal{P}}(x) \to 0$; (*i.e.* $l_{\mathcal{P}}(x) = o(1)$).

Now suppose $l_{\mathcal{P}}(x) = o(1)$. We would like to show that $m_{\mathcal{P}}(x) = o(1)$. Let $\varepsilon > 0$. Then $|l_{\mathcal{P}}(x)| < \varepsilon$ for $x \geq x_0$, some x_0. Thus $|l_{\mathcal{P}}(\frac{x}{d^2})| < \varepsilon$ if $\frac{x}{d^2} \geq x_0$. Hence (4.2) gives

$$\begin{aligned} |m_{\mathcal{P}}(x)| = \sum_{\substack{d^2 \leq x \\ d \in \mathcal{N}}} \frac{|l_{\mathcal{P}}(\frac{x}{d^2}) \cdot \mu_{\mathcal{P}}(d)|}{d^2} &\leq \varepsilon \sum_{\substack{d^2 \leq \frac{x}{x_0} \\ d \in \mathcal{N}}} \frac{1}{d^2} + \sum_{\substack{\frac{x}{x_0} < d^2 \leq x \\ d \in \mathcal{N}}} \frac{|l_{\mathcal{P}}(\frac{x}{d^2})|}{d^2} \quad (\text{since } |\mu_{\mathcal{P}}(d)| \leq 1) \\ &\leq \varepsilon \sum_{\substack{d^2 \leq \frac{x}{x_0} \\ d \in \mathcal{N}}} \frac{1}{d^2} + A\,\frac{x_0}{x} \sum_{\substack{\frac{x}{x_0} < d^2 \leq x \\ d \in \mathcal{N}}} 1 \quad (\text{since } |l_{\mathcal{P}}(\tfrac{x}{d^2})| \leq A) \\ &\leq \varepsilon\, \zeta_{\mathcal{P}}(2) + A x_0\, \frac{N_{\mathcal{P}}(\sqrt{x})}{x}. \end{aligned}$$

Letting $x \longrightarrow \infty$ and using $N_{\mathcal{P}}(\sqrt{x}) = o(x)$ (by Lemma 4.3), we find

$$\limsup_{x \longrightarrow \infty} |m_{\mathcal{P}}(x)| \leq \varepsilon\, \zeta_{\mathcal{P}}(2).$$

This is true for all $\varepsilon > 0$. Since ε is arbitrary, then $m_{\mathcal{P}}(x) \to 0$; (*i.e.* $m_{\mathcal{P}}(x) = o(1)$). □

Remark 4.5. In particular, if the abscissa of the g-prime system $\mathcal{P}$ is 1, then $m_{\mathcal{P}}(x)$ and $l_{\mathcal{P}}(x)$ tend to zero together.

In Chapters 5 and 6 we are interested in how quickly the partial sum of $\lambda_{\mathcal{P}}$ over $n \leq x$ tends to zero. The following theorem establishes a useful correspondence between $l_{\mathcal{P}}$ and $m_{\mathcal{P}}$ which we will use in later calculations to link these estimates.

Theorem 4.6. *Let $\mathcal{P}$ be a g-prime system with abscissa 1 and let $0 < a < \frac{1}{2}$. Then $l_{\mathcal{P}}(x) = O\left(\frac{1}{x^a}\right)$ if and only if $m_{\mathcal{P}}(x) = O\left(\frac{1}{x^a}\right)$.*

Proof. Suppose $m_{\mathcal{P}}(x) = O\left(\frac{1}{x^a}\right)$. We would like to show that $l_{\mathcal{P}}(x) = O\left(\frac{1}{x^a}\right)$. Using (4.1), we have

$$l_{\mathcal{P}}(x) = \sum_{\substack{d^2 \leq x \\ d \in \mathcal{N}}} \frac{m_{\mathcal{P}}(\frac{x}{d^2})}{d^2} \ll \sum_{\substack{d \leq \sqrt{x} \\ d \in \mathcal{N}}} \frac{d^{2a-2}}{x^a} \leq \frac{1}{x^a} \sum_{d \in \mathcal{N}} \frac{1}{d^{2-2a}} = \frac{\zeta_{\mathcal{P}}(2-2a)}{x^a}$$

since $2 - 2a > 1$, so $\zeta_{\mathcal{P}}(2-2a)$ exists. Hence

$$l_{\mathcal{P}}(x) = O\left(\frac{1}{x^a}\right).$$

For the converse, suppose $l_{\mathcal{P}}(x) = O\left(\frac{1}{x^a}\right)$. We want to show that $m_{\mathcal{P}}(x) = O\left(\frac{1}{x^a}\right)$. Using (4.2), and since $|\mu_{\mathcal{P}}(d)| \leq 1$, we have

$$m_{\mathcal{P}}(x) = \sum_{\substack{d^2 \leq x \\ d \in \mathcal{N}}} \frac{l_{\mathcal{P}}(\frac{x}{d^2})}{d^2} \mu_{\mathcal{P}}(d) \ll \frac{1}{x^a} \sum_{d \in \mathcal{N}} \frac{1}{d^{2-2a}} = \frac{\zeta_{\mathcal{P}}(2-2a)}{x^a}.$$

Hence

$$m_{\mathcal{P}}(x) = O\left(\frac{1}{x^a}\right).$$

□

For the following, we recall that

$$M_{\mathcal{P}}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \mu_{\mathcal{P}}(n) \quad \text{and} \quad L_{\mathcal{P}}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n).$$

In the same way of Theorem 4.6 we can prove:

Theorem 4.7. *Let $\mathcal{P}$ be a g-prime system with abscissa 1 and let $\frac{1}{2} < a \leq 1$. Then $L_{\mathcal{P}}(x) = O(x^a)$ if and only if $M_{\mathcal{P}}(x) = O(x^a)$.*

Proof. Assume $M_{\mathcal{P}}(x) = O(x^a)$. Then, in the same way that we obtained (4.1), we have

$$L_{\mathcal{P}}(x) = \sum_{\substack{d^2 \le x \\ d \in \mathcal{N}}} M_{\mathcal{P}}\left(\frac{x}{d^2}\right) \ll \sum_{\substack{d \le \sqrt{x} \\ d \in \mathcal{N}}} \frac{x^a}{d^{2a}} \le x^a \zeta_{\mathcal{P}}(2a)$$

since $2a > 1$, so $\zeta_{\mathcal{P}}(2a)$ exists.

For the converse, assume $L_{\mathcal{P}}(x) = O(x^a)$. Since $|\mu_{\mathcal{P}}(d)| \le 1$, in the same way that we obtained (4.2), we have

$$M_{\mathcal{P}}(x) = \sum_{\substack{d^2 \le x \\ d \in \mathcal{N}}} L_{\mathcal{P}}\left(\frac{x}{d^2}\right) \mu_{\mathcal{P}}(d) \ll x^a \sum_{\substack{d \le \sqrt{x} \\ d \in \mathcal{N}}} \frac{1}{d^{2a}} = x^a \zeta_{\mathcal{P}}(2a).$$

□

Remark 4.8.

(i) It immediately follows from Theorem 4.7 that if $0 \le a \le \frac{1}{2}$ and

(a) $M_{\mathcal{P}}(x) = O(x^a)$, then $L_{\mathcal{P}}(x) = O(x^{\frac{1}{2}+\varepsilon})$ for all $\varepsilon > 0$.

(b) $L_{\mathcal{P}}(x) = O(x^a)$, then $M_{\mathcal{P}}(x) = O(x^{\frac{1}{2}+\varepsilon})$ for all $\varepsilon > 0$.

(ii) We can ask whether Theorem 4.7 extends to $a \le \frac{1}{2}$. But as we see below this cannot be expected in general as it depends on $\zeta_{\mathcal{P}}(s)$ having pole at $s = \frac{1}{2}$.

Propostion 4.9. *Let $\mathcal{P}$ be a g-prime system with abscissa 1. Suppose $L_{\mathcal{P}}(x) = O(x^c)$ for some $c < \frac{1}{2}$. Then $\zeta_{\mathcal{P}}(s)$ has a pole at $\frac{1}{2}$.*

Proof. Since $L_{\mathcal{P}}(x) = O(x^c)$ for some $c < \frac{1}{2}$, then this implies $(Z_{\mathcal{P}}(s) =)\frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}$ is holomorphic for $\Re s > c$. Thus $Z_{\mathcal{P}}(s)$ has analytic continuation to $\Re s > c$. On the other hand, we know that $\zeta_{\mathcal{P}}(s)$ is holomorphic and has no zeros for $\Re s > 1$ since $\mathcal{P}$ has abscissa 1. Thus $\zeta_{\mathcal{P}}(2s)$ is holomorphic and has no zeros for $\Re s > \frac{1}{2}$. This shows that $\frac{1}{\zeta_{\mathcal{P}}(2s)}$ is holomorphic for $\Re s > \frac{1}{2}$ and $\frac{Z_{\mathcal{P}}(s)}{\zeta_{\mathcal{P}}(2s)}$ is also holomorphic for $\Re s > \frac{1}{2}$. Thus $\frac{1}{\zeta_{\mathcal{P}}(s)}$ is holomorphic for $\Re s > \frac{1}{2}$. This implies $\frac{1}{\zeta_{\mathcal{P}}(2s)}$ is holomorphic for $\Re s > \frac{1}{4}$. Therefore $\frac{Z_{\mathcal{P}}(s)}{\zeta_{\mathcal{P}}(2s)}$ is holomorphic for $\Re s > \alpha = \max\{c, \frac{1}{4}\}$ since $Z_{\mathcal{P}}(s)$ is holomorphic for $\Re s > c$. This means $\frac{1}{\zeta_{\mathcal{P}}(s)}$ is holomorphic for $\Re s > \alpha$ as well as implying that $\zeta_{\mathcal{P}}(s)$ is meromorphic and has no zeros for $\Re s > \alpha$.

Also, $\zeta_{\mathcal{P}}(s)$ has a pole at $s = 1$ since it has a singularity at the abscissa of convergence which is $s = 1$. Thus $\zeta_{\mathcal{P}}(2s)$ has a pole at $s = \frac{1}{2}$. However, we know that $\frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}$ has to be holomorphic for $\Re s > c$, and so $\zeta_{\mathcal{P}}(s)$ must have a pole at $\frac{1}{2}$ since the pole of $\zeta_{\mathcal{P}}(s)$ must be cancelled with the pole of $\zeta_{\mathcal{P}}(2s)$.

$\square$

Remark 4.10. The only way which would get $L_{\mathcal{P}}(x) = o(\sqrt{x})$ is if $\zeta_{\mathcal{P}}(s)$ has a pole at $\frac{1}{2}$. In other words, in order to have $L_{\mathcal{P}}(x) = \Omega(x^{\frac{1}{2}})$, $\zeta_{\mathcal{P}}(s)$ must have no pole at $\frac{1}{2}$ (see also Proposition 5.13 for more precise result).

4.2 Partial Sums of the Möbius function over $\mathcal{N}$

In this section, we are interested in Beurling prime systems for which the counting functions $\psi_{\mathcal{P}}(x)$, $N_{\mathcal{P}}(x)$ and $M_{\mathcal{P}}(x)$ are asymptotically well-behaved, in the sense that

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon}), \tag{4.3}$$

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta+\varepsilon}) \quad \text{for some } \rho > 0 \tag{4.4}$$

and

$$M_{\mathcal{P}}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \mu_{\mathcal{P}}(n) = O(x^{\gamma+\varepsilon}) \tag{4.5}$$

hold for all $\varepsilon > 0$, but for no $\varepsilon < 0$, where $\rho > 0$, $\alpha, \beta, \gamma < 1$ respectively. We shall explore which values of α, β, γ are feasible. We show that it is impossible to have both β and γ less than $\frac{1}{2}$ or both α and γ less than $\frac{1}{2}$. We also rule out some possible orders for α, β, γ and show that out of the three numbers $\{\alpha, \beta, \gamma\}$, the largest two must be equal and at least $\frac{1}{2}$. Clearly, we need $\alpha, \beta, \gamma \ge 0$ since $\psi_{\mathcal{P}}(x)$, $N_{\mathcal{P}}(x)$ and $M_{\mathcal{P}}(x)$ are $\Omega(1)$.

Theorem 4.11. *Given a g-prime system $\mathcal{P}$ satisfying (4.4) for some $\beta < 1$ and (4.13) for some $\gamma < 1$, we have* $\max\{\beta, \gamma\} \ge \frac{1}{2}$.

Proof. Assume $\Theta := \max\{\beta, \gamma\} < \frac{1}{2}$.

Assumption (4.13) implies that $s \int_1^\infty \frac{M_{\mathcal{P}}(x)}{x^{s+1}}\, dx$ converges to a holomorphic function for $\Re s > \gamma$. This implies $(U_{\mathcal{P}}(s) =)\frac{1}{\zeta_{\mathcal{P}}(s)}$ is holomorphic for $\Re s > \gamma$. However, the assumption (4.4) for some $\beta < \frac{1}{2}$ implies that $\zeta_{\mathcal{P}}(s)$ is holomorphic for $\Re s > \beta$ except for a simple pole at $s = 1$. Thus (4.4) and (4.13) together show $\zeta_{\mathcal{P}}(s)$ is holomorphic except for a simple pole at $s = 1$ and has no zeros for $\Re s > \Theta$. But, this gives a contradiction with Corollary 3.30 (part (ii)) since this says that $\zeta_{\mathcal{P}}(s)$ has infinitely many zeros in the strip $\{s \in \mathbb{C} : \eta' < \Re s < 1\}$ for every $\eta' \in (\beta, \frac{1}{2})$. Hence $\Theta \geq \frac{1}{2}$.

□

Theorem 4.12. *Given a g-prime system $\mathcal{P}$ satisfying (4.3) for some $\alpha < 1$ and (4.13) for some $\gamma < 1$, we have* $\max\{\alpha, \gamma\} \geq \frac{1}{2}$.

Proof. Assume $\Theta := \max\{\alpha, \gamma\} < \frac{1}{2}$.

Assumption (4.13) implies that $s \int_1^\infty \frac{M_{\mathcal{P}}(x)}{x^{s+1}}\, dx$ converges to a holomorphic function for $\Re s > \gamma$. This implies $U_{\mathcal{P}}(s)$ is holomorphic for $\Re s > \gamma$. However, the assumption (4.3) for some $\alpha < \frac{1}{2}$ implies that $\zeta_{\mathcal{P}}(s)$ is holomorphic for $\Re s > \alpha$ except for a simple pole at $s = 1$ and $\zeta_{\mathcal{P}}(s) \neq 0$ in this region by Theorem 3.28. Thus (4.3) and (4.13) together show $\zeta_{\mathcal{P}}(s)$ is holomorphic and has no zeros for $\Re s > \Theta$ except for a simple pole at $s = 1$. Let $s = \sigma + it$, with $\Re s > \Theta$. From the Mellin transform, we initially have

$$V_{\mathcal{P}}(\sigma + it) = O(|t|) \quad \text{for } \Re s > \alpha, \text{ and} \tag{4.6}$$

$$U_{\mathcal{P}}(\sigma + it) = O(|t|) \quad \text{for } \Re s > \gamma. \tag{4.7}$$

We have that for all $\delta > 0$, (4.6) holds uniformly as $|t| \longrightarrow \infty$ for $\sigma \geq \alpha + \delta$, and equally, as $|t| \longrightarrow \infty$ (4.7) holds uniformly for $\sigma \geq \gamma + \delta$.

We know that $U_{\mathcal{P}}(s)$ is non-zero for $\Re s > \Theta$ except for a simple pole at $s = 1$, and so $\log U_{\mathcal{P}}(s)$ is well-defined and holomorphic on $\{s \in \mathbb{C} : \Re s > \Theta\} \setminus (\Theta, 1]$. Thus for $\sigma > \Theta$, and uniformly for $\sigma \geq \Theta + \delta$,

$$\Re(-\log \zeta_{\mathcal{P}}(\sigma + it)) = \log |U_{\mathcal{P}}(\sigma + it)| \leq C \log |t|, \quad |t| \geq 2,$$

for some C. Now the Borel-Carathéodory Theorem (see Theorem 1.26) can be applied to the function $\log U_{\mathcal{P}}(z)$ and the circles with centre $3 + it$ and radii $r = 3 - \Theta - 2\delta$ and $R = 3 - \Theta - \delta$. Hence on the smaller circle, we have

$$\begin{aligned} |\log U_{\mathcal{P}}(z)| &\leq \frac{2r}{R - r} \sup_{|z| \leq R} \Re \log U_{\mathcal{P}}(z) + \frac{R + r}{R - r} |\log U_{\mathcal{P}}(3 + it)| \\ &\leq \frac{6 - 2\Theta - 4\delta}{\delta} (C \log |t|) + \frac{6 - 2\Theta - 3\delta}{\delta} |\log U_{\mathcal{P}}(3 + it)| = O(\log |t|). \end{aligned}$$

It follows that $|\log U_{\mathcal{P}}(\sigma+it)| = |\log \zeta_{\mathcal{P}}(\sigma+it)| = O(\log|t|)$ uniformly for $\sigma \geq \Theta + \delta$ since δ is arbitrary. Thus

$$\log|\zeta_{\mathcal{P}}(\sigma+it)| = \Re \log \zeta_{\mathcal{P}}(\sigma+it) \leq |\log \zeta_{\mathcal{P}}(\sigma+it)| = O(\log|t|)$$

Hence, for $\sigma > \Theta$,

$$|\zeta_{\mathcal{P}}(\sigma+it)| = O(|t|^k) \quad \text{for some } k.$$

But, this gives a contradiction with Corollary 3.30 (part (i)) since this says that $\zeta_{\mathcal{P}}(s)$ does not have finite order throughout the strip $\{s \in \mathbb{C} : \eta < \Re s < 1\}$ for every $\eta \in (\alpha, \frac{1}{2})$. Hence $\Theta \geq \frac{1}{2}$.

□

Remark 4.13. As a consequence of Theorems 4.11 and 4.12 if we want $M_{\mathcal{P}}(x) = O(x^c)$ for some $c < \frac{1}{2}$, then $N_{\mathcal{P}}(x) - \rho x$ and $\psi_{\mathcal{P}}(x) - x$ must both be $\Omega(x^{\frac{1}{2}-\varepsilon})$ for all $\varepsilon > 0$.

The following theorem shows that $U_{\mathcal{P}}(s)$ and $V_{\mathcal{P}}(s)$ are of zero order in a strip to the left of 1 if (4.3) and (4.13) hold.

Theorem 4.14. *Given a g-prime system $\mathcal{P}$ satisfying (4.3) for some $\alpha < 1$ and (4.13) for some $\gamma < 1$, we have for $1 \geq \sigma > \Theta = \max\{\alpha, \gamma\}$ and uniformly for $1 - \delta \geq \sigma \geq \Theta + \delta$ (any $\delta > 0$)*

$$V_{\mathcal{P}}(\sigma+it) = O\Big((\log|t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}\Big) \quad \textit{and} \quad U_{\mathcal{P}}(\sigma+it) = O\Big((\log|t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}\Big),$$

for all $\varepsilon > 0$. In particular, for $\sigma > \Theta$ we have $U_{\mathcal{P}}(\sigma+it) = O(|t|^{\varepsilon})$ for all $\varepsilon > 0$.

