University of Reading

Wittgenstein’s Intermediate Period: Grammar, Verification, Infinity, Inductive Proof, and Set Theory

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Abstract

There is a gap in explaining the *interrelationships* between Wittgenstein's philosophy of mathematics and the other areas of his thought in the intermediate period. With special attention paid to Wittgenstein's philosophy of mathematics, this thesis is meant as a first step in outlining some of these important interconnections. Chapter 1 sets the stage by presenting Wittgenstein's views in the philosophy of mathematics in the *Tractatus*, with a focus on his analysis of infinity. Chapter 2 outlines the principal aspects of Wittgenstein's philosophy of mathematics, and the phenomenological language and its demise. Against the background of the *Tractatus*, the phenomenological language, and Wittgenstein's relationship with the Vienna Circle, Chapter 3 reconstructs the development of the verification principle, before examining the extensive application he makes of it to the philosophy of mathematics. Chapter 4 examines Wittgenstein's analyses of infinity, with special attention given to how his evolving views either contain the seeds of later insights or exemplify more general aspects of his philosophy. In Chapters 5 and 6, the results of the previous chapter are considered in relation to two important topics in Wittgenstein's thought: inductive proof and set theory, respectively. The discussion of Wittgenstein's views on inductive proof culminates in outlining how they influenced his philosophy of mathematics and philosophy generally. The examination of Wittgenstein's views on set theory concludes with a re-evaluation of an important debate within Wittgenstein studies on the extent of Wittgenstein's criticisms of set theory. The conflict arises as the result of an unduly narrow focus on (seemingly) contradictory elements of Wittgenstein's philosophy of mathematics, but disappears with a comprehensive understanding of Wittgenstein's mature philosophy of mathematics.

Declaration: I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Harry Tomany

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Abbreviations

Ludwig WITTGENSTEIN:

AL *Wittgenstein's Lectures, Cambridge, 1932-1935*, ed. A. Ambrose, Oxford: Blackwell, 1979.

BT *The Big Typescript: TS 213*, ed. & tr. C.G. Luckhardt & M.A.E. Aue, Oxford: Blackwell, 2005.

LFM *Wittgenstein's Lectures on the Foundations of Mathematics Cambridge, 1939,* ed. C. Diamond, Hassocks, Sussex: Harvester Press, 1976.

LL *Wittgenstein's Lectures, Cambridge, 1930-1932*, ed. D. Lee, Oxford: Blackwell, 1980.

MS Manuscript in *Wittgenstein's Nachlass: The Bergen Electronic Edition*, Oxford: OUP, 2000.

NB *Notebooks 1914-16*, ed. G.H. von Wright & G.E.M. Anscombe, tr. G.E.M. Anscombe, rev. ed. Oxford: Blackwell 1979.

NM *Wittgenstein: Lectures, Cambridge 1930-1933, from the Notes of G.E.Moore*, eds David G. Stern, Brian Rogers, and Gabriel Citron. Cambridge: Cambridge University Press, 2016.

OC *On Certainty*, eds G.E.M. Anscombe and G.H. von Wright, tr. Denis Paul and G.E.M. Anscombe, New York: Harper & Row, 1969.

PG *Philosophical Grammar*, ed. R. Rhees, tr. A.J.P. Kenny, Oxford: Blackwell, 1974.

PI *Philosophical Investigations*, eds P. M. S. Hacker & J. Schulte, tr. G.E.M. Anscombe; P.M.S. Hacker, J. Schulte, Oxford: Wiley-Blackwell, 2009.

PR *Philosophical Remarks*, ed. R. Rhees, tr. R. Hargreaves & R. White, Oxford: Blackwell, 1975.

RFM *Remarks on the Foundations of Mathematics*, eds G.H. von Wright, R. Rhees, G.E.M. Anscombe; tr. G.E.M. Anscombe, rev. ed., Oxford: Blackwell, 1978.