Proof. From the proof of Theorem 4.12, we have

$$|\log U_{\mathcal{P}}(\sigma+it)| = O(\log|t|) \quad \text{for } \sigma > \Theta.$$

Now we apply Cauchy's differentiation formula (see Theorem 1.28) to the function $\log U_{\mathcal{P}}(z)$, we have

$$V_{\mathcal{P}}(\sigma+it) = \frac{1}{2\pi i} \int_{\gamma} \frac{\log U_{\mathcal{P}}(z)}{(z-\sigma-it)^2} dz,$$

where $\sigma + it$ is the centre of the circle γ with radius ε. Selecting $\varepsilon > 0$ such that $\sigma > \Theta + \varepsilon$, yields

$$|V_{\mathcal{P}}(\sigma+it)| = \frac{1}{2\pi}\Big|\int_{\gamma} \frac{\log U_{\mathcal{P}}(z)}{(z-\sigma-it)^2} dz\Big| \leq \frac{1}{2\pi}\frac{2\pi\varepsilon}{\varepsilon^2} \sup_{z\in\gamma} |\log U_{\mathcal{P}}(z)| = O(\log|t|). \quad (4.8)$$

Now the Hadamard's Three-Circles Theorem (see Theorem 1.27) can be applied to the function $V_{\mathcal{P}}(z)$ on the circles C_1, C_2, C_3 with centre $a + it$ such that $a > 1 + \omega$ passing through the points $1 + \omega + it, \sigma + it, \Theta + \delta + it$, where $\delta, \omega > 0$. The radii are thus

$$r_1 = a - 1 - \omega, \quad r_2 = a - \sigma, \quad r_3 = a - \Theta - \delta.$$

Let M_1, M_2, M_3 be the maxima of $|V_{\mathcal{P}}(z)|$ on the three circles C_1, C_2, C_3. Then

$$M_2 \leq M_1^{1-\kappa} M_3^{\kappa}, \quad \text{where} \quad \kappa = \frac{\log\left(\frac{r_2}{r_1}\right)}{\log\left(\frac{r_3}{r_1}\right)}.$$

Now $M_3 = O(\log|t|)$ by estimate (4.8), and

$$\begin{aligned} M_1 &= \max_{z \in C_1} |V_{\mathcal{P}}(z)| = \max_{z \in C_1} \Big| \sum_{n \in \mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^z} \Big| \leq \max_{z \in C_1} \sum_{n \in \mathcal{N}} \frac{|\Lambda_{\mathcal{P}}(n)|}{|n^z|} = \max_{z \in C_1} \sum_{n \in \mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^{\Re z}} \\ &= \max_{z \in C_1} V_{\mathcal{P}}(\Re z) = V_{\mathcal{P}}(1 + \omega) = O(1). \end{aligned}$$

Thus $M_2 = O((\log|t|)^{\kappa})$. In particular, for $z \in C_2$ we have

$$|V_{\mathcal{P}}(\sigma + it)| \leq M_2 \leq M_1^{1-\kappa} M_3^{\kappa} = O((\log|t|)^{\kappa}).$$

We can make κ, the exponent, as close to $\frac{1-\sigma}{1-\Theta}$ as we like through selection of ω, δ small and a large, since

$$\begin{aligned} \kappa &= \frac{\log\left(\frac{r_2}{r_1}\right)}{\log\left(\frac{r_3}{r1}\right)} = \frac{\log\left(\frac{a-\sigma}{a-1-\omega}\right)}{\log\left(\frac{a-\Theta-\delta}{a-1-\omega}\right)} = \frac{\log(1 - \frac{\sigma}{a}) - \log\left(1 - \frac{(1-\omega)}{a}\right)}{\log\left(1 - \frac{(\Theta+\delta)}{a}\right) - \log\left(1 - \frac{(1-\omega)}{a}\right)} \\ &= \frac{-\frac{\sigma}{a} + \frac{1+\omega}{a} + O\left(\frac{1}{a^2}\right)}{-\frac{\Theta+\delta}{a} + \frac{1+\omega}{a} + O\left(\frac{1}{a^2}\right)} = \frac{\frac{1}{a}\left(1 + \omega - \sigma + O(\frac{1}{a})\right)}{\frac{1}{a}\left(1 + \omega - (\Theta + \delta) + O(\frac{1}{a})\right)} \\ &= \frac{1 - \sigma + O(\omega) + O(\frac{1}{a})}{1 - \Theta + O(\omega) + O(\delta) + O(\frac{1}{a})} = \frac{(1-\sigma)(1 + O(\frac{\omega}{1-\sigma}) + O(\frac{1}{(1-\sigma)a}))}{(1-\Theta)(1 + O(\frac{\omega}{1-\sigma}) + O(\frac{\delta}{1-\sigma}) + O(\frac{1}{(1-\sigma)a}))} \\ &= \frac{1-\sigma}{1-\Theta}\left(1 + O(\omega) + O\left(\frac{1}{a}\right)\right)\left(1 + O(\omega) + O(\delta) + O\left(\frac{1}{a}\right)\right) \\ &= \frac{1-\sigma}{1-\Theta} + O(\omega) + O(\delta) + O\left(\frac{1}{a}\right). \end{aligned}$$

Hence

$$|V_{\mathcal{P}}(\sigma + it)| = O((\log|t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}) \quad \text{for any } \varepsilon > 0.$$

Finally, applying Theorem 1.29 to the function $\log U_{\mathcal{P}}(z)$ from the point $\sigma + it$ to the point $2 + it$ and taking the real part, we have

$$\log |U_{\mathcal{P}}(\sigma + it)| = \log |U_{\mathcal{P}}(2 + it)| - \Re\Big\{ \int_{\sigma+it}^{2+it} V_{\mathcal{P}}(z)\ dz \Big\} \ll A + \int_{\sigma}^{2} |V_{\mathcal{P}}(y + it)|\, dy$$
$$\ll \int_{\sigma}^{2} (\log |t|)^{\frac{1-y}{1-\Theta}+\varepsilon}\, dy \le (\log |t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon} \int_{\sigma}^{2} dy \ll (\log |t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon},$$

and therefore $|U_{\mathcal{P}}(\sigma + it)| = O(\exp(\log |t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon})$. Choosing ε sufficiently small, so that the exponent $\frac{1-\sigma}{1-\Theta} + \varepsilon < 1$ for $\sigma > \Theta$ enables us to write for t sufficiently large and any given $\varepsilon > 0$

$$-\varepsilon \log |t| < \log |U_{\mathcal{P}}(\sigma + it)| < \varepsilon \log |t|,$$

so $U_{\mathcal{P}}(\sigma + it) = O(|t|^{\varepsilon})$ for all $\varepsilon > 0$.

□

Theorem 4.15. *Let $\mathcal{P}$ be a g-prime system satisfying (4.3), (4.4) and (4.13) for some α, β and $\gamma < 1$ respectively. Then out of the three numbers $\{\alpha, \beta, \gamma\}$, the largest two must be equal and at least $\frac{1}{2}$.*

Proof. We rule out the three cases where one is strictly larger than the others.

(i) If $\alpha, \gamma < \beta$, then Theorem 4.14 tells us that for $\Re s > \max\{\alpha, \gamma\}$, we have $\zeta_{\mathcal{P}}(s)$ and $U_{\mathcal{P}}(s)$ are of zero order. But this gives a contradiction as $\zeta_{\mathcal{P}}(s)$ has infinite order in the strip $\alpha < \Re s < \beta$ by Remark 3.33 (part (i)).

(ii) If $\beta, \gamma < \alpha$, then $\zeta_{\mathcal{P}}(s)$ is holomorphic and has no zeros for $\Re s > \max\{\beta, \gamma\}$ except for a simple pole at $s = 1$. But this gives a contradiction as $\zeta_{\mathcal{P}}(s)$ has zeros in the strip $\beta < \Re s < \alpha$ by Remark 3.33 (part (ii)).

(iii) If $\alpha, \beta < \gamma$, we know from Theorem 3.32 that $\zeta_{\mathcal{P}}(s)$ and $U_{\mathcal{P}}(s)$ have analytic continuations to $\{s \in \mathbb{C} : \Re s > \min\{\alpha, \beta\}\}$ except for simple pole of $\zeta_{\mathcal{P}}(s)$ at $s = 1$ and for $\Re s > \max\{\alpha, \beta\}$, they are of zero order. Now we apply the converse part of Theorem 3.35 with $U_{\mathcal{P}}(s)$, so that $M_{\mathcal{P}}(x) = O(x^{\alpha+\varepsilon})$ or $M_{\mathcal{P}}(x) = O(x^{\beta+\varepsilon})$ for all $\varepsilon > 0$. This contradicts assumption (4.13) since $\alpha, \beta < \gamma$.

Therefore we have the other cases which are either $\beta < \alpha = \gamma$, $\alpha < \beta = \gamma$ or $\gamma < \alpha = \beta$. These show that the largest two of these three numbers must be equal.

Furthermore, Theorems 3.31, 4.11 and 4.12 also reveal us that the largest two must be at least $\frac{1}{2}$. □

4.2.1 Examples

From this theorem, we are motivated to explore whether it really is possible that each of β, α and γ can be strictly less than the other two and whether it can be less than 1, and as such, we provide some examples of g-prime systems.

Example 4.16. (i) Let $\mathcal{P} = \mathbb{P}$, so that $\mathcal{N} = \mathbb{N}$, then (4.4) holds with $\beta = 0$ (and $\rho = 1$) and if the RH is true, (4.3) and (4.13) hold for α and γ equal to $\frac{1}{2}$. This is a conditional example where $\beta < \alpha = \gamma$.

Example 4.17. (ii) Let $\mathcal{P} = \{p_1, p_2, p_3, \cdots\}$ be a g-prime system where $p_n = R^{-1}(n)$ for all $n \in \mathbb{N}$ and R is the function defined by

$$R(x) := \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!k\zeta(k+1)},$$

where $\zeta(\cdot)$ is the Riemann zeta function. Note R is strictly increasing and continuous on $[1, \infty)$. For then, $\pi_{\mathcal{P}}(x) \leq R(x) < \pi_{\mathcal{P}}(x) + 1$ (since if $p_n \leq x < p_{n+1}$, then $n = \pi_{\mathcal{P}}(x)$). Thus the function $\Pi_{\mathcal{P}}(x)$ (see Section 3.2.2) of this system can be calculated by

$$\begin{aligned}
\Pi_{\mathcal{P}}(x) &= \sum_{n=1}^{\infty} \frac{\pi_{\mathcal{P}}(x^{\frac{1}{n}})}{n} = \sum_{1 \leq n \leq A \log x} \frac{\pi_{\mathcal{P}}(x^{\frac{1}{n}})}{n} \quad (\text{since } \pi_{\mathcal{P}}(x^{\frac{1}{n}}) = 0 \text{ if } x^{\frac{1}{n}} < p_1) \\
&= \sum_{1 \leq n \leq A \log x} \frac{R(x^{\frac{1}{n}})}{n} + \sum_{1 \leq n \leq A \log x} \frac{O(1)}{n} \quad (\text{for some } A > 0) \\
&= \sum_{1 \leq n \leq A \log x} \frac{1}{n} \sum_{k=1}^{\infty} \frac{(\log x^{\frac{1}{n}})^k}{k!k\zeta(k+1)} + O\Big(\sum_{1 \leq n \leq A \log x} \frac{1}{n} \Big) \\
&= \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!k\zeta(k+1)} \sum_{1 \leq n \leq A \log x} \frac{1}{n^{k+1}} + O(\log\log x) \\
&= \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!k\zeta(k+1)} \bigg(\zeta(k+1) + O\Big(\frac{1}{k(A \log x)^k} \Big) \bigg) + O(\log\log x) \\
&= \mathrm{li}(x) + O(\log\log x).
\end{aligned}$$

For this system, the prime counting function $\psi_{\mathcal{P}}(x)$ can be found by integration as follows:

$$\begin{aligned}\psi_{\mathcal{P}}(x) &= \int_{p_1}^{x} \log t \, d\,\Pi_{\mathcal{P}}(t) = \int_{p_1}^{x} \log t \, d\,\mathrm{li}(t) + \int_{p_1}^{x} \log t \, d\,(\Pi_{\mathcal{P}}(t) - \mathrm{li}(t)) \\ &= \int_{p_1}^{x} dt + (\Pi_{\mathcal{P}}(t) - \mathrm{li}(t)) \log x - \int_{p_1}^{x} \frac{\Pi_{\mathcal{P}}(t) - \mathrm{li}(t)}{t} dt \\ &= x + O(\log x \log\log x) + O\Big(\int_{p_1}^{x} \frac{\log\log t}{t} dt \Big) = x + O(\log x \log\log x) \\ &= x + O(x^{\varepsilon}) \quad \text{for all } \varepsilon > 0.\end{aligned}$$

This shows $\alpha = 0$. Therefore, by Theorem 4.15, β must be equal to γ, but it is not clear what the common value is except that it lies in $[\frac{1}{2}, 1]$. Theorem 3.29 gives $N_{\mathcal{P}}(x) = \rho x + O(x^{-c\sqrt{\log x \log\log x}})$ for some $\rho, c > 0$. It may be the case that $N_{\mathcal{P}}(x) - \rho x = O(x^{\eta})$ for some $\eta < 1$, in which case the infimum of such η is β. But we do not know if this holds for any $\eta < 1$. Hence we get $0 = \alpha < \frac{1}{2} \leq \beta = \gamma \leq 1$ for this system.

Example 4.18. (iii) For $\frac{1}{2} \leq \beta < 1$, let $\mathcal{P} = \mathbb{P} \cup \mathbb{P}^{\frac{1}{\beta}}$, where we include any multiplicities. Assume that $M(x) = O(x^{\eta})$ for some $\frac{1}{2} \leq \eta < 1$, where $M(x)$ is the partial sum of the Möbius function not exceeding x. This is equivalent to a weaker version of the RH which is when $\zeta(s) \neq 0$ for $\Re s > \eta$. The associated Beurling zeta function of this system is

$$\zeta_{\mathcal{P}}(s) = \zeta(s)\zeta(s/\beta) = \sum_{m \geq 1} \frac{1}{m^s} \sum_{n \geq 1} \frac{1}{n^{\frac{s}{\beta}}} = \sum_{m,n \geq 1} \frac{1}{(mn^{\frac{1}{\beta}})^s}.$$

For this system, we have

$$N_{\mathcal{P}}(x) = \sum_{\substack{m,\, n \,\geq\, 1 \\ mn^{\frac{1}{\beta}} \,\leq\, x}} 1.$$

Applying Dirichlet's hyperbola theorem (see Theorem 1.11), we have

$$N_{\mathcal{P}}(x) = \sum_{n \leq a^{\beta}} \Big[\frac{x}{n^{1/\beta}} \Big] + \sum_{n \leq b} \Big[\Big(\frac{x}{n} \Big)^{\beta} \Big] - [a^{\beta}][b]$$

for any $ab = x$. Putting $a = x^{\lambda}$, we obtain

$$N_{\mathcal{P}}(x) = \sum_{n \leq x^{\lambda\beta}} \Big[\frac{x}{n^{1/\beta}} \Big] + \sum_{n \leq x^{1-\lambda}} \Big[\Big(\frac{x}{n} \Big)^{\beta} \Big] - [x^{\lambda\beta}][x^{1-\lambda}]$$

$$
\begin{aligned}
&= x\sum_{n\le x^{\lambda\beta}}\frac{1}{n^{1/\beta}} + x^{\beta}\sum_{n\le x^{1-\lambda}}\frac{1}{n^{\beta}} - x^{\lambda\beta+1-\lambda} + O(x^{\lambda\beta}) + O(x^{1-\lambda})\\
&= x\bigg(\zeta\Big(\frac{1}{\beta}\Big) - \frac{\beta}{1-\beta}x^{-\lambda\beta(\frac{1}{\beta}-1)} + O(x^{-\lambda\beta(\frac{1}{\beta})})\bigg)\\
&+ x^{\beta}\Big(\frac{x^{(1-\lambda)(1-\beta)}}{1-\beta} + \zeta(\beta) + O(x^{-(1-\lambda)\beta}\Big) - x^{\lambda\beta+1-\lambda} + O(x^{\lambda\beta}) + O(x^{1-\lambda})\\
&= \zeta\Big(\frac{1}{\beta}\Big)x + \zeta(\beta)x^{\beta} + O(x^{\lambda\beta}) + O(x^{1-\lambda}).
\end{aligned}
$$

Selecting $\lambda = \frac{1}{1+\beta}$, then $\lambda\beta = 1-\lambda$ minimises this quantity and gives

$$
N_{\mathcal{P}}(x) = \zeta\Big(\frac{1}{\beta}\Big)x + \zeta(\beta)x^{\beta} + O(x^{\frac{\beta}{1+\beta}}).
$$

Thus the "β" for this system is indeed β. Furthermore, α on our assumption can be calculated as follows:

$$
\psi_{\mathcal{P}}(x) = \psi(x) + \psi(x^{\beta}) = x + x^{\beta} + O(x^{\eta+\varepsilon}),\ \text{where } \eta \in [\tfrac{1}{2}, 1).
$$

Thus, if we assume that $\eta < \beta$, then $\alpha = \beta$. Now we would like to estimate $M_{\mathcal{P}}(x)$ for this system. We have

$$
\frac{1}{\zeta_{\mathcal{P}}(s)} = \frac{1}{\zeta(s)\zeta(s/\beta)} = \sum_{m\ge 1}\frac{\mu(m)}{m^s}\sum_{n\ge 1}\frac{\mu(n)}{n^{\frac{s}{\beta}}} = \sum_{m,n\ge 1}\frac{\mu(m)\mu(n)}{(mn^{\frac{1}{\beta}})^s}.
$$

Therefore

$$
M_{\mathcal{P}}(x) = \sum_{\substack{mn^{\frac{1}{\beta}} \le x\\ m,\, n \ge 1}}\mu(m)\mu(n) = \sum_{n\le x^{\beta}}\mu(n)\sum_{m\le \frac{x}{n^{\beta}}}\mu(m) = \sum_{n\le x^{\beta}}\mu(n)M\Big(\frac{x}{n^{1/\beta}}\Big),
$$

where $M(x)$ is the partial sum of the Möbius function not exceeding x. Now using the bound $M(x) = O(x^{\eta})$ for some $\frac{1}{2} \le \eta < 1$ and applying Dirichlet's hyperbola theorem again, we have

$$
\begin{aligned}
M_{\mathcal{P}}(x) = \sum_{mn^{\frac{1}{\beta}}\le x}\mu(m)\mu(n) &= \sum_{n\le a^{\beta}}\mu(n)M\Big(\frac{x}{n^{1/\beta}}\Big) + \sum_{n\le b}\mu(n)M\Big(\Big(\frac{x}{n}\Big)^{\beta}\Big) - M(b)M(a^{\beta}),\\
&\ll \sum_{n\le a^{\beta}}\frac{x^{\eta}}{n^{\frac{\eta}{\beta}}} + \sum_{n\le b}\frac{x^{\beta\eta}}{n^{\beta\eta}} + b^{\eta}a^{\beta\eta}
\end{aligned}
$$

for any $ab = x$. Putting $a = x^{\lambda}$, we obtain

$$M_{\mathcal{P}}(x) \ll x^{\eta} \sum_{n \leq x^{\beta\lambda}} \frac{1}{n^{\frac{\eta}{\beta}}} + x^{\beta\eta} \sum_{n \leq x^{1-\lambda}} \frac{1}{n^{\beta\eta}} + x^{\eta(1-\lambda)} x^{\beta\eta\lambda}.$$

Now if we assume that $\eta < \beta$, then $M_{\mathcal{P}}(x)$ is

$$\begin{aligned} &\ll x^{\eta} x^{\lambda\beta(1-\frac{\eta}{\beta})} + x^{\beta\eta} x^{(1-\lambda)(1-\beta\eta)} + x^{\lambda\beta\eta+\eta-\lambda\eta} \\ &\ll x^{\eta+\lambda\beta-\lambda\eta} + x^{1-\lambda+\lambda\beta\eta} + x^{\lambda\beta\eta+\eta-\lambda\eta}. \end{aligned}$$

The third exponent is less than or equal to the second exponent since

$$\lambda\beta\eta + \eta - \lambda\eta = \lambda\beta\eta + \eta(1-\lambda) \leq 1 - \lambda + \lambda\beta\eta.$$

Thus choosing $\lambda = \frac{1}{1+\beta}$, so that $1 - \lambda + \lambda\beta\eta = \eta + \lambda\beta - \lambda\eta$ minimises the error. This gives

$$M_{\mathcal{P}}(x) \ll x^{\beta\frac{(1+\eta)}{1+\beta}} + x^{\beta\frac{(2\eta)}{1+\beta}} \ll x^{\beta\frac{(1+\eta)}{1+\beta}}.$$

Hence, $\gamma \leq \beta\frac{(1+\eta)}{1+\beta} < \beta$ and we get $\gamma < \alpha = \beta < 1$ for this system and $\gamma \geq \frac{1}{2}$ since $\frac{1}{\zeta_{\mathcal{P}}(s)} = \frac{1}{\zeta(s)\zeta(s/\beta)}$ has poles on the $\frac{1}{2}$-line. Again, it is a conditional example as a version of RH.

Propostion 4.19. *Let $\mathcal{P}$ be a g-prime system satisfying (4.4), (4.3) and (4.13) for some α, β, $\gamma < 1$ respectively and define ξ via*

$$L_{\mathcal{P}}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) = O(x^{\xi+\varepsilon}) \tag{4.9}$$

holds for all $\varepsilon > 0$, but for no $\varepsilon < 0$. Suppose $\gamma \geq \frac{1}{2}$. Then $\gamma = \xi$ and the largest two of the numbers α, β, ξ are equal and at least $\frac{1}{2}$. On the other hand, if $\gamma < \frac{1}{2}$, then $\xi > \gamma$.

Proof. Theorem 4.7 and Remark 4.8, as well as using Theorem 4.15 give the required result.

□

Notice that we do not know if there is a system with abscissa 1 which satisfies (4.13) with $\gamma < \frac{1}{2}$.

4.3 Infinite sums involving the functions $\lambda_{\mathcal{P}}$ and $\mu_{\mathcal{P}}$

In this section, we investigate the value of the sums $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ and $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n}$ of g-prime systems under some assumptions over $\mathcal{P}$. We also provide special examples of these sums when the g-prime system $\mathcal{P}$ is as in Example 3.3.

Propostion 4.20. *Let $\mathcal{P}$ be a g-prime system for which $\sum_{n\in\mathcal{N}} \frac{1}{n}$ converges to A. Then*

(i) $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ *converges to* $\frac{1}{A}$.

(ii) $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n}$ *converges to* $\frac{\zeta_{\mathcal{P}}(2)}{A}$.

Proof. (i) We know, $\sum_{n\in\mathcal{N}} \frac{1}{n}$ converges absolutely to A since $\sum_{n\in\mathcal{N}} \frac{1}{n}$ converges and $\frac{1}{n}$ is positive for all $n \in \mathcal{N}$. Thus, by Theorem 3.22 when $f(n) = \frac{1}{n}$, we have

$$\sum_{n\in\mathcal{N}} \frac{1}{n} = \prod_{p\in\mathcal{P}} \frac{1}{1-\frac{1}{p}} = A. \tag{4.10}$$

Now since $\frac{|\mu_{\mathcal{P}}(n)|}{n} \le \frac{1}{n}$ for all $n \in \mathcal{N}$ and $\sum_{n\in\mathcal{N}} \frac{1}{n}$ converges, then $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ converges absolutely.

By using Theorem 3.22 again, we have

$$\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n} = \prod_{p\in\mathcal{P}} \Big(1-\frac{1}{p}\Big) = \frac{1}{A}.$$

Hence $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ converges to $\frac{1}{A}$.

(ii) Now since $\frac{|\lambda_{\mathcal{P}}(n)|}{n} = \frac{1}{n}$ for all $n \in \mathcal{N}$ and $\sum_{n\in\mathcal{N}} \frac{1}{n}$ converges (by (4.10)), then $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n}$ converges absolutely.

By using Theorem 3.22 again, we have

$$\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n} = \prod_{p\in\mathcal{P}} \Big(1+\frac{1}{p}\Big)^{-1} = \prod_{p\in\mathcal{P}} \frac{1-\frac{1}{p}}{1-\frac{1}{p^2}} = \frac{1}{A}\prod_{p\in\mathcal{P}} \frac{1}{1-\frac{1}{p^2}} = \frac{\zeta_{\mathcal{P}}(2)}{A}.$$

Hence $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n}$ converges to $\frac{\zeta_{\mathcal{P}}(2)}{A}$.

□

Propostion 4.21. *Let $\mathcal{P}$ be a g-prime system for which $\sum_{n\in\mathcal{N}} \frac{1}{n}$ diverges and $\sum_{n\in\mathcal{N}} \frac{1}{n^s}$ converges for every s with $\Re s > 1$. If*

(i) $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ converges to S, then S must be zero.

(ii) $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n}$ converges to S, then S must be zero.

Proof. (i) Using Example 3.24 and that $\sum_{n\in\mathcal{N}} \frac{1}{n^s}$ converges for every s with $\Re s > 1$, we have

$$U_{\mathcal{P}}(s) = \sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n^s} = \frac{1}{\zeta_{\mathcal{P}}(s)} \quad \text{for } \Re s > 1.$$

By Theorem 1.13 and the assumption that $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ converges to S, we have

$$U_{\mathcal{P}}(1+\varepsilon) = \frac{1}{\zeta_{\mathcal{P}}(1+\varepsilon)} \longrightarrow S \quad \text{as } \varepsilon \longrightarrow 0^+.$$

Thus, if $S \neq 0$, then

$$\zeta_{\mathcal{P}}(1+\varepsilon) \longrightarrow \frac{1}{S} \text{ as } \varepsilon \longrightarrow 0^+.$$

On the other hand, we know that $\sum_{n\in\mathcal{N}} \frac{1}{n}$ diverges, therefore $\zeta_{\mathcal{P}}(1+\varepsilon)$ diverges as $\varepsilon \longrightarrow 0^+$, since

$$\lim_{\varepsilon\to 0^+} \zeta_{\mathcal{P}}(1+\varepsilon) = \lim_{\varepsilon\to 0^+} \sum_{n\in\mathcal{N}} \frac{1}{n^{1+\varepsilon}} \geq \lim_{\varepsilon\to 0^+} \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \frac{1}{n^{1+\varepsilon}} = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \lim_{\varepsilon\to 0^+} \frac{1}{n^{1+\varepsilon}} = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \frac{1}{n}$$

for any x. This gives a contradiction since the right hand side can be made arbitrarily large. Hence S must be zero.

(ii) Using Example 3.24 and that $\sum_{n\in\mathcal{N}} \frac{1}{n^s}$ converges for every s with $\Re s > 1$, we have

$$Z_{\mathcal{P}}(s) = \sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n^s} = \frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}.$$

By Theorem 1.13 and the assumption that $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n}$ converges to S, we have

$$Z_{\mathcal{P}}(1+\varepsilon) = \frac{\zeta_{\mathcal{P}}(2(1+\varepsilon))}{\zeta_{\mathcal{P}}(1+\varepsilon)} \longrightarrow S \quad \text{as } \varepsilon \longrightarrow 0^+.$$

Thus, if $S \neq 0$, then

$$\frac{\zeta_{\mathcal{P}}(1+\varepsilon)}{\zeta_{\mathcal{P}}(2(1+\varepsilon))} \longrightarrow \frac{1}{S} \text{ as } \varepsilon \longrightarrow 0^+.$$

Hence $\zeta_{\mathcal{P}}(1+\varepsilon)$ converges to a limit since $\zeta_{\mathcal{P}}(2(1+\varepsilon)) \longrightarrow \zeta_{\mathcal{P}}(2) \neq 0$ as $\varepsilon \longrightarrow 0^+$. By above this gives a contradiction. Hence S must be zero.

□

Remark 4.22.

(i) From above results if the abscissa of $\mathcal{P}$ is 1 and $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ converges, then $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n} = 0$ if and only if $\sum_{n\in\mathcal{N}} \frac{1}{n}$ diverges. But if $\mathcal{P}$ has abscissa greater than 1, then $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ may not be convergent (see Example 4.23).

(ii) Is it possible to find a g-prime system $\mathcal{P}$ with abscissa 1 and $\sum_{n\in\mathcal{N}} \frac{1}{n}$ diverges and either $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ or $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n}$ diverges?

This question has been partially answered by Tao [50], by assuming $\mathcal{P}$ is subset of the usual primes. He showed that $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ converges if and only if $\sum_{n\in\mathcal{N}} \frac{1}{n}$ diverges. This means that it is impossible to have such a system if $\mathcal{P}$ is a set of primes.

Example 4.23. Let $\mathcal{P} = \{\sqrt{p} : p \in \mathbb{P}\}$. Then $\mathcal{N} = \{\sqrt{n} : n \in \mathbb{N}\}$ and $\sum_{n\in\mathcal{N}} \frac{1}{n}$ diverges and

$$\zeta_{\mathcal{P}}(s) = \sum_{n\in\mathcal{N}} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{s}{2}}} = \zeta(\tfrac{s}{2}) \quad \text{for } \Re s > 2.$$

But $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n}$ is divergent since

$$\begin{aligned}\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n} &= \sum_{m=1}^{\infty} \frac{\mu_{\mathcal{P}}(\sqrt{m})}{\sqrt{m}}, \quad \text{where } n = \sqrt{m} \text{ and } m \in \mathbb{N} \\ &= \sum_{m=1}^{\infty} \frac{\mu(m)}{\sqrt{m}} \text{ is not convergent.}\end{aligned}$$

For if $\sum_{m=1}^{\infty} \frac{\mu(m)}{\sqrt{m}}$ converges, then put $m(x) = \sum_{m\leq x} \frac{\mu(m)}{\sqrt{m}} = C + o(1)$, so that

$$\begin{aligned}
M(x) = \sum_{m\leq x} \mu(m) = m(x)\cdot\sqrt{x} - \frac{1}{2}\int_1^x \frac{m(t)}{t^{\frac{1}{2}}}dt \\
&= (C+o(1))\sqrt{x} - \frac{1}{2}\int_1^x \frac{C+o(1)}{t^{\frac{1}{2}}}dt \\
&= C\sqrt{x} + o(\sqrt{x}) - \frac{C}{2}\int_1^x \frac{1}{t^{\frac{1}{2}}}dt + \frac{1}{2}\int_1^x o\Big(\frac{1}{t^{\frac{1}{2}}}\Big)dt \\
&= C\sqrt{x} + o(\sqrt{x}) - C[\sqrt{x}-1] + o(\sqrt{x}) \\
&= o(\sqrt{x}).
\end{aligned}$$

But this gives a contradiction with the fact that $M(x) = \Omega(\sqrt{x})$. Hence

$$\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n} = \sum_{m=1}^{\infty} \frac{\mu(m)}{\sqrt{m}} \text{ is divergent.}$$

Example 4.24. Let $\mathcal{P} = \{2, 2, 3, 3, 5, 5, \cdots\}$. Then $\mathcal{N} = \mathbb{N}$ with multiplicity $d(n)$ and for $\Re s > 1$, we have

$$\frac{1}{\zeta_{\mathcal{P}}(s)} = \sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n^s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)^2 = \frac{1}{\zeta^2(s)} = \sum_{n=1}^{\infty} \frac{(\mu*\mu)(n)}{n^s}.$$

We would like to show that

$$\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n} = 0.$$

It is equivalent to show that

$$\sum_{n=1}^{\infty} \frac{(\mu*\mu)(n)}{n} = 0. \tag{4.11}$$

Applying Dirichlet's hyperbola theorem with $a = b = \sqrt{x}$ (see Theorem 1.11) to $J(x) := \sum_{n\leq x}(\mu*\mu)(n)$, we have

$$J(x) = 2\sum_{n\leq\sqrt{x}} \mu(n) M\Big(\frac{x}{n}\Big) - (M(\sqrt{x}))^2, \tag{4.12}$$

where $M(x)$ is the partial sum of Möbius function up to and including x. Since

$M(x) = O\left(\frac{x}{(\log x)^A}\right)$ for any $A > 2$ as $x \longrightarrow \infty$, then $|M(x)| \leq \frac{Cx}{(\log x)^A}$ for all A [37]. The sum on the right of (4.12) satisfies

$$\Big| \sum_{n \leq \sqrt{x}} \mu(n) M\Big(\frac{x}{n}\Big)\Big| \leq \sum_{n \leq \sqrt{x}} |\mu(n)| \frac{Cx}{n(\log \frac{x}{n})^A} \leq \frac{Cx}{(\log \sqrt{x})^A} \sum_{n \leq \sqrt{x}} \frac{1}{n} = \frac{Cx}{(\log \sqrt{x})^{A-1}}.$$

The last term of the right hand side of (4.12) is bounded by

$$|(M(\sqrt{x}))^2| \leq \Big(\frac{C\sqrt{x}}{(\log \sqrt{x})^A}\Big)^2 = \Big(\frac{C^2 x}{(\log \sqrt{x})^{2A}}\Big).$$

Therefore

$$|J(x)| \leq \frac{2Cx}{(\log \sqrt{x})^{A-1}} + \frac{C^2 x}{(\log \sqrt{x})^{2A}} < \frac{2Cx}{(\log \sqrt{x})^{A-1}} \quad \text{for all } A.$$

Hence

$$J(x) = O\Big(\frac{x}{(\log x)^{A-1}}\Big) \quad \text{for all } A.$$

Now, by Abel summation,

$$\begin{aligned}
\sum_{n \leq x} \frac{(\mu * \mu)(n)}{n} &= J(x)\frac{1}{x} + \int_2^x J(t)\frac{1}{t^2}\,dt \\
&= \frac{O\big(\frac{x}{(\log x)^{A-1}}\big)}{x} + \int_2^\infty \frac{J(t)}{t^2}dt - \int_x^\infty \frac{J(t)}{t^2}dt \\
&= \frac{O\big(\frac{x}{(\log x)^{A-1}}\big)}{x} + \int_2^\infty \frac{J(t)}{t^2}dt - \int_x^\infty \frac{O\big(\frac{t}{(\log t)^{A-1}}\big)}{t^2}dt \\
&= O\Big(\frac{1}{(\log x)^{A-1}}\Big) + C + O\Big(\frac{1}{(\log x)^{A-2}}\Big), \quad \text{where } C \text{ is constant}
\end{aligned}$$

since $\int_2^\infty \frac{(\log t)^{1-A}}{t} dt$ converges for $A > 2$ and $\big| \int_x^\infty \frac{(\log t)^{1-A}}{t} dt \big| = \frac{1}{(A-2)(\log x)^{A-2}}$. Thus

$$\sum_{n \leq x} \frac{(\mu * \mu)(n)}{n} = O\Big(\frac{1}{(\log x)^{A-2}}\Big) \quad \text{for all } A.$$

Letting $x \longrightarrow \infty$, we have (4.11).

Notice that Theorem 4.4 also gives $\sum_{n \in \mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n} = 0$.

4.4 Open problems

(i) From Examples (4.16) and (4.17) we have systems with $(\alpha, \beta, \gamma) = (A, 0, A)$ and $(0, B, B)$ for some $\frac{1}{2} \leq A, B \leq 1$. Can we find, unconditionally, such systems with $A < 1$ and $B < 1$?

(ii) In Example (4.18) we have a system with $(\alpha, \beta, \gamma) = (C, C, D)$ with $\frac{1}{2} \leq D < C < 1$. Can we find one unconditionally, with $D < 1$. Furthermore, can we find one with $D < \frac{1}{2}$?

(iii) The findings of the second section may suggest the following conjecture.

Conjecture 4.25. Given a g-prime systems $\mathcal{P}$ with abscissa 1 for which

$$M_{\mathcal{P}}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \mu_{\mathcal{P}}(n) = O(x^{\gamma}),$$

it is impossible to have $\gamma < \frac{1}{2}$.

Chapter 5

Completely Multiplicative Functions of Zero Sum over $\mathcal{N}$

In this chapter, we generalise the notion of CMO functions to Beurling g-prime systems. We give some properties and examples of these functions. In particular, we provide some examples of the function $\frac{\lambda_{\mathcal{P}}(n)}{n}$ for different g-prime systems $\mathcal{P}$ where $\lambda_{\mathcal{P}}(n)$ is Liouville function over $\mathcal{P}$. Kahane and Saïas used examples of such functions $\frac{\lambda_{\mathcal{P}}(n)}{n}$ with $\mathcal{P}$ being a subset of usual primes $\mathbb{P}$ (see [30], [32]).

5.1 $CMO_{\mathcal{P}}$ functions

Let $\mathcal{P}$ be a g-prime system. We say that $f : \mathcal{N} \longrightarrow \mathbb{C}$ is a $\boldsymbol{CMO_{\mathcal{P}}}$ function if it satisfies the following conditions:

$$(i)\ f \text{ is a completely multiplicative function} \qquad (ii) \sum_{n\in\mathcal{N}} f(n) = 0.$$

This is a generalisation of a CMO function. We investigate some properties of $CMO_{\mathcal{P}}$ functions. For instance, let f be a $CMO_{\mathcal{P}}$ function and g a completely multiplicative function "close" to f. We shall show that g is also a $CMO_{\mathcal{P}}$ function under some extra condition on f. Furthermore, the same questions that were asked by Kahane and Saïas [31] about CMO functions can be discussed for $CMO_{\mathcal{P}}$ functions. For example, how quickly the partial sum of $f(n)$ not exceeding x, (*i.e.* $\sum_{n\leq x} f(n)$) can tend to zero. In particular, we would like to investigate how quickly the partial sum of $\lambda_{\mathcal{P}}(n)$ over n up to and including x tends to zero with different type of systems where $\lambda_{\mathcal{P}}(n)$ is Liouville's function over $\mathcal{N}$. Specially, we discuss O-Results of $\sum_{n\leq x} \frac{\lambda_{\mathcal{P}}(n)}{n}$

over $\mathcal{N}$ with a system which satisfies

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta+\varepsilon}) \text{ (for some } \rho > 0) \quad \text{and} \quad \psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon}) \tag{5.1}$$

for all $\varepsilon > 0$, but for no $\varepsilon < 0$ and $0 \leq \alpha, \beta < 1$. As special case we treat Zhang's system (see Theorem 3.34) with error term $O(x^{\frac{1}{2}}e^{(c\log x)^{\frac{2}{3}}})$ for the counting function $N_{\mathcal{P}}(x)$ and $O(x^{\frac{1}{2}})$ for $\psi_{\mathcal{P}}(x)$. We show that $\sum_{n\leq x}\frac{\lambda_{\mathcal{P}}(n)}{n}$ for the system which satisfies (5.1) is $O\big(\frac{1}{x^{1-\Theta-\varepsilon}}\big)$, where Θ is the maximum value between α and β, whereas Zhang's system gives

$$L_{\mathcal{P}}(x) = O(x^{\frac{1}{2}}e^{(c\log x)^{\frac{2}{3}}}), \tag{5.2}$$

where $L_{\mathcal{P}}(x)$ is the partial sum of the Liouville function on $\mathcal{N}$ as defined previously. This can be compared to the conditional result of M. Balazard and A. de Roton [3] concerning the Möbius function of the standard integers. They showed that assuming RH,

$$M(x) = O\Big(x^{\frac{1}{2}}e^{(\log x)^{\frac{1}{2}}(\log\log x)^{\frac{5}{2}+\varepsilon}}\Big) \text{ for all } \varepsilon > 0,$$

where $M(x)$ is the partial sum of the Möbius function. Following the above result, Theorem 4.7 can be used to show that

$$L(x) = O\Big(x^{\frac{1}{2}}e^{c(\log x)^{\frac{1}{2}}(\log\log x)^{\frac{5}{2}+\varepsilon}}\Big) \text{ for all } \varepsilon > 0.$$

We notice that the right hand side of (5.2) can be automatically improved if one would be able to improve the error term in Zhang's system.

We also explore $\Omega-$Results for the behaviour of $\sum_{n\leq x}\frac{\lambda_{\mathcal{P}}(n)}{n}$ for a system $\mathcal{P}$ which satisfies either the assumption

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta}) \quad \text{for some } \rho > 0 \quad \text{or} \quad \psi_{\mathcal{P}}(x) = x + O(x^{\alpha}),$$

for some $\alpha, \beta < \frac{1}{2}$. The aim of this chapter is to find a completely multiplicative function f over $\mathcal{N}$ with abscissa 1 such that

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} f(n) = O\Big(\frac{1}{x^c}\Big) \text{ for some } c > \tfrac{1}{2}.$$

5.2 Some properties of $CMO_{\mathcal{P}}$ functions

In this section, we derive some preliminary properties of $CMO_{\mathcal{P}}$ functions.

Propostion 5.1. *Let f be a $CMO_{\mathcal{P}}$ function. Then $\sum_{n\in\mathcal{N}} |f(n)|$ diverges. Indeed $\sum_{p\in\mathcal{P}} |f(p)|$ diverges.*

Proof. Let us assume the converse, so that

$$\sum_{n\in\mathcal{N}} |f(n)| \text{ converges.}$$

Then, by completely multiplicativity,

$$\sum_{n\in\mathcal{N}} |f(n)| = \prod_{p\in\mathcal{P}} \frac{1}{1-|f(p)|} \quad \text{and} \quad \sum_{n\in\mathcal{N}} f(n) = \prod_{p\in\mathcal{P}} \frac{1}{1-f(p)} \neq 0.$$

This contradiction implies

$$\sum_{n\in\mathcal{N}} |f(n)| \text{ diverges.}$$

Furthermore, Proposition 3.26 gives $\sum_{p\in\mathcal{P}} |f(p)|$ diverges, as required.

□

5.2.1 Partial sums of $CMO_{\mathcal{P}}$ functions

By definition the partial sum of a $CMO_{\mathcal{P}}$ function not exceeding x tends to zero when x tends to infinity. As we asked in Chapter 2, we can ask how small can we make $g(x)$, so that (5.3) is true for all $CMO_{\mathcal{P}}$ functions f?

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} f(n) = \Omega(g(x)) \tag{5.3}$$

To answer this we need some assumptions on g-prime systems.