SRLF 'Some Remarks on Logical Form.' *Proceedings of the Aristotelian Society, Supplementary Volumes 9* (1929): 162-171.

TLP *Tractatus Logico-Philosophicus*, tr. D.F. Pears & B.F. McGuinness, London: Routledge & Kegan Paul, 1961.

VW *The Voices of Wittgenstein*, tr. Gordon Baker, Michael Mackert, John Connolly and Vasilis Politis, London: Routledge, 2003.

WC *Wittgenstein in Cambridge: Letters and Documents, 1911-1951*, ed. B. McGuinness, Oxford: Blackwell, 2008.

WL *Wittgenstein's Whewell's Court Lectures, Cambridge, 1938-1941: from the Notes of Yorick Smithies*, eds. Volker A. Munz and Bernhard Ritter, Chichester, England: Wiley-Blackwell, 2017.

WVC *Ludwig Wittgenstein and the Vienna Circle*. Conversations recorded by Friedrich Waismann, ed. B. McGuinness, tr. J. Schulte & B. McGuinness, Oxford: Blackwell, 1979.

Introduction

There has already been admirable scholarship on Wittgenstein's intermediate period. Setting the standard for scholarship on Wittgenstein's philosophy generally is P.M.S. Hacker's work, who devoted a chapter to Wittgenstein's intermediate period in his influential book *Insight and Illusion* (the revised edition). Hacker also touches on various parts of Wittgenstein's intermediate period thought, as well as some specifics of how his thought developed, in his brilliant commentary on Wittgenstein's *Philosophical Investigations* as well as in a few essays. This work has stood as the standard for understanding Wittgenstein's later philosophy, but it does not deal with the more technical topics in the philosophy of mathematics, or with many of the *details* of Wittgenstein's thought in the intermediate period. The first area has been subsequently investigated, beginning around the time of the *Analytical Commentary*, with the work of Stuart Shanker, Pasquale Frascolla, Mathieu Marion, and Victor Rodych. The works of Frascolla (1994) and Rodych (1997, 2000a, and 2008) have been *especially* important contributions to understanding Wittgenstein's intermediate philosophy of mathematics.

It is also noteworthy that there are few attempts to give a detailed analysis of the intermediate period of Wittgenstein's philosophy outside of the philosophy of mathematics. Until recently, other than what has already been mentioned, David Stern's *Wittgenstein on Mind and Language* was one of the more detailed accounts of this period. One of the few other works that dealt with Wittgenstein's intermediate period is Wolfgang Kienzler's *Wittgensteins Wende zu seiner Spätphilosophie 1930-1932: Eine Historische und Systematische Darstellung* [*Wittgenstein's Turn to his Later Philosophy 1930-1932: A Historical and Systematic Exposition*]. However, they both also either focused on specific parts of the intermediate period (e.g. Kienzler's analysis of the *Wiederaufnahme* ['resumption' or 'taking-up-again']) or relatively specific topics or themes in it. They did not, for the most part, try to trace the *details* of the development of the *major parts* of Wittgenstein's philosophy in the intermediate period, nor the interrelationships between the various elements of Wittgenstein's thought.

This changed with the work of Mauro Luiz Engelmann (2013), which importantly supplemented the aforementioned ones by giving a rigorous and detailed account of the development of Wittgenstein's thought from when he returned to philosophy in 1929 to the composition of the *Philosophical Investigations*. This included considerable original material, such as a detailed explanation of the phenomenological language, how this project, which was designed to preserve the *Tractatus*, transformed into the calculus conception of language, and how this in turn developed into the genetic method and anthropological viewpoint. Engelmann's work nicely blends the overall themes in the genesis of Wittgenstein's thought with the nuances of Wittgenstein's development, as well as relevant historical and scholarly details; for example, Engelmann provides a brilliant reinterpretation of the famous story involving Sraffa's Neopolitan gesture. Most importantly, Engelmann showed the significant continuity between the *Tractatus* and Wittgenstein's intermediate period. Indeed, in most, if not all respects, Wittgenstein's work in 1929 picks up right where he left off and moves by incremental steps into several distinct philosophies prior to his developed later philosophy in the *Philosophical Investigations*.