Propostion 5.2. *Let $\mathcal{P}$ be a g-prime system with unique representation (all the multiplicities are 1) for which $\sum_{p\in\mathcal{P}} \frac{1}{p\log p}$ converges. If f is a $CMO_{\mathcal{P}}$ function, then*

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} f(n) = \Omega\Big(\frac{1}{x\log x}\Big).$$

Proof. Let us assume that the statement is false, so that

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} f(n) = O\Big(\frac{1}{x\log x}\Big).$$

We know that for every $p \in \mathcal{P}$, we have

$$f(p) = \sum_{m\le p} f(m) - \sum_{m<p} f(m) = O\Big(\frac{1}{p\log p}\Big). \tag{5.4}$$

Now it follows that $\sum_{p\in\mathcal{P}} |f(p)|$ converges and $|f(p)| < 1$ for all $p \in \mathcal{P}$ by the assumptions. Hence, by Proposition 3.26, we have $\sum_{n\in\mathcal{N}} |f(n)|$ converges. But by Proposition 5.1 we have a contradiction, and so it follows that

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} f(n) = \Omega\Big(\frac{1}{x\log x}\Big).$$

□

Remark 5.3.

(i) Similarly, we can get different results by having different assumptions on the g-prime systems with unique representation. For Example, if f is a $CMO_{\mathcal{P}}$ function and $\mathcal{P}$ is a g-prime system with unique representation for which $\sum_{p\in\mathcal{P}} \frac{1}{p(\log\log p)^2}$ converges, then

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} f(n) = \Omega\Big(\frac{1}{x(\log\log x)^2}\Big). \tag{5.5}$$

(ii) As mentioned in Chapter 2, Kahane and Saïas [31] have shown that if f is a CMO function, then

$$\sum_{n\le x} f(n) = \Omega(\frac{1}{x}).$$

Now if we assume $\sum_{p\in\mathcal{P}} \frac{1}{p}$ converges, then the sum on the left of (5.5) is $\Omega(\frac{1}{x})$.

(iii) Without unique representations, it can be more complicated. For instance, if $\mathcal{P}$ is as in Example 3.5, then (5.4) does not work.

5.2.2 Closeness relation between two completely multiplicative functions which are defined over $\mathcal{N}$

Let $\mathcal{CM}_{\mathcal{P}} := \{f : \mathcal{N} \longrightarrow \mathbb{C}$ completely multiplicative$\}$, and let us define an ***(extended) metric*** on $\mathcal{CM}_{\mathcal{P}}$ to be the distance function

$$D(f,g) := \sum_{p\in\mathcal{P}} |g(p) - f(p)|.$$

Then $\mathcal{CM}_{\mathcal{P}}$ is an ***extended metric space*** since $D(f,g)$ can attain the value ∞. It is straightforward to check for all $f, g, h \in \mathcal{CM}_{\mathcal{P}}$

(i) $D(f,g) = 0$ if and only if $f = g$,

(ii) $D(f,g) = D(g,f)$,

(iii) $D(f,h) \leq D(f,g) + D(g,h)$

hold. We aim to generalise Theorem 3 of Kahane and Saïas in [31] over Beurling prime systems. The following theorem shows that if f is a $CMO_{\mathcal{P}}$ function and g is a nearby completely multiplicative function on $\mathcal{N}$, then g is also a $CMO_{\mathcal{P}}$ function. In other words, under extra conditions on the values of g-primes for two completely multiplicative functions if one is a $CMO_{\mathcal{P}}$, then so is the other.

Theorem 5.4. *Let $\mathcal{P}$ be a g-prime system with abscissa 1, f a $CMO_{\mathcal{P}}$ function and g a completely multiplicative function on $\mathcal{P}$ such that*

$$|g(p)| < 1 \quad \textit{for all } p \in \mathcal{P} \tag{5.6}$$

and

$$D(f,g) < \infty. \tag{5.7}$$

Then g is a $CMO_{\mathcal{P}}$ function.

Proof. Let $F(s) := \sum_{n\in\mathcal{N}} \frac{f(n)}{n^s}$ and $G(s) := \sum_{n\in\mathcal{N}} \frac{g(n)}{n^s}$. Then the series for $F(s)$ is absolutely convergent for $\Re s > 1$ and it is convergent for $\Re s > 0$ and $s = 0$ since $\sum_{n\in\mathcal{N}} f(n) = 0$. Assumption (5.6) and that g is completely multiplicative function imply $|g(n)| \leq 1$. Thus the series for $G(s)$ also converges for $\Re s > 1$ since g is bounded and the abscissa of $\mathcal{P}$ is 1. Therefore $F(s)$ and $G(s)$ can be written as follows:

$$F(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{f(p)}{p^s}} \quad \text{and} \quad G(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{g(p)}{p^s}} \quad \Re s > 1.$$

Now

$$H(s) := \prod_{p \in \mathcal{P}} \left(\frac{1 - \frac{f(p)}{p^s}}{1 - \frac{g(p)}{p^s}} \right) = \prod_{p} \left(1 + \frac{\frac{g(p)-f(p)}{p^s}}{1 - \frac{g(p)}{p^s}} \right)$$

converges absolutely for $\Re s \geq 0$ if and only if

$$\sum_{p \in \mathcal{P}} \frac{\left| \frac{g(p)-f(p)}{p^s} \right|}{\left| 1 - \frac{g(p)}{p^s} \right|} \tag{5.8}$$

converges for $\Re s \geq 0$. But

$$\frac{\left| \frac{g(p)-f(p)}{p^s} \right|}{\left| 1 - \frac{g(p)}{p^s} \right|} \leq 2 \, \frac{|g(p) - f(p)|}{p^{\Re s}}$$

since $\frac{|g(p)-f(p)|}{p^{\Re s}} \geq \frac{1}{2}$ for p sufficiently large and $\Re s > 0$. Thus, since (5.7) and $|1 - \frac{g(p)}{p^s}| \sim 1$ as $p \longrightarrow \infty$ and $\Re s \geq 0$, (5.8) converges for $\Re s \geq 0$ and $H(s)$ converges absolutely to a holomorphic function for $\Re s > 0$. However, $H(s) = (G/F)(s)$ for $\Re s > 1$ then $G(s) = F(s)H(s)$, where the series for $F(s)$ converges for $\Re s > 0$ and $s = 0$ since f is a $CMO_{\mathcal{P}}$ function, and $H(s)$ converges absolutely for $\Re s \geq 0$. Therefore $G(s)$ converges for $\Re s > 0$ and $s = 0$ using the extension of Theorem 1.18. Thus we have $G(0) = F(0)H(0) = 0$. Hence g is a $CMO_{\mathcal{P}}$ function. $\square$

The proof of Theorem 5.4 implies the following result.

Corollary 5.5. *Let $\mathcal{P}$ be a g-prime system with abscissa 1, f and g both be completely multiplicative functions on $\mathcal{P}$ such that $D(f, g)$ is finite and satisfying*

$$|f(p)|, \, |g(p)| < 1 \quad \text{for all } p \in \mathcal{P}.$$

Then the following two assertions are equivalent:

$$\sum_{n \in \mathcal{N}} f(n) = 0 \quad \text{and} \quad \sum_{n \in \mathcal{N}} g(n) = 0.$$

5.3 The function $\frac{\lambda_{\mathcal{P}}(n)}{n}$ over different systems $\mathcal{P}$

In this section, we provide some examples of $CMO_{\mathcal{P}}$ functions. In particular, we introduce some examples of the function $\frac{\lambda_{\mathcal{P}}(n)}{n}$ with different g-prime systems where $\lambda_{\mathcal{P}}(n)$ is the Liouville function over the g-prime system $\mathcal{P}$. We emphasize that we are only interested in those systems for which the abscissa of convergence of the Dirichlet series for $\zeta_{\mathcal{P}}$ is 1.

Example 5.6. As shown in [10],[16], [17], if $\mathcal{P}$ satisfies one of the following assumptions:

$$\psi_{\mathcal{P}}(x) = \int_1^x \frac{d(\psi_{\mathcal{P}}(t))}{t} = \log x + c + o(1) \text{ for some constant } c, \tag{5.9}$$

or

$$M_{\mathcal{P}}(x) = o(x) \text{ and } N_{\mathcal{P}}(x) - \rho x = O\big(\tfrac{x}{\log^{\gamma} x}\big) \text{ holds for some } \rho > 0 \text{ and } \gamma > 1, \tag{5.10}$$

or

$$N_{\mathcal{P}}(x) \sim \rho x \text{ and } \log \zeta_{\mathcal{P}}(s) - \log\big(\tfrac{1}{s-1}\big) \text{ has a continuous extension to } \Re s = 1, \tag{5.11}$$

or

$$\int_2^{\infty} \big|\Pi_{\mathcal{P}}(x) - \frac{x}{\log x}\big| \frac{dx}{x^2} < \infty, \quad \text{where} \quad \Pi_{\mathcal{P}}(x) = \sum_{k=1}^{\infty} \frac{1}{k}\pi_{\mathcal{P}}(x^{\frac{1}{k}}), \tag{5.12}$$

then $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n} = 0$. Therefore, by Theorem 4.4, $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n} = 0$. Hence $\frac{\lambda_{\mathcal{P}}(n)}{n}$ is a $CMO_{\mathcal{P}}$ function since it is completely multiplicative with sum zero.

In fact, we do not even need $\pi_{\mathcal{P}}(x) \sim \frac{x}{\log x}$ for $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n} = 0$ to be true since it is shown in [10] that the following condition

$$\int_2^{\infty} \Big|\Pi_{\mathcal{P}}(x) - \frac{x}{\log x}\Big(1 + \sum_{j=0}^{k} b_j \cos(t_j \log x + y_j)\Big)\Big| \frac{dx}{x^2} < \infty \tag{5.13}$$

with distinct $t_j > 0$ and $(1+t_j^2)^{1/2}\,|b_j \cos(y_j + \arctan t_j)| < 2,\ j = 1, \cdots, k$, is also sufficient to have $\sum_{n\in\mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n} = 0$.

From Example 5.6, we see that $\sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n} = 0$, but how quickly does it converge?

5.3.1 O-Results for Euler's example over $\mathcal{N}$

In this section, we shall be interested in the behaviour of $\sum_{n\le x}\frac{\lambda_{\mathcal{P}}(n)}{n}$ over g-prime system $\mathcal{P}$ which satisfies

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta+\varepsilon}) \quad \text{(for some } \rho > 0) \tag{5.14}$$

and

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon}) \tag{5.15}$$

as $x \longrightarrow \infty$ for all $\varepsilon > 0$, but for no $\varepsilon < 0$ and $0 \le \alpha, \beta < 1$ (see section 3.7).

Theorem 5.7. *Given a g-prime system $\mathcal{P}$ satisfying (5.14) and (5.15) for some $\beta, \alpha < 1$, and let $\lambda_{\mathcal{P}}$ as defined before, we have*

$$\sum_{\substack{n\,\le\, x\\ n\,\in\,\mathcal{N}}} \frac{\lambda_{\mathcal{P}}(n)}{n} = O\Big(\frac{1}{x^{1-\Theta-\varepsilon}}\Big) \quad \textit{for all } \varepsilon > 0,$$

where $\Theta = \max\{\alpha, \beta\}$.

Proof. We let $\mathcal{P}$ denote a g-prime system satisfying (5.14) and (5.15). Then, for all $\Re s > 1$, we have

$$Z_{\mathcal{P}}(s) = \sum_{n\in\mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n^s} = \frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}.$$

The idea is to find a bound for $\sum_{n\le x}\lambda_{\mathcal{P}}(n)$. This bound will be used with Abel Summation to show that

$$l_{\mathcal{P}}(x) = O\Big(\frac{1}{x^{1-\Theta-\varepsilon}}\Big) \quad \text{for all } \varepsilon > 0.$$

In order to do that, we use Perron's formula. For this we need a bound for $Z_{\mathcal{P}}(s)$ on vertical line $s = \sigma + it$ with $|t|$ large and $\sigma > \Theta$. To find such bound of $Z_{\mathcal{P}}(s)$, we start with

$$|\zeta_{\mathcal{P}}(2s)| = |\zeta_{\mathcal{P}}(2\sigma + 2it)| = \Bigg|\sum_{n\in\mathcal{N}} \frac{1}{n^{2\sigma+2it}}\Bigg| \le \sum_{n\in\mathcal{N}} \frac{1}{n^{2\sigma}} = \zeta_{\mathcal{P}}(2\sigma) = O(1)$$

for $2\sigma > 1$; (*i.e.* $\sigma > \frac{1}{2}$).

From the proof of Theorem 2.3 of [27], for $\sigma > \Theta$, we have

$$\log|\zeta_{\mathcal{P}}(\sigma+it)| = O((\log|t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}),$$

which implies

$$\log\frac{1}{|\zeta_{\mathcal{P}}(\sigma+it)|} = O((\log|t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}).$$

In particular, for $\sigma > \Theta$,

$$\frac{1}{|\zeta_{\mathcal{P}}(\sigma+it)|} = O(|t|^{\varepsilon}) \quad \text{for all } \varepsilon > 0.$$

Hence, for $\sigma > \Theta$,

$$|Z_{\mathcal{P}}(s)| = \left|\frac{\zeta_{\mathcal{P}}(2\sigma+2it)}{\zeta_{\mathcal{P}}(\sigma+it)}\right| = O(|t|^{\varepsilon}) \text{ for all } \varepsilon > 0 \text{ and for } |t| \geq 1. \tag{5.16}$$

Using the inverse Mellin transform (see Theorem 1.24) we have for $x > 0$, $x \notin \mathcal{N}$

$$L_{\mathcal{P}}(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{Z_{\mathcal{P}}(s)}{s} x^s ds \quad (c > 1).$$

Now split the range into $(c-i\infty, c-iT]$, $[c-iT, c+iT]$ and $[c+iT, c+i\infty)$, where $T > 0$ is a suitable function of x which will be chosen later, we obtain

$$L_{\mathcal{P}}(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{Z_{\mathcal{P}}(s)}{s} x^s ds + \frac{1}{2\pi i}\left(\int_{c+iT}^{c+i\infty} + \int_{c-i\infty}^{c-iT}\right)\frac{Z_{\mathcal{P}}(s)}{s} x^s ds. \tag{5.17}$$

Denote the left integral by I_1. We note that assumption (5.14) implies $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \Re s > \beta\}$ except for a simple pole at $s = 1$. This implies $(U_{\mathcal{P}}(s) =)\frac{1}{\zeta_{\mathcal{P}}(s)}$ is holomorphic for $\Re s > \beta$. However, assumption (5.15) implies that $\zeta_{\mathcal{P}}(s)$ is holomorphic for $\Re s > \alpha$ except for a simple pole at $s = 1$ and $\zeta_{\mathcal{P}}(s) \neq 0$ in this region by Theorem 3.28. Thus (5.14) and (5.15) together show $\zeta_{\mathcal{P}}(s)$ is holomorphic and has no zeros for $\Re s > \Theta$ except for a simple pole at $s = 1$. Hence $U_{\mathcal{P}}(s)$ has a simple zero at $s = 1$ and an analytic continuation to $\{s \in \mathbb{C} : \Re s > \Theta\}$.

Thus $(Z_{\mathcal{P}}(s) =)\frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}$ has an analytic continuation to $\{s \in \mathbb{C} : \Re s > \Theta\}$ with a simple zero at $s = 1$. Hence $Z_{\mathcal{P}}(s)$ is holomorphic for $\sigma > \Theta$ since $\zeta_{\mathcal{P}}(2s)$ is holomorphic for $\sigma > \Theta$. Now move the contour past the line $s = 1$ to the line $\Re s = \sigma$ for any $\sigma > \Theta$ since $Z_{\mathcal{P}}(s)$ is holomorphic in this region, as in the figure below.

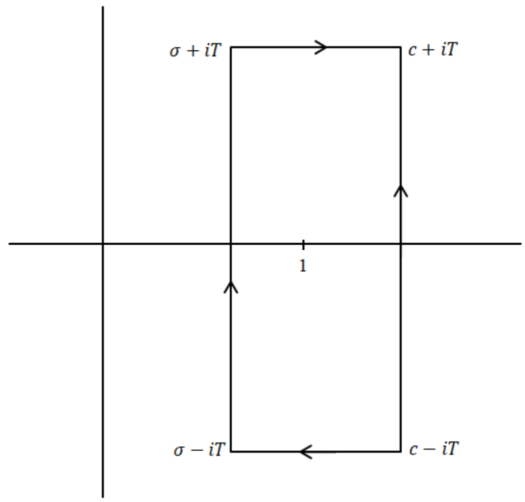

Figure 5.1: rectangular contour

Hence

$$I_1 = \frac{1}{2\pi i}\left(\int_{c-iT}^{\sigma-iT} + \int_{\sigma-iT}^{\sigma+iT} + \int_{\sigma+iT}^{c+iT}\right)\frac{Z_{\mathcal{P}}(s)}{s}x^s ds.$$

These integrals will be estimated by using the bound $|Z_{\mathcal{P}}(s)| = O(t^{\varepsilon})$ for all $\varepsilon > 0$. The integral over the horizontal path $[\sigma + iT, c + iT]$ is

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\sigma+iT}^{c+iT}\frac{Z_{\mathcal{P}}(s)}{s}x^s ds\right| &= \left|\frac{1}{2\pi i}\int_{\sigma}^{c}\frac{Z_{\mathcal{P}}(y+iT)}{y+iT}x^{y+iT}dy\right| \\ &\leq \frac{x^c}{2\pi T}\int_{\sigma}^{c}|Z_{\mathcal{P}}(y+iT)|dy \\ &= O\Big(\frac{x^c}{T}T^{\varepsilon}\Big) = O\Big(\frac{x^c}{T^{1-\varepsilon}}\Big) \text{ for all } \varepsilon > 0,\end{aligned}$$

by (5.16). Similarly for the integral over $[c - iT, \sigma - iT]$. On the line $\Re s = \sigma$, we will have

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{Z_{\mathcal{P}}(s)}{s}x^s ds\right| =& \left|\frac{1}{2\pi i}\int_{-T}^{T}\frac{Z_{\mathcal{P}}(\sigma+it)}{\sigma+it}x^{\sigma+it}dt\right| \\ &\leq \frac{x^\sigma}{2\pi}\int_{-T}^{T}\frac{|Z_{\mathcal{P}}(\sigma+it)|}{|\sigma+it|}dt = \frac{x^\sigma}{\pi}\int_{0}^{T}\frac{|Z_{\mathcal{P}}(\sigma+it)|}{|\sigma+it|}dt \\ &= \frac{x^\sigma}{\pi}\int_{0}^{1}\frac{|Z_{\mathcal{P}}(\sigma+it)|}{|\sigma+it|}dt + \frac{x^\sigma}{\pi}\int_{1}^{T}\frac{|Z_{\mathcal{P}}(\sigma+it)|}{t}dt \\ &= O(x^\sigma) + O(x^\sigma T^\varepsilon) = O(x^\sigma T^\varepsilon) \text{ for all } \varepsilon > 0,\end{aligned}$$

by (5.16). Hence

$$I_1 = O\Big(\frac{x^c}{T^{1-\varepsilon}}\Big) + O(x^\sigma T^\varepsilon).$$

Now the right integral of (5.17) is

$$I_2 = \frac{1}{2\pi i}\left(\int_{c+iT}^{c+i\infty} + \int_{c-i\infty}^{c-iT}\right)\sum_{n\in\mathcal{N}}\frac{\lambda_{\mathcal{P}}(n)}{s}\left(\frac{x}{n}\right)^s ds.$$

This integral will be estimated as follows:

$$|I_2| \leq \sum_{n\in\mathcal{N}}|\lambda_{\mathcal{P}}(n)|\cdot\left|\frac{1}{2\pi i}\left(\int_{c+iT}^{c+i\infty} + \int_{c-i\infty}^{c-iT}\right)\left(\frac{x}{n}\right)^s\frac{ds}{s}\right|.$$

Using Lemma 1.23 and $|\lambda_{\mathcal{P}}(n)| = 1$, we get

$$|I_2| = O\left(\sum_{n\in\mathcal{N}}\frac{(\frac{x}{n})^c}{T\,|\log\frac{x}{n}\,|}\right) = O\left(\frac{x^c}{T}\sum_{n\in\mathcal{N}}\frac{1}{n^c\,|\log\frac{x}{n}\,|}\right).$$

The range is split into ($n \geq 2x$ and $n \leq \frac{x}{2}$) and ($\frac{x}{2} < n < 2x$) in order to use the bound $|\log\frac{x}{n}| \geq \log 2$ for the first range. This gives

$$I_2 = O\left(\frac{x^c}{T}\sum_{\substack{n\,\geq\,2x\,\&\,n\,\leq\,\frac{x}{2}\\ n\,\in\,\mathcal{N}}}\frac{1}{n^c|\log\frac{x}{n}|}\right) + O\left(\frac{x^c}{T}\sum_{\substack{\frac{x}{2}\,<\,n\,<\,2x\\ n\,\in\,\mathcal{N}}}\frac{1}{n^c|\log\frac{x}{n}|}\right).$$

Using $\left|\log\frac{x}{n}\right| = \left|\log\left(1+\frac{n-x}{x}\right)\right| \asymp \frac{|n-x|}{x}$ for the second range, we obtain

$$I_2 = O\left(\frac{x^c}{T} \sum_{\substack{n \geq 2x \,\&\, n \leq \frac{x}{2} \\ n \in \mathcal{N}}} \frac{1}{n^c}\right) + O\left(\frac{x}{T} \sum_{\substack{\frac{x}{2} < n < 2x \\ n \in \mathcal{N}}} \frac{1}{|n-x|}\right) = O\left(\frac{x^c}{T} \cdot \zeta_{\mathcal{P}}(c)\right) + O\left(\frac{x}{T} \sum_{\substack{\frac{x}{2} < n < 2x \\ n \in \mathcal{N}}} \frac{1}{|n-x|}\right).$$

Therefore

$$L_{\mathcal{P}}(x) = I_1 + I_2 = O\left(\frac{x^c}{T^{1-\varepsilon}}\right) + O(x^\sigma T^\varepsilon) + O\left(\frac{x^c}{T} \cdot \zeta_{\mathcal{P}}(c)\right) + O\left(\frac{x}{T} \sum_{\substack{\frac{x}{2} < n < 2x \\ n \in \mathcal{N}}} \frac{1}{|n-x|}\right).$$

Taking $c = 1 + \frac{1}{\log x}$ and using $\zeta_{\mathcal{P}}(1+\delta) = O(\frac{1}{\delta})$, gives

$$L_{\mathcal{P}}(x) = O\left(\frac{x}{T^{1-\varepsilon}}\right) + O(x^\sigma T^\varepsilon) + O\left(\frac{x \log x}{T}\right) + O\left(\frac{x}{T} \sum_{\substack{\frac{x}{2} < n < 2x \\ n \in \mathcal{N}}} \frac{1}{|n-x|}\right)$$

for $x \notin \mathcal{N}$ and for all $\varepsilon > 0$. We need to bound the term on the right hand side, which is difficult for general x when n is an integer close to x, as then $|n-x|^{-1}$ could be very large. To take into account this eventuality we choose x here such that $|n-x| < \frac{1}{x^2}$. This ensures that it stays away from these integer n; *i.e.*

$$\left(x - \frac{d}{x}, x + \frac{d}{x}\right) \cap \mathcal{N} = \phi.$$

Then, for such x,

$$\sum_{\substack{\frac{x}{2} < n < 2x \\ n \in \mathcal{N}}} \frac{1}{|n-x|} \leq \frac{x}{d} \cdot \sum_{\substack{\frac{x}{2} < n < 2x \\ n \in \mathcal{N}}} 1 < \frac{x}{d} \cdot N(2x) = O(x^2).$$

Hence, for such x,

$$L_{\mathcal{P}}(x) = O\left(\frac{x}{T^{1-\varepsilon}}\right) + O(x^\sigma T^\varepsilon) + O\left(\frac{x \log x}{T}\right) + O\left(\frac{x^3}{T}\right).$$

Taking $T = x^3$, then

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) = O(x^{\sigma + 3\varepsilon}) \quad \text{for all } \varepsilon > 0.$$

Taking $\sigma = \Theta + \varepsilon$ for any $\varepsilon > 0$, then

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) = O(x^{\Theta+4\varepsilon})$$

whenever x is such that $\left(x - \frac{d}{x}, x + \frac{d}{x}\right) \cap \mathcal{N} = \phi$ for some $d > 0$. Now we follow the method used in the proof of Theorem 2.2, originally given in [27]. We will show for every x sufficiently large, there exist $x_1 \in (x-2, x)$ and $x_2 \in (x, x+2)$ such that

$$\left(x_1 - \frac{d}{x_1}, x_1 + \frac{d}{x_1}\right) \cap \mathcal{N} = \phi \text{ and } \left(x_2 - \frac{d}{x_2}, x_2 + \frac{d}{x_2}\right) \cap \mathcal{N} = \phi \text{ for some } d > 0. \quad (5.18)$$

Then the result will follow since

$$\sum_{\substack{n \le x_1 \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) = O(x_1^{\Theta+4\varepsilon}) = O(x^{\Theta+4\varepsilon}) \text{ for all } \varepsilon > 0.$$

Hence

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) = \sum_{\substack{n \le x_1 \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) + \sum_{\substack{x_1 < n \le x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) = O(x^{\Theta+4\varepsilon}) + O(x^{\beta+\varepsilon}) = O(x^{\Theta+4\varepsilon}) \text{ for all } \varepsilon > 0$$

since

$$\Bigg|\sum_{\substack{x_1 < n \le x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n)\Bigg| \le \sum_{\substack{x_1 < n \le x \\ n \in \mathcal{N}}} 1 = N_{\mathcal{P}}(x) - N_{\mathcal{P}}(x_1) \le N_{\mathcal{P}}(x) - N_{\mathcal{P}}(x-2) = O(x^{\beta+\varepsilon}) \text{ for all } \varepsilon > 0,$$

by (5.14). It remains to prove (5.18).

Assume x is sufficiently large, so that $N_{\mathcal{P}}(x) < L$, where $L = [b \cdot x]$ such that $b > \rho > 0$ since $N_{\mathcal{P}}(x) \sim \rho x$. Divide $(x, x+2)$ into L intervals of equal length. Then one of them contains no elements of $\mathcal{N}$. Let its midpoint be x_1. Then $\left(x_1 - \frac{1}{2L}, x_1 + \frac{1}{2L}\right) \cap \mathcal{N} = \phi$. Thus the equation (5.18) holds with such x_1 when $\frac{d}{x_1} \le \frac{1}{2L}$; (*i.e.* $L \le \frac{1}{2d} x_1$).

Similarly $(x, x+2)$ contains suitable x_2. Thus

$$L_{\mathcal{P}}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) = O(x^{\Theta+4\varepsilon}) \text{ for all } \varepsilon > 0.$$

Now using Abel Summation, we find $\sum_{n\le x}\frac{\lambda_{\mathcal{P}}(n)}{n^s}$ as follows: we note that the integral $\int_1^\infty \frac{L_{\mathcal{P}}(x)}{x^{s+1}}\,dx$ converge for $\sigma > \Theta$. Thus

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in\mathcal{N}}}\frac{\lambda_{\mathcal{P}}(n)}{n^s} &= \frac{L_{\mathcal{P}}(x)}{x^s} + s\int_1^x \frac{L_{\mathcal{P}}(t)}{t^{s+1}}dt \\ &= \frac{O(x^{\Theta+4\varepsilon})}{x^s} + s\int_1^\infty \frac{L_{\mathcal{P}}(t)}{t^{s+1}}dt - s\int_x^\infty \frac{L_{\mathcal{P}}(t)}{t^{s+1}}dt \\ &= O\Big(\frac{1}{x^{\sigma-\Theta-4\varepsilon}}\Big) + s\int_1^\infty \frac{L_{\mathcal{P}}(t)}{t^{s+1}}dt - s\int_x^\infty \frac{O(t^{\Theta+4\varepsilon})}{t^{s+1}}dt \\ &= s\int_1^\infty \frac{L_{\mathcal{P}}(t)}{t^{s+1}}dt + O\Big(\frac{1}{x^{\sigma-\Theta-4\varepsilon}}\Big)\end{aligned}$$

since

$$\Big|\int_x^\infty \frac{O(t^{\Theta+4\varepsilon})}{t^{s+1}}dt\Big| = O\bigg(\int_x^\infty \frac{1}{t^{\sigma+1-\Theta-4\varepsilon}}dt\bigg) = O\Big(\frac{1}{x^{\sigma-\Theta-4\varepsilon}}\Big).$$

This shows $\sum_{n\in\mathcal{N}}\frac{\lambda_{\mathcal{P}}(n)}{n^s}$ converges for $\sigma > \Theta$ and it is holomorphic. Since it equals $\frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}$ for $\Re s > 1$, by analytic continuation it is true for $\Re s > \Theta$. Thus

$$Z_{\mathcal{P}}(s) = s\int_1^\infty \frac{L_{\mathcal{P}}(t)}{t^{s+1}}dt \quad \text{for } \sigma > \Theta.$$

In particular for $s = 1$, this means $\sum_{n\in\mathcal{N}}\frac{\lambda_{\mathcal{P}}(n)}{n^1} = Z_{\mathcal{P}}(1) = 0$. Hence

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}}\frac{\lambda_{\mathcal{P}}(n)}{n} = O\Big(\frac{1}{x^{1-\Theta-4\varepsilon}}\Big) \text{ for all } \varepsilon > 0.$$

□

Remark 5.8.

(i) It was shown in Theorem 3.31 that $\Theta \ge \frac{1}{2}$. Thus, $1-\Theta \le \frac{1}{2}$ and in Theorem 5.7, we can therefore only have an example with exponent $\le \frac{1}{2}$ for such systems.

(ii) If $\Theta = \frac{1}{2}$ in Theorem 5.7, then we have $l_{\mathcal{P}}(x) = O\big(\frac{e^{\varepsilon\log x}}{x^{\frac{1}{2}}}\big)$ for all $\varepsilon > 0$. We can do slightly better than this bound if we take $\mathcal{P} = \mathcal{P}_Z$ (Zhang's system) which is a special case of a well-behaved system. In this case

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\frac{1}{2}}e^{c(\log x)^{\frac{2}{3}}}) \quad \text{for some } \rho, c > 0 \tag{5.19}$$

and

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(x^{\frac{1}{2}}) \tag{5.20}$$

hold. The existence of $\mathcal{P}_Z$ was shown by Zhang [58]. The proof of the following theorem is roughly identical with the previous theorem except that we need a strictly stronger bound on $Z_{\mathcal{P}}(s)$ than (5.16).

Theorem 5.9. *For Zhang's system $\mathcal{P}_Z$, we have*

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{\lambda_{\mathcal{P}}(n)}{n} = O\Big(\frac{e^{C(\log x)^{\frac{2}{3}}}}{x^{\frac{1}{2}}}\Big) \quad \text{for some constant } C.$$

Remark 5.10. In order to prove this result, we let $\mathcal{P}$ denote a g-prime system satisfying (5.19) and (5.20). Then, for all $\Re s > 1$, we have

$$Z_{\mathcal{P}}(s) = \sum_{n \in \mathcal{N}} \frac{\lambda_{\mathcal{P}}(n)}{n^s} = \frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}.$$

As in the proof of Theorem 5.7 the idea is to find a bound for $\sum_{n \le x} \lambda_{\mathcal{P}}(n)$. This bound will be used with Abel Summation to show that

$$l_{\mathcal{P}}(x) = O\left(\frac{e^{C(\log x)^{\frac{2}{3}}}}{x^{\frac{1}{2}}}\right).$$

As mentioned above, we need a stronger bound for $Z_{\mathcal{P}}(s)$. This can be found by using Theorem 3.34 of Zhang [16], [58].

Lemma 5.11. *Let $F(x,t)$ be a function defined for $1 \le x < \infty$ and $t \ge 0$ that is locally of bounded variation in x and satisfies $F(1,t) = 0$ as well as*

$$F(x,t) \ll \sqrt{x}\left(1 + \sqrt{\frac{\log(t+1)}{\log x}}\right).$$

Then

$$\int_{v=1}^{\infty} v^{-\sigma} dF(v,t) \ll \frac{\sigma}{\sigma - \frac{1}{2}} + \sigma\sqrt{\frac{\log(t+1)}{\sigma - \frac{1}{2}}}$$

for all $\sigma > \frac{1}{2}$ and $t \ge 0$.

Proof. See Lemma 17.13 of [16].

□

Proof of Theorem 5.9.

We know from the proof of Theorem 3.34 of Zhang that there is a sequence of real numbers which represents a set of g-primes, ($i.e$ $\mathcal{P} = \{p_j\}$), $j = 1, 2, \cdots$, such that

$$\sum_{p_j \le x} p_j^{-it} - \int_1^x v^{-it} f(v)dv \ll \sqrt{x}\left(1 + \sqrt{\frac{\log(t+1)}{1+\log x}}\right) \tag{5.21}$$

for $1 \le x < \infty$ and $t \ge 0$, where

$$f(v) := \frac{1 - v^{-1}}{\log v} \text{ for } v \ge 1.$$

In particular, when $t = 0$,

$$\sum_{p_j \le x} 1 - \int_1^x f(v)dv = O(x^{\frac{1}{2}}), \tag{5.22}$$

and hence

$$\pi_{\mathcal{P}}(x) := \sum_{p_j \le x} 1 = \mathrm{li}(x) + O(x^{\frac{1}{2}}). \tag{5.23}$$

Further Zhang showed that $\zeta_{\mathcal{P}}(s)$ can be written as

$$\zeta_{\mathcal{P}}(s) = \frac{s}{s-1} \exp\{F_2(s) - F_1(s)\},$$

where F_1 and F_2 are Riemann-Stieltjes integrals (see Section 1.1.3) as follows:

$$F_1(s) = \int_1^\infty (v^{-s} + \log(1 - v^{-s}))d\pi_{\mathcal{P}}(v)$$

and

$$F_2(s) = \int_1^\infty v^{-s}(d\pi_{\mathcal{P}}(v) - f(v)dv).$$

We know that $\log(1 - v^{-s}) = -v^{-s} + O(v^{-2\sigma})$ for $v > 1$. Therefore

$$F_1(s) = \int_1^\infty (v^{-s} + \log(1 - v^{-s}))d\pi_{\mathcal{P}}(v)$$
$$= \int_1^\infty O(v^{-2\sigma})d\pi_{\mathcal{P}}(v) = O\biggl(\int_1^\infty \frac{d\pi_{\mathcal{P}}(v)}{v^{2\sigma}}\biggr).$$

The right integral converges uniformly for $\sigma > \frac{1}{2}$. Thus $F_1(s)$ is holomorphic for $\sigma > \frac{1}{2}$. We know from (5.22) and (5.23) that

$$\pi_{\mathcal{P}}(v) - \int_1^v f(w)dw = O(v^{\frac{1}{2}}).$$

Therefore

$$F_2(s) = \int_1^\infty v^{-s}(d\pi_{\mathcal{P}}(v) - f(v)dv) = \left[\frac{\pi_{\mathcal{P}}(v) - \int_1^v f(w)dw}{v^{s+1}}\right]_1^\infty + s\int_1^\infty \frac{\pi_{\mathcal{P}}(v) - \int_1^v f(w)dw}{v^{s+1}}\,dv$$
$$= s\int_1^\infty \frac{O(v^{\frac{1}{2}})}{v^{s+1}}\,dv = O\biggl(\int_1^\infty \frac{v^{\frac{1}{2}}}{v^{\sigma+1}}\,dv\biggr) \quad \left(\text{since } \left[\frac{O(v^{\frac{1}{2}})}{v^{s+1}}\right]_1^\infty = 0\right).$$

This integral converges for $\sigma > \frac{1}{2}$. Thus $F_2(s)$ is also holomorphic for $\sigma > \frac{1}{2}$. Hence $\zeta_{\mathcal{P}}(s)$ has an analytic continuation in the half plane $\sigma > \frac{1}{2}$ except for a simple pole at $s = 1$ with residue $k = \exp\{F_2(1) - F_1(1)\} > 0$. From (5.21), we have

$$\sum_{p_j \le x} p_j^{-it} - \int_1^x v^{-it} f(v)dv = \int_1^x v^{-it} d\pi_{\mathcal{P}}(v) - \int_1^x v^{-it} f(v)dv$$
$$= \int_1^x v^{-it}(d\pi_{\mathcal{P}}(v) - f(v)dv) \ll \sqrt{x}\Biggl(1 + \sqrt{\frac{\log(t+1)}{1+\log x}}\Biggr).$$

Now let $F(v,t) = \int_1^x v^{-it}(d\pi_{\mathcal{P}}(v) - f(v)dv)$ and applying Lemma 5.11, we have for $t \in \mathbb{R}$

$$F_2(\sigma + it) = \int_1^\infty v^{-s}(d\pi_{\mathcal{P}}(v) - f(v)dv) = \int_1^\infty v^{-\sigma}(v^{-it}(d\pi_{\mathcal{P}}(v) - f(v)dv))$$
$$= \int_1^\infty v^{-\sigma} dF(v,t) \quad (\text{since } dF(v,t) = v^{-it}(d\pi_{\mathcal{P}}(v) - f(v)dv)$$

$$\ll \frac{\sigma}{\sigma - \frac{1}{2}} + \sigma\sqrt{\frac{\log|t|}{\sigma - \frac{1}{2}}}.$$

This also means that

$$F_2(\sigma+it) \ll \frac{1}{\sigma-\frac{1}{2}} + \sqrt{\frac{\log|t|}{\sigma-\frac{1}{2}}}$$

uniformly for $\frac{1}{2} < \sigma \leq 2$. Furthermore, we know that

$$\log(1-v^{-s}) = -v^{-s} - \frac{1}{2}v^{-2s} + O(v^{-3\sigma}) \quad \text{for } v > 1.$$

Hence we have

$$\begin{aligned}
F_1(s) &= \int_1^\infty (v^{-s} + \log(1-v^{-s}))d\pi_{\mathcal{P}}(v) = \int_1^\infty (-\tfrac{1}{2}v^{-2s} + O(v^{-3\sigma}))d\pi_{\mathcal{P}}(v) \\
&= -\tfrac{1}{2}\int_1^\infty v^{-2s}d\pi_{\mathcal{P}}(v) + O\bigg(\int_1^\infty v^{-3\sigma}d\pi_{\mathcal{P}}(v)\bigg) \\
&= -\tfrac{1}{2}\int_1^\infty v^{-2s}d\pi_{\mathcal{P}}(v) + O\bigg(\int_1^\infty \frac{\pi_{\mathcal{P}}(v)}{v^{3\sigma+1}}dv\bigg) \quad \bigg(\text{since } \Big[\frac{\pi_{\mathcal{P}}(v)}{v^{3s}}\Big]_1^\infty = 0\bigg) \\
&= -\tfrac{1}{2}\int_1^\infty v^{-2s}d\pi_{\mathcal{P}}(v) + O(1) \quad \bigg(\text{since } \int_1^\infty \frac{\pi_{\mathcal{P}}(v)}{v^{3\sigma+1}}dv < \infty\bigg).
\end{aligned}$$

Therefore

$$F_1(\sigma+it) = O(1) - \tfrac{1}{2}\int_1^\infty v^{-2(\sigma+it)}d\pi_{\mathcal{P}}(v) \ll 1 + \int_1^\infty v^{-2\sigma}d\pi_{\mathcal{P}}(v).$$

Applying the method of integration by parts to the right hand side we obtain

$$\begin{aligned}
2\sigma\int_1^\infty v^{-2\sigma-1}\pi_{\mathcal{P}}(v)dv &\ll \int_1^\infty v^{-2\sigma}dv \quad (\text{since } [v^{-2\sigma}\pi_{\mathcal{P}}(v)]_1^\infty = 0) \\
&= \frac{1}{2\sigma-1} \ll \frac{\sigma}{\sigma-\frac{1}{2}}.
\end{aligned}$$

This also means that

$$F_1(\sigma+it) \ll \frac{1}{\sigma-\frac{1}{2}}$$

uniformly for $\frac{1}{2} < \sigma \leq 2$. Hence $\zeta_{\mathcal{P}}$ is holomorphic for $\frac{1}{2} < \sigma \leq 2$ except for a pole at $s = 1$, and for these values satisfies uniformly

$$F_2(s) - F_1(s) \ll \left(\frac{1}{\sigma-\frac{1}{2}} + \sqrt{\frac{\log|t|}{\sigma-\frac{1}{2}}}\right).$$

Then, for $\frac{1}{2} < \sigma \leq 2$ and $|t| \geq 1$, we have

$$\log|\zeta_{\mathcal{P}}(\sigma+it)| \ll \frac{1}{\sigma-\frac{1}{2}} + \sqrt{\frac{\log|t|}{\sigma-\frac{1}{2}}}.$$

Also, for $\frac{1}{2} < \sigma \leq 2$, we have

$$\log\frac{1}{|\zeta_{\mathcal{P}}(\sigma+it)|} \ll \frac{1}{\sigma-\frac{1}{2}} + \sqrt{\frac{\log|t|}{\sigma-\frac{1}{2}}}.$$

Thus, for $\frac{1}{2} < \sigma \leq 2$ and $|t| \geq 1$, we have

$$\frac{1}{|\zeta_{\mathcal{P}}(\sigma+it)|} \ll \exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right) + C\sqrt{\frac{\log|t|}{\sigma-\frac{1}{2}}}\right\}$$

and

$$|\zeta_{\mathcal{P}}(2s)| = |\zeta_{\mathcal{P}}(2\sigma+2it)| = \left|\sum_{n\in\mathcal{N}}\frac{1}{n^{2\sigma+2it}}\right| \leq \sum_{n\in\mathcal{N}}\frac{1}{n^{2\sigma}} = \zeta_{\mathcal{P}}(2\sigma) = O(1)$$

for $2\sigma > 1$; (*i.e.* for $\sigma > \frac{1}{2}$). Hence

$$|Z_{\mathcal{P}}(s)| = \left|\frac{\zeta_{\mathcal{P}}(2\sigma+2it)}{\zeta_{\mathcal{P}}(\sigma+it)}\right| \ll \exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right) + C\sqrt{\frac{\log|t|}{\sigma-\frac{1}{2}}}\right\}. \qquad (5.24)$$

We know from Theorem 1 of Zhang in [58] that the function $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$ except for a simple pole at $s = 1$. Also, $\zeta_{\mathcal{P}}(s)$ has no zeros for $\sigma > \frac{1}{2}$ but $\frac{1}{\zeta_{\mathcal{P}}(s)}$ has a simple zero at $s = 1$ and analytic continuation to $\{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$. Thus $Z_{\mathcal{P}}(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$ with a simple zero at $s = 1$. Thus $Z_{\mathcal{P}}(s)$ is holomorphic for $\sigma > \frac{1}{2}$ since $\zeta_{\mathcal{P}}(2s)$ is holomorphic for $\sigma > \frac{1}{2}$. Now move the contour past the line $s = 1$ to the line $\Re s = \sigma$ for any $\sigma > \frac{1}{2}$ since $Z_{\mathcal{P}}(s)$ is holomorphic in this region (see Figure 5.1). We obtain

$$I_1 = \frac{1}{2\pi i}\left(\int_{c-iT}^{\sigma-iT} + \int_{\sigma-iT}^{\sigma+iT} + \int_{\sigma+iT}^{c+iT}\right)\frac{Z_{\mathcal{P}}(s)}{s}x^s ds.$$

These integrals will be estimated by using the bound (5.24). The integral over the horizontal path $[\sigma+iT, c+iT]$ is

$$\left|\frac{1}{2\pi i}\int_{\sigma+iT}^{c+iT}\frac{Z_{\mathcal{P}}(s)}{s}x^s ds\right| \le \frac{x^c}{2\pi T}\int_{\sigma}^{c}|Z_{\mathcal{P}}(y+iT)|dy$$

$$\ll \frac{x^c}{T}\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\} \quad \text{(using (5.24))}.$$

Similarly for the integral over $[c-iT, \sigma-iT]$. On the line $\Re s=\sigma$, we will have

$$\left|\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{Z_{\mathcal{P}}(s)}{s}x^s ds\right| = \frac{x^\sigma}{\pi}\int_0^1\frac{|Z_{\mathcal{P}}(\sigma+it)|}{|\sigma+it|}dt+\frac{x^\sigma}{\pi}\int_1^T\frac{|Z_{\mathcal{P}}(\sigma+it)|}{t}dt$$

$$\ll x^\sigma + x^\sigma\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\} \quad \text{(using (5.24))}$$

$$\ll x^\sigma\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\}.$$

Hence

$$I_1 \ll \frac{x^c}{T}\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\}+x^\sigma\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\},$$

and I_2 is exactly the same as in the proof of Theorem 5.7. Therefore

$$L_{\mathcal{P}}(x) = I_1+I_2 \ll \frac{x^c}{T}\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\}+x^\sigma\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\}$$
$$+O\left(\frac{x^c}{T}\cdot\zeta_{\mathcal{P}}(c)\right)+O\left(\frac{x}{T}\sum_{\substack{\frac{x}{2}<n<2x\\ n\in\mathcal{N}}}\frac{1}{|n-x|}\right).$$

Taking $c=1+\frac{1}{\log x}$ and using $\zeta_{\mathcal{P}}(1+\delta)=O(\frac{1}{\delta})$ gives

$$L_{\mathcal{P}}(x) \ll \frac{x}{T}\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\}+x^\sigma\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\}$$
$$+O\left(\frac{x\log x}{T}\right)+O\left(\frac{x}{T}\sum_{\substack{\frac{x}{2}<n<2x\\ n\in\mathcal{N}}}\frac{1}{|n-x|}\right).$$

By following the same argument shown in the proof of Theorem 5.7, we have

$$L_{\mathcal{P}}(x) \ll \frac{x}{T}\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\}+x^{\sigma}\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{\log|T|}{\sigma-\frac{1}{2}}}\right\}$$
$$+O\left(\frac{x\log x}{T}\right)+O\left(\frac{x^3}{T}\right).$$

Taking $T = x^3$, then

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}}\lambda_{\mathcal{P}}(n)\ll x^{\sigma}\exp\left\{C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{3\log x}{\sigma-\frac{1}{2}}}\right\}$$

since the O(1) are smaller than the main term. This means

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}}\lambda_{\mathcal{P}}(n)\ll \exp\left\{\sigma\log x+C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{3\log x}{\sigma-\frac{1}{2}}}\right\}$$
$$\ll x^{\frac{1}{2}}\exp\left\{\left(\sigma-\frac{1}{2}\right)\log x+C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\sqrt{\frac{3\log x}{\sigma-\frac{1}{2}}}\right\}.$$

Now we want to minimise this quantity over $\frac{1}{2}<\sigma<2$; *i.e.* minimise

$$\left(\sigma-\frac{1}{2}\right)\log x+C\left(\frac{1}{\sigma-\frac{1}{2}}\right)+C\big(\sigma-\tfrac{1}{2}\big)^{-\frac{1}{2}}(3\log x)^{\frac{1}{2}}.$$

Put $\sigma=\frac{1}{2}+\frac{1}{(\log x)^{\alpha}}$ for some $\alpha>0$. This gives

$$(\log x)^{1-\alpha}+C(\log x)^{\alpha}+C(\log x)^{\frac{1}{2}+\frac{\alpha}{2}}.$$

This is optimal when $\alpha=\frac{1}{3}$. For $\sigma=\frac{1}{2}+\frac{1}{(\log x)^{\frac{1}{3}}}$, we have

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}}\lambda_{\mathcal{P}}(n)=\ O\big(x^{\frac{1}{2}}e^{(\log x)^{\frac{2}{3}}+C(\log x)^{\frac{1}{3}}+C(\log x)^{\frac{1}{6}}(3\log x)^{\frac{1}{2}}}\big)=O\big(x^{\frac{1}{2}}e^{C(\log x)^{\frac{1}{3}}+(1+\sqrt{3}C)(\log x)^{\frac{2}{3}}}\big)$$
$$=O\big(x^{\frac{1}{2}}e^{C(\log x)^{\frac{1}{3}}+C'(\log x)^{\frac{2}{3}}}\big)=O\big(x^{\frac{1}{2}}e^{C''(\log x)^{\frac{2}{3}}}\big).$$

Hence, by following the same argument shown in the proof of Theorem 5.7 again, we have

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{\lambda_{\mathcal{P}}(n)}{n} = O\Big(\frac{e^{C''(\log x)^{\frac{2}{3}}}}{x^{\frac{1}{2}}}\Big) \quad \text{for some constant } C''.$$

□

5.3.2 Ω-Results for Euler's example over $\mathcal{N}$

We now consider Ω-results of $\sum_{n\le x} \frac{\lambda_{\mathcal{P}}(n)}{n}$ for a system $\mathcal{P}$ which satisfies either

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta}) \quad \text{for some } \rho > 0 \tag{5.25}$$

or

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha}) \tag{5.26}$$

for some $\alpha, \beta < \frac{1}{2}$. Both of which give the lower bound $\Omega\big(\frac{1}{\sqrt{x}}\big)$ for the sum.

Propostion 5.12. *Let $\mathcal{P}$ be a g-prime system satisfying (5.25) for some $\beta < \frac{1}{2}$. Then*

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{\lambda_{\mathcal{P}}(n)}{n} = \Omega\Big(\frac{1}{\sqrt{x}}\Big). \tag{5.27}$$

Proof. We wish to show that (5.27) is true. It is enough to show that

$$L_{\mathcal{P}}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) = \Omega(\sqrt{x}).$$

Let us assume the converse, so that $L_{\mathcal{P}}(x) = o(\sqrt{x})$. We know that for all $\Re s > \frac{1}{2}$,

$$Z_{\mathcal{P}}(s) = s \int_1^{\infty} \frac{L_{\mathcal{P}}(x)}{x^{s+1}}\, dx \tag{5.28}$$

is holomorphic. But for $\Re s > 1$, $(Z_{\mathcal{P}}(s) =)\frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}$ and $\zeta_{\mathcal{P}}(2s)$ is holomorphic for $\Re s > \frac{1}{2}$ with no zeros for $\Re s > \frac{1}{2}$ since $\zeta_{\mathcal{P}}(s)$ is holomorphic for $\Re s > 1$ and has no zeros here. For $\Re s > 1$, we have $\zeta_{\mathcal{P}}(s) = \frac{\zeta_{\mathcal{P}}(2s)}{Z_{\mathcal{P}}(s)}$ which has a meromorphic continuation for $\Re s > \frac{1}{2}$ except for a pole at $s = 1$ and no zeros. Therefore $\zeta_{\mathcal{P}}(2s)$ has a meromorphic continuation for $\Re s > \frac{1}{4}$ and pole at $s = \frac{1}{2}$. Now we know from the assumption (5.25) that $\zeta_{\mathcal{P}}(s)$ has an analytic continuation for $\sigma > \beta$ except for a simple pole at $s = 1$. Thus $Z_{\mathcal{P}}(s)$ is holomorphic for $\sigma = \Re s > \frac{1}{2}$ with pole at $s = \frac{1}{2}$. On the one hand, we know that $Z_{\mathcal{P}}(s)$ has a pole at $\frac{1}{2}$ since $\zeta_{\mathcal{P}}(2s)$ has a simple pole at $\frac{1}{2}$. Thus

$$Z_{\mathcal{P}}(\tfrac{1}{2}+\epsilon) \sim \frac{C}{\epsilon^k} \quad \text{for some } k \geq 1 \text{ and } C \neq 0. \tag{5.29}$$

On the other hand, from the right integral of (5.28) we have to check as $s = \frac{1}{2} + \epsilon \to \frac{1}{2}$ when $\epsilon \longrightarrow 0^+$ as follows:

$$|Z_{\mathcal{P}}(\tfrac{1}{2}+\epsilon)| = \left|(\tfrac{1}{2}+\epsilon)\int_1^\infty \frac{L_{\mathcal{P}}(x)}{x^{\frac{3}{2}+\epsilon}}\,dx\right| \leq \left|\tfrac{1}{2}+\epsilon\right| \int_1^\infty \frac{|L_{\mathcal{P}}(x)|}{x^{\frac{3}{2}+\epsilon}}\,dx$$

$|L_{\mathcal{P}}(x)|$ can be written as $g(x)\sqrt{x}$, where $g(x) \geq 0$ and $g(x) \longrightarrow 0$ as $x \longrightarrow \infty$. Hence we have

$$|Z_{\mathcal{P}}(\tfrac{1}{2}+\epsilon)| \leq B\int_1^\infty \frac{g(x)\sqrt{x}}{x^{\frac{3}{2}+\epsilon}}\,dx = B\int_1^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx, \quad \text{where } B \text{ is constant.}$$

Given $\delta > 0$, there exists a constant A such that $0 \leq g(x) < \delta$ for $x \geq A$ in order to split the right integral into $(1 < x < A)$ and $(A < x < \infty)$ ranges. Hence

$$\begin{aligned}\int_1^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx = \int_1^A \frac{g(x)}{x^{1+\epsilon}}\,dx + \int_A^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx &\leq \int_1^A \frac{g(x)}{x}\,dx + \delta\int_A^\infty x^{-1-\epsilon}\,dx \\ &\leq C + \frac{\delta}{\epsilon}.\end{aligned}$$

Thus

$$\epsilon\int_1^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx \leq C\epsilon + \delta.$$

For $\epsilon \longrightarrow 0^+$ we have

$$\limsup_{\epsilon\longrightarrow 0^+} \epsilon\int_1^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx \leq \delta \quad \text{for all } \delta > 0.$$

Thus

$$\int_1^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx = o\Big(\frac{1}{\epsilon}\Big).$$

This also implies that

$$\int_1^\infty \frac{L_{\mathcal{P}}(x)}{x^{\frac{3}{2}+\epsilon}}\,dx = o\Big(\frac{1}{\epsilon}\Big),$$

so that $Z_{\mathcal{P}}(\frac{1}{2}+\epsilon) = o(\frac{1}{\epsilon})$. Therefore this gives a contradiction with (5.29) and hence

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{\lambda_{\mathcal{P}}(n)}{n} = \Omega\Big(\frac{1}{\sqrt{x}}\Big).$$

□

In order to show (5.26) implies (5.27), we first need to prove a more general result.

Propostion 5.13. *If $\mathcal{P}$ is a g-prime system for which*

$$N_{\mathcal{P}}(x) \sim \rho x \quad \textit{for some } \rho > 0,$$

and $\zeta_{\mathcal{P}}(s)$ has an analytic continuation past $\Re s = 1$ to a region containing a neighborhood of $s = \frac{1}{2}$, then (5.27) holds.

Proof. To show (5.27) is true, it is enough to show that

$$L_{\mathcal{P}}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \lambda_{\mathcal{P}}(n) = \Omega(\sqrt{x}).$$

Let us again assume that $L_{\mathcal{P}}(x) = o(\sqrt{x})$. We know that

$$Z_{\mathcal{P}}(s) = s\int_1^{\infty} \frac{L_{\mathcal{P}}(x)}{x^{s+1}}\,dx$$

is holomorphic for all $\Re s > \frac{1}{2}$. But for $\Re s > 1$, $(Z_{\mathcal{P}}(s) =)\frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}$ and $\zeta_{\mathcal{P}}(2s)$ is holomorphic for $\Re s > \frac{1}{2}$ with no zeros for $\Re s > \frac{1}{2}$ since $\zeta_{\mathcal{P}}(s)$ is holomorphic for $\Re s > 1$ and has no zeros here. For $\Re s > 1$, we have $\zeta_{\mathcal{P}}(s) = \frac{\zeta_{\mathcal{P}}(2s)}{Z_{\mathcal{P}}(s)}$ which has a meromorphic continuation for $\Re s > \frac{1}{2}$ except for a pole at $s = 1$ and no zeros. Thus $\zeta_{\mathcal{P}}(2s)$ has a meromorphic continuation for $\Re s > \frac{1}{4}$ and pole at $s = \frac{1}{2}$. Thus $Z_{\mathcal{P}}(s)$ is holomorphic for $\sigma = \Re s > \frac{1}{2}$ with pole at $s = \frac{1}{2}$ because $\zeta_{\mathcal{P}}(s)$ is holomorphic at $\frac{1}{2}$.

Hence, by following the same argument shown in the proof of Proposition 5.12, we obtain the required result.

□

Remark 5.14. With Zhang's system which was previously detailed in the thesis, then (5.27) is also true if we assume $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to a neighborhood of $s = \frac{1}{2}$.

In the following corollary we consider the effect of the assumption (5.26) for some $\alpha < \frac{1}{2}$.

Corollary 5.15. *Let $\mathcal{P}$ be a g-prime system satisfying (5.26) for some $\alpha < \frac{1}{2}$. Then (5.27) holds.*

Proof. By Theorems 3.28 and 3.29 the assumption (5.26) for some $\alpha < \frac{1}{2}$ implies $\zeta_{\mathcal{P}}(s)$ is holomorphic for $\Re s > \alpha$ except for a simple pole at $s = 1$ and that it has no zeros in this region, and

$$N_{\mathcal{P}}(x) = \rho x + O(x^{-c\sqrt{\log x \log\log x}})$$

for some $\rho > 0$ (see [27]). Hence, by Proposition 5.13, (5.27) holds.

□

Remark 5.16. Let $\mathcal{P}$ be a g-prime system satisfying (4.4), (4.3) and (4.9) for some $\alpha,\ \beta,\ \xi < 1$ respectively. Then Proposition 5.12 and Corollary 5.15 imply that $\max\{\beta, \xi\} \geq \frac{1}{2}$ and $\max\{\alpha, \xi\} \geq \frac{1}{2}$.

5.4 Open problem

As mentioned in Chapter 2, Kahane and Saïas proposed that for all CMO functions, one has $\sum_{n \leq x} f(n) = \Omega\left(\frac{1}{\sqrt{x}}\right)$. They also showed that GRH-RH (Generalised Riemann Hypothesis-Riemann Hypothesis) would follow from this result. In our findings, we did not find any $CMO_{\mathcal{P}}$ functions f such that $\sum_{n \leq x} f(n) = O\left(\frac{1}{x^c}\right)$ for $c > \frac{1}{2}$. This may suggest the following conjecture.

Conjecture 5.17. Let $\mathcal{P}$ be a g-prime system with abscissa 1. Then, for all completely multiplicative functions on $\mathcal{N}$, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} f(n) = \Omega\left(\frac{1}{\sqrt{x}}\right).$$

Chapter 6

Multiplicative Functions of Zero Sum over $\mathcal{N}$

In this chapter, we extend the concept of $CMO_{\mathcal{P}}$ functions to multiplicative functions using the generalisation of MO functions. Firstly, we introduce these functions, while in the second part we discuss some properties of these functions. Finally, we construct some examples of such functions. In particular, we give a different type of example to those which have been previously discussed.

6.1 $MO_{\mathcal{P}}$ functions

Let $\mathcal{P}$ be g-prime system. A function $f : \mathcal{N} \longrightarrow \mathbb{C}$ is called an $\boldsymbol{MO_{\mathcal{P}}}$ function if it is multiplicative and satisfies

$$(i)\ \sum_{n\in\mathcal{N}} f(n) = 0 \quad \text{and} \quad (ii)\ \sum_{k=0}^{\infty} f(p^k) \neq 0 \text{ for all } p \in \mathcal{P}.$$

The extra (ii) condition is put in order to avoid trivial examples such as: if $f(1) = 1$, $f(p_1) = -1$ and $f(n) = 0$ for all $n \in \mathcal{N} \setminus \{1, p_1\}$, then $\sum_{n\in\mathcal{N}} f(n) = f(n_1) + f(n_2) + f(n_3) + \cdots = 0$ but $\sum_{k=1}^{\infty} f(p_1^k) = f(1) + f(p_1) + f(p_1^2) + \cdots = 0$, and so does not satisfy the extra condition.

For the convenience of exposition, we define $MO_{\mathcal{P}}$ functions to be

$$MO_{\mathcal{P}} := \{f : \mathcal{N} \longrightarrow \mathbb{C} \text{ multiplicative and } (i) \text{ and } (ii)\}$$

We can view such functions as a generalisation of $CMO_{\mathcal{P}}$ functions with an extra condition or a generalisation of MO functions over Beurling prime systems. Such a generalisation gives more scope and allows us to look for other types of functions. For instance, we consider the function $\frac{a_{\mathcal{P}}(n)}{n^{\alpha}}$ with a g-prime system satisfying

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta}) \quad \text{for some } \rho > 0 \text{ and } \beta < 1,$$

where $a_{\mathcal{P}}(n)$ is $1 - p_0$ if p_0 divides $n \in \mathcal{N}$ and 1 if p_0 does not divide $n \in \mathcal{N}$. In particular, we show that if α with $\Re\alpha > \beta$ is a zero of $\zeta_{\mathcal{P}}$, then

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \frac{a_{\mathcal{P}}(n)}{n^{\alpha}} = O\Big(\frac{1}{x^{\Re\alpha - \beta}}\Big).$$

We develop the theory of $CMO_{\mathcal{P}}$ functions which have been studied in Chapter 5 to multiplicative functions. Moreover, we derive O and Ω results of the partial sum of $\mu_{\mathcal{P}}(n)$ over n up to and including x on $\mathcal{N}$, ($i.e$ $\sum_{n\leq x} \frac{\mu_{\mathcal{P}}(n)}{n}$) with various g-prime systems. For all examples which are found, we have $\sum_{n\leq x} f(n) = \Omega\big(\frac{1}{x^{\frac{1}{2}+\varepsilon}}\big)$ for all $\varepsilon > 0$. In fact, this may suggest that for all functions f which are multiplicative functions over $\mathcal{N}$, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} f(n) = \Omega\Big(\frac{1}{\sqrt{x}}\Big).$$

Furthermore, we discuss repercussion of this conjecture.

6.2 Some properties of $MO_{\mathcal{P}}$ functions

In this section, we drive some preliminary properties of $MO_{\mathcal{P}}$ functions.

Propostion 6.1. *If f is a $CMO_{\mathcal{P}}$ function, then f is an $MO_{\mathcal{P}}$ function;* ($i.e$ $CMO_{\mathcal{P}} \subset MO_{\mathcal{P}}$).

Proof. It is clear that f is multiplicative and $\sum_{n\in\mathcal{N}} f(n) = 0$. It remains to show that $\sum_{k=0}^{\infty} f(p^k) \neq 0$ for every $p \in \mathcal{P}$. Now since f is completely multiplicative, then $f(p^k) = f(p)^k$ for every $p \in \mathcal{P}$. Therefore

$$\sum_{k=0}^{\infty} f(p^k) = \sum_{k=0}^{\infty} f(p)^k = \frac{1}{1 - f(p)} \neq 0.$$

This series converges since $|f(p)| < 1$. Hence f is an $MO_{\mathcal{P}}$ function. □

Propostion 6.2. *Let f be an $MO_{\mathcal{P}}$ function. Then $\sum_{n\in\mathcal{N}}|f(n)|$ diverges. Indeed $\sum_{p\in\mathcal{P}}\sum_{k\in\mathbb{N}}|f(p^k)|$ diverges.*

Proof. Let us assume the converse, i.e. that

$$\sum_{n\in\mathcal{N}}|f(n)| \text{ converges.}$$

Then, by multiplicative property,

$$\sum_{n\in\mathcal{N}} f(n) = \prod_{p\in\mathcal{P}}\sum_{k=0}^{\infty} f(p^k) \neq 0 \text{ since } \sum_{k=0}^{\infty} f(p^k) \neq 0.$$

Thus this gives a contradiction since f is an $MO_{\mathcal{P}}$ function and hence

$$\sum_{n\in\mathcal{N}}|f(n)| \text{ diverges.}$$

Furthermore, Proposition 3.27 gives $\sum_{p\in\mathcal{P}}\sum_{k\in\mathbb{N}}|f(p^k)|$ diverges, as required.

□

6.2.1 Partial sums of $MO_{\mathcal{P}}$ functions

By definition the partial sum of an $MO_{\mathcal{P}}$ function not exceeding x tends to zero when x tends to infinity.

As mentioned in Chapter 5, we can also ask how small can we make $g(x)$, so that (5.3) is true for all $MO_{\mathcal{P}}$ functions f?

We can do this under assumptions on g-prime systems as follows:

Propostion 6.3. *Let $\mathcal{P}$ be a g-prime system with unique representation (all the multiplicities are 1) for which $\sum_{p\in\mathcal{P}}\frac{1}{p\log p}$ converges. If f is an $MO_{\mathcal{P}}$ function, then*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} f(n) = \Omega\Big(\frac{1}{x\log x}\Big).$$

Proof. Let us assume the converse, so that

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} f(n) = O\Big(\frac{1}{x\log x}\Big).$$

We know that for every p^k, where $p \in \mathcal{P}$ and $k \in \mathbb{N}$, we have

$$f(p^k) = \sum_{m \le p^k} f(m) - \sum_{m < p^k} f(m) = O\Big(\frac{1}{p^k \log p^k}\Big). \tag{6.1}$$

Now it follows that $\sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{N}} |f(p^k)|$ converges since

$$\begin{aligned}\sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{N}} \frac{1}{p^k \log p^k} &\le \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{N}} \frac{1}{p^k \log p} = \sum_{p\in\mathcal{P}} \frac{1}{\log p} \sum_{k\in\mathbb{N}} \frac{1}{p^k} \quad (\text{since } \log p^k \ge \log p)\\ &= \sum_{p\in\mathcal{P}} \frac{1}{(p-1)\log p} \quad \text{converges (since } \textstyle\sum_{p\in\mathcal{P}} \frac{1}{p\log p} \text{ converges).}\end{aligned}$$

Hence, by Proposition 3.27, $\sum_{n\in\mathcal{N}} |f(n)|$ converges. However, by Proposition 6.2, we have a contradiction, and so it follows that

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} f(n) = \Omega\Big(\frac{1}{x \log x}\Big).$$

$\square$

Remark 6.4.

(i) Similarly, we can get different results by having different assumptions on the g-prime systems. For Example, if f is an $MO_{\mathcal{P}}$ function and $\mathcal{P}$ is a g-prime system with unique representation for which $\sum_{p\in\mathcal{P}} \frac{1}{p(\log\log p)^2}$ converges, then

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} f(n) = \Omega\Big(\frac{1}{x(\log\log x)^2}\Big).$$

(ii) We also can make the above sum to $\Omega(\frac{1}{x})$ if we assume $\sum_{p\in\mathcal{P}} \frac{1}{p}$ converges.

(iii) As mentioned in Chapter 5, without unique representations, it can also be more complicated. For instance, if $\mathcal{P}$ is as in Example 3.5, then (6.1) does not work.

6.2.2 Closeness relation between two multiplicative functions which are defined over $\mathcal{N}$

Let $\mathcal{M}_{\mathcal{P}} := \{f : \mathcal{N} \longrightarrow \mathbb{C} \text{ multiplicative}\}$, and let us define an ***(extended) metric*** on $\mathcal{M}_{\mathcal{P}}$ to be the distance function

$$D(f, g) := \sum_{p \in \mathcal{P}} \sum_{k=0}^{\infty} |g(p^k) - f(p^k)|.$$

Then $\mathcal{M}_{\mathcal{P}}$ is an ***extended metric space*** since $D(f, g)$ can attain the value ∞. It is straightforward to check for all $f, g, h \in \mathcal{M}_{\mathcal{P}}$

(i) $D(f, g) = 0$ if and only if $f = g$,

(ii) $D(f, g) = D(g, f)$,

(iii) $D(f, h) \leq D(f, g) + D(g, h)$,

hold. We aim to generalise Theorem 2.11 in Chapter 2 over g-prime systems. We aim to show that if f is an $MO_{\mathcal{P}}$ function and g is a multiplicative function "close" to f, then g is also an $MO_{\mathcal{P}}$ function. We can do this under an extra condition on f.

Theorem 6.5. *Let $\mathcal{P}$ be a g-prime system with abscissa 1, f an $MO_{\mathcal{P}}$ function for which*

$$\Big| \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \Big| \geq a \quad \text{for some } a > 0, \text{ for all } p \in \mathcal{P} \text{ and all } \Re s \geq 0, \tag{6.2}$$

and let g be a multiplicative function such that $D(f, g)$ is finite and

$$\sum_{k=0}^{\infty} g(p^k) \neq 0 \quad \text{for all } p \in \mathcal{P}. \tag{6.3}$$

Then g is an $MO_{\mathcal{P}}$ function.

Proof. Let $F(s) := \sum_{n \in \mathcal{N}} \frac{f(n)}{n^s}$ and $G(s) := \sum_{n \in \mathcal{N}} \frac{g(n)}{n^s}$. Then the series for $F(s)$ is absolutely convergent for $\Re s > 1$ and it is convergent for $\Re s > 0$ and $s = 0$ since $\sum_{n \in \mathcal{N}} f(n) = 0$. We note that $D(f, g)$ is finite and the fact that f is an $MO_{\mathcal{P}}$ function imply $|g(p^k)| \longrightarrow 0$ as $p^k \longrightarrow \infty$. Then, by Theorem 1.9, $g(n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Therefore the series for $G(s)$ converges for $\Re s > 1$ since g is bounded and the abscissa of $\mathcal{P}$ is 1. Therefore $F(s)$ and $G(s)$ can be written as follows:

$$F(s) = \prod_{p\in\mathcal{P}} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \quad \text{and} \quad G(s) = \prod_{p\in\mathcal{P}} \sum_{k=0}^{\infty} \frac{g(p^k)}{p^{ks}} \quad \Re s > 1.$$

Now

$$H(s) := \prod_{p\in\mathcal{P}} \left(\frac{\sum_{k=0}^{\infty} \frac{g(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}} \right) = \prod_{p\in\mathcal{P}} \left(1 + \frac{\sum_{k=0}^{\infty} \frac{g(p^k)-f(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}} \right)$$

converges absolutely for $\Re s \geq 0$ if and only if

$$\sum_{p\in\mathcal{P}} \frac{\left| \sum_{k=0}^{\infty} \frac{g(p^k)-f(p^k)}{p^{ks}} \right|}{\left| \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right|} \tag{6.4}$$

converges for $\Re s \geq 0$. But

$$\sum_{p\in\mathcal{P}} \frac{\left| \sum_{k=0}^{\infty} \frac{g(p^k)-f(p^k)}{p^{ks}} \right|}{\left| \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right|} \leq \frac{1}{a} \sum_{p\in\mathcal{P}} \sum_{k=0}^{\infty} |g(p^k) - f(p^k)|$$

by (6.2) so, since $D(f,g)$ is finite, (6.4) converges for $\Re s \geq 0$ and $H(s)$ converges absolutely to holomorphic function for $\Re s > 0$. However, $H(s) = (G/F)(s)$ for $\Re s > 1$ then $G(s) = F(s)H(s)$, where the series for $F(s)$ converges for $\Re s > 0$ and $s = 0$ since f is an $MO_{\mathcal{P}}$ function, and $H(s)$ absolutely converges for $\Re s \geq 0$. Therefore $G(s)$ converges for $\Re s > 0$ and $s = 0$ using the extension of Theorem 1.18. Thus we have $G(0) = F(0)H(0) = 0$. Hence, by assumption (6.3) and $G(0) = 0$, g is an $MO_{\mathcal{P}}$ function.

□

The proof of Theorem 6.5 implies the following result.

Corollary 6.6. *Let $\mathcal{P}$ be a g-prime system with abscissa 1, f and g both be multiplicative functions on $\mathcal{P}$ such that $D(f,g)$ is finite and satisfies*

$$\Big| \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \Big| \geq a \quad \textit{for some } a > 0, \ \textit{for all } p \in \mathcal{P} \textit{ and all } \Re s \geq 0,$$

$$\Big|\sum_{k=0}^{\infty}\frac{g(p^k)}{p^{ks}}\Big| \geq b \quad \textit{for some } b>0, \textit{ for all } p\in\mathcal{P} \textit{ and all } \Re s\geq 0.$$

Then the following two assertions are equivalent:

$$\sum_{n\in\mathcal{N}} f(n)=0 \quad \textit{and} \quad \sum_{n\in\mathcal{N}} g(n)=0.$$

6.3 The function $\frac{\mu_{\mathcal{P}}(n)}{n}$ over different systems $\mathcal{P}$

In this section, we present some examples of $MO_{\mathcal{P}}$ functions. In particular, we provide some examples of the function $\frac{\mu_{\mathcal{P}}(n)}{n}$ associated with various g-prime systems where $\mu_{\mathcal{P}}(n)$ is the Möbius function over the g-prime system $\mathcal{P}$.

Example 6.7. As mentioned in Example 5.6 that if $\mathcal{P}$ satisfies one of the conditions (5.9), (5.10), (5.11), (5.12) or (5.13), then $\sum_{n\in\mathcal{N}}\frac{\mu_{\mathcal{P}}(n)}{n}=0$. Hence $\frac{\mu_{\mathcal{P}}(n)}{n}$ is a $MO_{\mathcal{P}}$ function since it is multiplicative with sum zero and

$$\sum_{k=0}^{\infty} f(p^k)=\sum_{k=0}^{\infty}\frac{\mu_{\mathcal{P}}(p^k)}{p^k}=1-\frac{1}{p}\neq 0 \quad \text{for all } p\in\mathcal{P}.$$

From Example 6.7, we see that $\sum_{n\in\mathcal{N}}\frac{\mu_{\mathcal{P}}(n)}{n}=0$, but how quickly does it converge?

6.3.1 O-Results for Möbius's example over $\mathcal{N}$

In this part, we shall be interested in the behaviour of the partial sum of $\mu_{\mathcal{P}}(n)$ over n up to and including x for either "well-behaved" systems $\mathcal{P}$ which satisfies (5.14) and (5.15) or its special case Zhang's system which satisfies (5.19) and (5.20).

Theorem 6.8. *Let $\mathcal{P}$ be a g-prime system satisfying (5.14) and (5.15) for some $\beta,\alpha<1$. Then*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{N}}}\frac{\mu_{\mathcal{P}}(n)}{n}=O\Big(\frac{1}{x^{1-\Theta-\varepsilon}}\Big) \quad \textit{for all } \varepsilon>0,$$

where $\Theta=\max\{\alpha,\beta\}$.

Proof. We employ a similar approach to that of Theorem 5.7 but now with $\mu_{\mathcal{P}}$. Instead of $\frac{\zeta_{\mathcal{P}}(2s)}{\zeta_{\mathcal{P}}(s)}$ and $|\lambda_{\mathcal{P}}(n)|=1$ we now have $\frac{1}{\zeta_{\mathcal{P}}(s)}$ and $|\mu_{\mathcal{P}}(n)|\leq 1$ which does not make a difference throughout the proof. □

Remark 6.9. It was shown in Theorem 3.31 that $\Theta \geq \frac{1}{2}$. Thus $1-\Theta \leq \frac{1}{2}$ and in Theorem 6.8 we can therefore only have an example with exponent $\leq \frac{1}{2}$ with such systems.

Theorem 6.10. *For Zhang's system $\mathcal{P}_Z$, we have*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(n)}{n} = O\Big(\frac{e^{C(\log x)^{\frac{2}{3}}}}{x^{\frac{1}{2}}}\Big) \quad \textit{for some constant } C.$$

Proof. Theorem 4.6 gives the required result.

□

6.3.2 Ω-Results for Möbius's example over $\mathcal{N}$

We now consider Ω-results of the partial sum $\mu_{\mathcal{P}}(n)$ over n up to and including x for a system $\mathcal{P}$ which satisfies either assumption (5.25) or (5.26) for some $\alpha, \beta < \frac{1}{2}$. From assumption (5.25) for some $\beta < \frac{1}{2}$, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(n)}{n} = \Omega\Big(\frac{1}{x^{\frac{1}{2}+\varepsilon}}\Big) \quad \textit{for all } \varepsilon > 0. \tag{6.5}$$

The term on the right hand side of (6.5) can be improved to be $\Omega\big(\frac{1}{\sqrt{x}}\big)$ by means of Conjecture 3.37 which is a stronger than Corollary 3.30. In addition, (6.5) can be also attained from assumption (5.26) for some $\alpha < \frac{1}{2}$.

Corollary 6.11. *Let $\mathcal{P}$ be a g-prime system satisfying (5.25) for some $\beta < \frac{1}{2}$. Then (6.5) holds.*

Proof. Let us assume the converse, so that (6.5) is false.

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(n)}{n} = O\Big(\frac{1}{x^d}\Big) \quad \text{for some } d > \tfrac{1}{2}.$$

By Abel summation, we have $M_{\mathcal{P}}(x) = O(x^{\gamma})$ for some $\gamma < \frac{1}{2}$, where $\gamma = 1-d$. But by Theorem 4.11 this contradicts our initial assumption, and hence the result follows.

□

If Conjecture 3.37 is true, then the bound of (6.5) will improve to $\Omega\left(\frac{1}{\sqrt{x}}\right)$, as the following Proposition shows.

Propostion 6.12. *Let $\mathcal{P}$ be a g-prime system satisfying (5.25) for some $\beta < \frac{1}{2}$. Then conjecture 3.37 implies*

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(n)}{n} = \Omega\left(\frac{1}{\sqrt{x}}\right). \tag{6.6}$$

Proof. We know that for all $\Re s > 1$,

$$U_{\mathcal{P}}(s) = \frac{1}{\zeta_{\mathcal{P}}(s)} = \sum_{n \in \mathcal{N}} \frac{\mu_{\mathcal{P}}(n)}{n^s} = s \int_1^{\infty} \frac{M_{\mathcal{P}}(x)}{x^{s+1}}\, dx, \text{ where } M_{\mathcal{P}}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \mu_{\mathcal{P}}(n). \tag{6.7}$$

We wish to show that (6.6) is true. It is enough to show that

$$M_{\mathcal{P}}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \mu_{\mathcal{P}}(n) = \Omega(\sqrt{x}).$$

Let us assume the converse, so that $M_{\mathcal{P}}(x) = o(\sqrt{x})$. We know that from our assumption (5.25) $\zeta_{\mathcal{P}}(s)$ has an analytic continuation for $\sigma > \beta$ except for a simple pole at $s = 1$. Therefore $U_{\mathcal{P}}(s)$ is holomorphic for $\Re s > \frac{1}{2}$. This also means $U_{\mathcal{P}}(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$.

On the one hand, Conjecture 3.37 implies that there exist zero of $\zeta_{\mathcal{P}}(s)$ such that $s_0 = \frac{1}{2} + it_0$. Thus

$$U_{\mathcal{P}}(s_0 + \epsilon) \sim \frac{C}{\epsilon^k} \quad \text{for some } k \ge 1 \text{ and } C \ne 0. \tag{6.8}$$

On the other hand, from the right integral of (6.7) we have to check as $s \longrightarrow s_0$ when $\epsilon \longrightarrow 0^+$ as follows:

$$|U_{\mathcal{P}}(s_0 + \epsilon)| = \left| \left(\tfrac{1}{2} + it_0 + \epsilon\right) \int_1^{\infty} \frac{M_{\mathcal{P}}(x)}{x^{\frac{3}{2} + it_0 + \epsilon}}\, dx \right| \le \left|\tfrac{1}{2} + it_0 + \epsilon\right| \int_1^{\infty} \frac{|M_{\mathcal{P}}(x)|}{x^{\frac{3}{2} + \epsilon}}\, dx.$$

$|M_{\mathcal{P}}(x)|$ can be written as $g(x)\sqrt{x}$, where $g(x) \ge 0$ and $g(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

Therefore we have

$$|U_{\mathcal{P}}(s_0+\epsilon)| \le B\int_1^\infty \frac{g(x)\sqrt{x}}{x^{\frac{3}{2}+\epsilon}}\,dx = B\int_1^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx, \quad \text{where } B \text{ is constant.}$$

Given $\delta > 0$, there exists a constant A such that $0 \le g(x) < \delta$ for $x \ge A > 1$ in order to split the right integral into $(1 < x < A)$ and $(A < x < \infty)$ ranges. Hence

$$\begin{aligned}\int_1^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx = \int_1^A \frac{g(x)}{x^{1+\epsilon}}\,dx + \int_A^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx &\le \int_1^A \frac{g(x)}{x}\,dx + \delta\int_A^\infty x^{-1-\epsilon}\,dx\\ &\le C + \frac{\delta}{\epsilon}.\end{aligned}$$

Thus

$$\epsilon\int_1^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx \le C\epsilon + \delta.$$

For $\epsilon \longrightarrow 0^+$, we have

$$\limsup_{\epsilon\longrightarrow 0^+} \epsilon\int_1^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx \le \delta \quad \text{for all } \delta > 0.$$

Thus

$$\int_1^\infty \frac{g(x)}{x^{1+\epsilon}}\,dx = o\Big(\frac{1}{\epsilon}\Big).$$

This also implies that

$$\int_1^\infty \frac{M_{\mathcal{P}}(x)}{x^{\frac{3}{2}+\epsilon}}\,dx = o\Big(\frac{1}{\epsilon}\Big).$$

Hence $U_{\mathcal{P}}(s_0+\epsilon) = o(\frac{1}{\epsilon})$ as $s \longrightarrow s_0$. Therefore this gives a contradiction with (6.8) and hence

$$\sum_{\substack{n \le x\\ n \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(n)}{n} = \Omega\Big(\frac{1}{\sqrt{x}}\Big).$$

□

In the following corollary we consider the effect of the assumption (5.26) for some $\alpha < \frac{1}{2}$.

Corollary 6.13. *Let $\mathcal{P}$ be a g-prime system satisfying (5.26) for some $\alpha < \frac{1}{2}$. Then (6.5) holds.*

Proof. Let us assume the converse, so that (6.5) is false. Then

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{\mu_{\mathcal{P}}(n)}{n} = O\Big(\frac{1}{x^d}\Big) \quad \text{for some } d > \tfrac{1}{2}.$$

By Abel summation, we have $M_{\mathcal{P}}(x) = O(x^c)$ for some $c < \frac{1}{2}$, where $c = 1 - d$. But by Theorem 4.12 this contradicts our initial assumption, and hence the result follows.

□

6.4 The Example $\frac{a_{\mathcal{P}}(n)}{n^\alpha}$

In this section, we first define $a_{\mathcal{P}}(n)$ which is a generalisation of the function $(-1)^{n-1}$ over $\mathbb{N}$ as follows:

Fix $p_0 \in \mathcal{P}$ and $n \in \mathcal{N}$, then

$$a_{\mathcal{P}}(n) := \begin{cases} 1 - p_0 & \text{if } p_0 \mid n, \\ 1 & \text{if } p_0 \nmid n. \end{cases} \tag{6.9}$$

We shall be concerned with the behaviour of $\sum_{n \le x} \frac{a_{\mathcal{P}}(n)}{n^\alpha}$ for a system $\mathcal{P}$ which satisfies

$$N_{\mathcal{P}}(x) = \rho x + O(x^\beta) \quad \text{for some } \rho > 0 \text{ and } \beta < 1. \tag{6.10}$$

More precisely, we will show that for any α with $\Re\alpha > \beta$,

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{a_{\mathcal{P}}(n)}{n^\alpha} = (1 - {p_0}^{1-\alpha})\zeta_{\mathcal{P}}(\alpha) + O\Big(\frac{1}{x^{\Re\alpha - \beta}}\Big).$$

Lemma 6.14. *Let $\mathcal{P}$ be a g-prime system, and let $a_{\mathcal{P}}(n)$ defined as above. Then $\frac{a_{\mathcal{P}}(n)}{n^\alpha}$ is a multiplicative function for any $\alpha \in \mathbb{C}$.*

Proof. We wish to find all values of $\alpha \in \mathbb{C}$ for which $a_{\mathcal{P}}(n)$ is a multiplicative function as follows:

Assume $(m, n) = 1$ and consider $a(mn)$. If $m = n = 1$, then $a_{\mathcal{P}}(m)a_{\mathcal{P}}(n) = a_{\mathcal{P}}(mn)$. Now if $p_0 \mid mn$, then either $p_0 \mid m$ and $p_0 \nmid n$ or vice versa.

(i) If $p_0 \mid n$ and $p_0 \mid m$, then $(m, n) \neq 1$. We cannot have p_0 divides both m, n since we need $(m, n) = 1$.

(ii) If $p_0 | m$ and $p_0 \not| n$, then $a_{\mathcal{P}}(m)a_{\mathcal{P}}(n) = (1-p_0)(1) = 1-p_0 = a_{\mathcal{P}}(mn)$.

If $p_0 \not| mn$, then $p_0 \not| m$ and $p_0 \not| n$, and $a_{\mathcal{P}}(m)a_{\mathcal{P}}(n) = (1)(1) = 1 = a_{\mathcal{P}}(mn)$.

Thus $a_{\mathcal{P}}(n)$ and $\frac{a_{\mathcal{P}}(n)}{n^\alpha}$ are a multiplicative function for any $\alpha \in \mathbb{C}$.

□

Theorem 6.15. *Let $\mathcal{P}$ be a g-prime system for which (6.10) hold, and let $a_{\mathcal{P}}(n)$ defined as above. Then*

(i) For any α with $\Re\alpha > \beta$,

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{a_{\mathcal{P}}(n)}{n^\alpha} = (1 - {p_0}^{1-\alpha})\zeta_{\mathcal{P}}(\alpha) + O\Big(\frac{1}{x^{\Re\alpha-\beta}}\Big).$$

(ii) $\frac{a_{\mathcal{P}}(n)}{n^\alpha}$ is an $MO_{\mathcal{P}}$ function if and only if $\Re\alpha > \beta$ and $\zeta_{\mathcal{P}}(\alpha) = 0$.

Proof. We note from (6.10) that $\zeta_{\mathcal{P}}(\alpha)$ has an analytic continuation to $\Re(\alpha) > \beta$ except for a simple pole at $s = 1$.

(i) The series $\sum_{n\in\mathcal{N}} \frac{a_{\mathcal{P}}(n)}{n^\alpha}$ converges for $\Re\alpha > \beta$ since

$$\begin{aligned} A(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} a_{\mathcal{P}}(n) &= \sum_{\substack{n \le x \\ p_0 \not| n}} 1 + \sum_{\substack{n \le x \\ p_0 | n}} (1-p_0) = \sum_{\substack{n \le x \\ p_0 \not| n \text{ or } p_0 | n}} 1 - \sum_{\substack{n \le x \\ p_0 | n}} p_0 \\ &= N_{\mathcal{P}}(x) - p_0 \sum_{\substack{m \le \frac{x}{p_0} \\ m \in \mathcal{N}}} 1 = N_{\mathcal{P}}(x) - p_0\, N_{\mathcal{P}}\Big(\frac{x}{p_0}\Big) \\ &= \rho x + O(x^\beta) - p_0\bigg(\frac{\rho x}{p_0} + O\Big(\Big(\frac{x}{p_0}\Big)^\beta\Big)\bigg) \text{ (using (6.10))} \\ &= O(x^\beta). \end{aligned}$$

Then, by Abel summation,

$$\begin{aligned} \sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{a_{\mathcal{P}}(n)}{n^\alpha} &= \frac{A(x)}{x^\alpha} + \alpha\int_1^x \frac{A(t)}{t^{\alpha+1}}dt = O\Big(\frac{1}{x^{\Re\alpha-\beta}}\Big) + \alpha\int_1^\infty \frac{A(t)}{t^{\alpha+1}}dt - \alpha\int_x^\infty \frac{O(t^\beta)}{t^{\alpha+1}}dt \\ &= C_\alpha + O\Big(\frac{1}{x^{\Re\alpha-\beta}}\Big), \quad \text{where } C_\alpha \text{ is a constant} \end{aligned}$$

since $\big|\int_x^\infty \frac{O(t^\beta)}{t^{\alpha+1}}dt\big| = O\big(\int_x^\infty \frac{1}{t^{\Re\alpha+1-\beta}}dt\big) = O\big(\frac{1}{x^{\Re\alpha-\beta}}\big)$ and $\int_1^\infty \frac{A(t)}{t^{\alpha+1}}\,dt$ converges for

$\Re\alpha > \beta$. Hence, for $\Re\alpha > \beta$,

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{a_{\mathcal{P}}(n)}{n^\alpha} = C_\alpha + O\Big(\frac{1}{x^{\Re\alpha-\beta}}\Big).$$

In particular,

$$\sum_{n\in\mathcal{N}} \frac{a_{\mathcal{P}}(n)}{n^\alpha} \text{ converges to } C_\alpha = (1-{p_0}^{1-\alpha})\zeta_{\mathcal{P}}(\alpha).$$

Now, for $\Re\alpha > 1$, we have

$$\sum_{n\in\mathcal{N}} \frac{a_{\mathcal{P}}(n)}{n^\alpha} = \sum_{n\in\mathcal{N}} \frac{1}{n^\alpha} - \sum_{n\in\mathcal{N}} \frac{p_0}{(p_0 n)^\alpha} = (1-{p_0}^{1-\alpha})\zeta_{\mathcal{P}}(\alpha). \tag{6.11}$$

The sum on the left of (6.11) is C_α for $\Re\alpha > \beta$ by analytic continuation. Thus

$$\sum_{n\in\mathcal{N}} \frac{a_{\mathcal{P}}(n)}{n^\alpha} = 0 \text{ if and only if } p_0^\alpha = p_0 \text{ or } \zeta_{\mathcal{P}}(\alpha) = 0.$$

(ii) We know that $\frac{a_{\mathcal{P}}(n)}{n^\alpha}$ is multiplicative function and $\sum_{n\in\mathcal{N}} \frac{a_{\mathcal{P}}(n)}{n^\alpha} = 0$ if and only if $p_0^\alpha = p_0$ or $\zeta_{\mathcal{P}}(\alpha) = 0$. It remains to get all α for which $\sum_{k=0}^{\infty} a(p^k) \neq 0$ for all $p \in \mathcal{P}$. If $p \neq p_0$, then $a_{\mathcal{P}}(p^k) = 1$. Therefore

$$\sum_{k=0}^{\infty} \frac{a_{\mathcal{P}}(p^k)}{p^{k\alpha}} = \sum_{k=0}^{\infty} \frac{1}{p^{\alpha k}} = \frac{1}{1-\frac{1}{p^\alpha}}.$$

This is non-zero for any α with $\Re\alpha > \beta$. Now if $p = p_0$, then $a_{\mathcal{P}}(p^k) = 1 - p_0$ for all $k \ge 1$. Therefore

$$\sum_{k=0}^{\infty} \frac{a_{\mathcal{P}}(p^k)}{p^{k\alpha}} = 1 + (1-p_0)\sum_{k=1}^{\infty} \frac{1}{p^{\alpha k}} = 1 + (1-p_0)\frac{1}{p^\alpha - 1} = \frac{p_0^\alpha - p_0}{p_0^\alpha - 1}.$$

This is non-zero if and only if $p_0^\alpha \neq p_0$.

We see that $\frac{a_{\mathcal{P}}(n)}{n^\alpha}$ is not an $MO_{\mathcal{P}}$ function if $p_0^\alpha = p_0$ since Lemma 6.14 and (i) hold but (ii) fails. We can conclude that $\frac{a_{\mathcal{P}}(n)}{n^\alpha}$ is an $MO_{\mathcal{P}}$ function if and only if $\Re\alpha > \beta$ and $\zeta_{\mathcal{P}}(\alpha) = 0$ since Lemma 6.14, (i) and (ii) hold.

Furthermore, if $\zeta_{\mathcal{P}}(\alpha) = 0$ with $\Re\alpha > 0$, then

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{a_{\mathcal{P}}(n)}{n^{\alpha}} = O\Big(\frac{1}{x^{\Re\alpha - \beta}}\Big).$$

□

Remark 6.16. If $\mathcal{P}$ is the system which satisfies (6.10) but has oscillating primes in the sense that

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + \Omega(xe^{-c\sqrt{\log x}})$$

holds. Here $\zeta_{\mathcal{P}}(\alpha)$ has infinitely many zeros, α, of the corresponding zeta function close to $\Re\alpha = 1$ [14], [58]. Then $\frac{a_{\mathcal{P}}(n)}{n^{\alpha}} \in MO_{\mathcal{P}}$ and

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{a_{\mathcal{P}}(n)}{n^{\alpha}} = O\Big(\frac{1}{x^{c}}\Big) \text{ for any } \ c < \tfrac{1}{2}.$$

To get examples where we have $O\big(\frac{1}{x^c}\big)$ for some $c > \frac{1}{2}$, we need $\Re\alpha - \beta > \frac{1}{2}$. In other words, we need zeros α of $\zeta_{\mathcal{P}}$ with $\Re\alpha > \frac{1}{2}$ and $\beta < \frac{1}{2}$ (since $\Re\alpha < 1$). Therefore we wonder if there exist such systems with zeros of $\zeta_{\mathcal{P}}(\alpha)$ close to the 1-line and small β such that $\beta < \frac{1}{2}$?

6.4.1 Special case when $\mathcal{P} \subset \mathbb{P}$

As we mentioned in Chapter 2, Kahane and Saïas give the example $\frac{\chi(n)}{n^s}$, where χ is a non-principal Dirichlet character of a CMO function. This motivates us to find an "equivalent", (*i.e.* multiplicative and periodic) example of $MO_{\mathcal{P}}$ functions to their example. In order to find an example such as this, we are required to seek a function that is both multiplicative and periodic. As we know that the function (6.9) is multiplicative it remains to be shown for which g-prime system $\mathcal{P}$ the function (6.9) is periodic. One way we are able to meet this criteria and have the function (6.9) periodic is for it to be defined on a subset of the usual primes; (*i.e.* $\mathcal{P} \subseteq \mathbb{P}$). In this instance we find that the function $\frac{a_{\mathcal{P}}(n)}{n^{\alpha}}$ is in some sense equivalent to their example. We also show that $\sum_{n=1}^{\infty} \frac{a_{\mathcal{P}}(n)}{n^{\alpha}}$ converges to $C_{\alpha} = (1 - {p_0}^{1-\alpha})\zeta(\alpha)$.

Theorem 6.17. *Let $\mathcal{P} = \mathbb{P} \backslash F$, where F is a finite set of primes and p_0 is the smallest element in $\mathcal{P}$. Then* $\left(1 - \frac{p_0}{p_0^\alpha}\right)\zeta(\alpha)\prod_{p\in F}\left(1 - \frac{1}{p^\alpha}\right) = \sum_{n=1}^{\infty} \frac{b(n)}{n^\alpha}$, *where*

$$b(n) := \begin{cases} 1 & \text{if } (k,n) = 1, \\ 1 - p_0 & \text{if } (k,n) = p_0, \\ 0 & \text{otherwise,} \end{cases}$$

here $k = p_0 \prod_{p \in F} p$. *In particular, $b(n)$ has period k.*

Proof.

$$\left(1 - \frac{p_0}{p_0^\alpha}\right)\zeta(\alpha)\prod_{p\in F}\left(1 - \frac{1}{p^\alpha}\right) = \left(1 - \frac{p_0}{p_0^\alpha}\right)\prod_{p\notin F}\frac{1}{\left(1 - \frac{1}{p^\alpha}\right)} = \left(1 - \frac{p_0}{p_0^\alpha}\right)\sum_{n=1}^{\infty}\frac{\chi_1(n)}{n^\alpha},$$

where χ_1 is the principle character mod $r = \prod_{p\in F} p$; *i.e.*

$$\chi_1(n) = \begin{cases} 1 & \text{if } (r,n) = 1, \\ 0 & \text{if } (r,n) > 1. \end{cases}$$

Thus

$$\left(1 - \frac{p_0}{p_0^\alpha}\right)\zeta(\alpha)\prod_{p\in F}\left(1 - \frac{1}{p^\alpha}\right) = \sum_{n=1}^{\infty}\frac{\chi_1(n)}{n^\alpha} - p_0\sum_{n=1}^{\infty}\frac{\chi_1(n)}{(np_0)^\alpha} = \sum_{n=1}^{\infty}\frac{\chi_1(n)}{n^\alpha} - p_0\sum_{n=1}^{\infty}\frac{c(n)}{n^\alpha},$$

where

$$c(n) = \begin{cases} \chi_1(\frac{n}{p_0}) & \text{if } p_0 \mid n, \\ 0 & \text{if } p_0 \nmid n. \end{cases}$$

Hence

$$\left(1 - \frac{p_0}{p_0^\alpha}\right)\zeta(\alpha)\prod_{p\in F}\left(1 - \frac{1}{p^\alpha}\right) = \sum_{n=1}^{\infty}\frac{b(n)}{n^\alpha},$$

where

$$b(n) = \begin{cases} 1 & \text{if } (r,n) = 1 \text{ and } p_0 \nmid n, \\ 1 - p_0\chi_1(\frac{n}{p_0}) & \text{if } (r,n) = 1 \text{ and } p_0 \mid n, \\ -p_0\chi_1(\frac{n}{p_0}) & \text{if } (r,n) > 1 \text{ and } p_0 \mid n, \\ 0 & \text{if } (r,n) > 1 \text{ and } p_0 \nmid n. \end{cases}$$

Suppose $p_0 \,|\, n$. If $(r,n)=1$, then $(r,\frac{n}{p_0})=1$ and $\chi_1(\frac{n}{p_0})=1$. Also, if $(r,n)>1$, then $(r,\frac{n}{p_0})>1$ since $(r,p_0)=1$ and $\chi_1(\frac{n}{p_0})=0$. Note that $(r,n)=1$ and $p_0 \not| \, n$ if and only if $(k,n)=1$, while $(r,n)=1$ and $p_0 \,|\, n$ if and only if $(k,n)=p_0$. Hence

$$b(n)=\begin{cases} 1 & \text{if } (k,n)=1,\\ 1-p_0 & \text{if } (k,n)=p_0,\\ 0 & \text{otherwise.}\end{cases}$$

Since $(k,n)=(k,n+k)$. It follow that $b(n)=b(n+k)$. Therefore $b(n)$ has a period k. It remains to show that k is the smallest period. Suppose $b(n)$ has a smallest period d. We would like to show $k=d$. We know that $d\leq k$ by definition so we shall just show that $k\,|\,d$. Suppose there exist $p\in\mathbb{P}$ such that $p\,|\,k$ and $p \not| \, d$. We know that $b(1)=1=b(d+1)=b(yd+1)$, so $(k,1+d)=1=(k,1+yd)$ for all $y\in\mathbb{Z}$. But $xp-yd=1$ for some $x,y\in\mathbb{Z}$ since $(p,d)=1$. Therefore $1=(k,1+yd)=(k,xp)\geq p$. This means that every $p\,|\,k$ must divide into d. We conclude that $k\,|\,d$ since k is square-free. Hence $k=d$.

□

Remark 6.18. Let $a_{\mathcal{P}}(n)$ be as before and now taking $\mathcal{P}$ as in Theorem 6.17. Then

$$\sum_{n\in\mathcal{N}}\frac{a_{\mathcal{P}}(n)}{n^{\alpha}}=\sum_{\substack{n=1\\(r,n)=1}}^{\infty}\frac{a_{\mathcal{P}}(n)}{n^{\alpha}}=\sum_{n=1}^{\infty}\frac{b(n)}{n^{\alpha}}-\sum_{\substack{n=1\\(k,n)\neq 1 or p_0}}^{\infty}\frac{b(n)}{n^{\alpha}} \text{ for any } \alpha \text{ with } \Re\alpha>0,$$

where $k=p_0r$ and $r=\prod_{p\in F}p$. Furthermore, the period of $a_{\mathcal{P}}(n)$ is

$$k-\sum_{\substack{n=1\\(k,n)\neq 1 or p_0}}^{k}1=\sum_{n=1}^{k}1-\sum_{\substack{n=1\\(k,n)\neq 1 or p_0}}^{k}1=\sum_{\substack{n=1\\(k,n)=1 or p_0}}^{k}1=\sum_{\substack{n=1\\(k,n)=1}}^{k}1+\sum_{\substack{n=1\\(k,n)=p_0}}^{k}1=\phi(k)+\phi(\tfrac{k}{p_0}),$$

where ϕ is the Euler's totient function which is the number of positive integers below and including k that are relatively prime to k.

Corollary 6.19. *Let $\mathcal{P}=\mathbb{P}\setminus F$, where F is a finite set of primes and p_0 is the smallest element in $\mathcal{P}$. The function $a_{\mathcal{P}}(n)$ is defined as before. Then, for any α with $\Re\alpha>0$,*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{N}}}\frac{a_{\mathcal{P}}(n)}{n^{\alpha}}=(1-{p_0}^{1-\alpha})\zeta(\alpha)+O\Big(\frac{1}{x^{\Re\alpha}}\Big).$$

Proof. Proving this corollary can be obtained by using similar approach to Theorem 6.15 with different error term. Thus, if $\zeta(\alpha) = 0$, then $\frac{a_{\mathcal{P}}(n)}{n^\alpha}$ is an $MO_{\mathcal{P}}$ function and

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} \frac{a_{\mathcal{P}}(n)}{n^\alpha} = O\Big(\frac{1}{x^{\Re\alpha}}\Big).$$

□

6.5 Open problem

In this section, we suggest an interesting open problem which is related to RH in the sense that it would follow if the following conjecture is true.

Conjecture 6.20. Let $\mathcal{P}$ be a g-prime system with abscissa 1. Then, for all multiplicative functions f on $\mathcal{N}$, we have

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} f(n) = \Omega\Big(\frac{1}{\sqrt{x}}\Big).$$

Consequences of Conjecture 6.20

Consider $\mathcal{P}$ a g-prime system which satisfies $N_{\mathcal{P}}(x) = cx + O(x^\beta)$ for some $c > 0$, $\beta < \frac{1}{2}$. Now suppose there exists a zero, α, of the corresponding zeta function such that $\beta < \Re\alpha < 1$. Let $f(n) = \frac{a_{\mathcal{P}}(n)}{n^\alpha}$ be the function as defined in Theorem 6.15. Then $f \in MO_{\mathcal{P}}$ and

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}}} f(n) = O\Big(\frac{1}{x^{\Re\alpha - \beta}}\Big).$$

Therefore we have $\Re\alpha - \beta \le \frac{1}{2}$, (*i.e.* $\Re\alpha \le \beta + \frac{1}{2} < 1$) if Conjecture 6.20 is true. This means that any zeros of the zeta function must be to the left or on the line $\sigma = \beta + \frac{1}{2}$. This conjecture automatically implies that the zeros of the Beurling zeta function must be bounded away from the 1-line. In other words, there are no zeros of the Beurling's zeta function in the strip $\{s \in \mathbb{C} : \beta + \frac{1}{2} < \Re s \le 1\}$. This is a very strong form of Riemann Hypothesis.

In particular, the Riemann Hypothesis holds for the actual zeta function when $\beta = 0$. This conjecture not only implies Riemann Hypothesis but also implies an analogous Riemann Hypothesis for all these systems with $\beta < \frac{1}{2}$.

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