Even Engelmann admits (2013, 5), however, that he does not address the details of Wittgenstein's views in the philosophy of mathematics. And other scholars have noted the great importance of Wittgenstein's philosophy of mathematics to his intermediate period – and thus, insofar as insights live on, to his later work also. For example, Hacker[1] has written:

> To trace in detail the story of the change in views between 1929 and 1932/3 is a task for a book-length study. It would have to trace simultaneous developments on *many* fronts, noting how some lagged behind when Wittgenstein initially failed to realize the implications of some of his advances. And it would have to examine his extensive writings on the philosophy of mathematics in this period, for that work played an important role in the general change of his ideas. (Hacker 2001, 153)

Thus, this thesis is meant as a starting point to fill this gap in Wittgenstein scholarship. Where possible, the goal of this thesis will be to show any important connections and interrelations between Wittgenstein's philosophy of mathematics and other important developments in his thought. Most importantly, it will be shown that Wittgenstein's reflections on mathematics are the origin for two central ideas in his thought, namely verificationism and the family resemblance concept, as well as some less central insights; moreover, it will be seen that developments in other areas of his thought often have a correlate in the philosophy of mathematics, regardless of whether the order of genesis can be precisely determined. In addition, given the thesis' focus on the

[1] Frascolla also notes this (1994, 42-43).

philosophy of mathematics, where possible I have endeavoured to clarify further (from the work already mentioned) Wittgenstein's views in the philosophy of mathematics.

Chapter 1 sets the stage by examining Wittgenstein's philosophy of mathematics in the context of the philosophy of language and metaphysics in the *Tractatus*, insofar as this is necessary for understanding his views in the intermediate period. This includes a discussion of operation theory, which is developed as a way of giving a foundation to mathematics. Numbers are defined as the exponent of an operation, and Wittgenstein shows how arithmetical proofs can be reconstructed using this notation. Against this background, we then examine Wittgenstein's view on mathematical pseudo-propositions and logical propositions. Given its central role in the intermediate period, additional attention is given to his views on the concept of infinity, which concludes the chapter.

Chapter 2 begins by outlining Wittgenstein's new positions in the philosophy of mathematics. These changes include his new analysis of number, his comparison of arithmetic with geometry, and his rejection of a foundation for arithmetic. In the course of examining these topics, as well as comparing Wittgenstein's new view on numbers to his *Tractatus* view, we shall outline some main features of Wittgenstein's intermediate philosophy of mathematics, including: the autonomy of mathematics, the normative nature of mathematics, the idea of a grammatical rule, and his new emphasis upon the *numerals* (and rules) we actually use. The chapter then outlines the phenomenological project and its demise, and concludes with the most important continuities and discontinuities that characterize Wittgenstein's philosophy between the *Tractatus* and his intermediate period. The mathematical parts of this chapter are especially important for Chapters 5 and 6. The material on the phenomenological project is especially relevant for Chapter 3.

Chapter 3 outlines Wittgenstein's views on the verification principle. We begin by examining the early development of what would become the verification principle when Wittgenstein returned to doing philosophy in early 1929. After identifying an important point of continuity between the *Tractatus* and the verification principle, we examine this relationship in more detail, making use of additional relevant sources from 1929. This leads naturally to a discussion of the relationship between the development of the verification principle and the Vienna Circle. From here, we examine the details of how verificationism evolved against the background of the phenomenological language. Specifically, we shall see how the principle was used to preserve the application of language to the world, while bypassing the problem of specifying the shared form between language and the world. We then examine how the verification principle continued to be used by Wittgenstein to make logical distinctions in the philosophy of mathematics (even later into the intermediate period). This includes his use of it to delimit meaningful propositions, which is closely related to his views regarding conjectures, problems, and questions in mathematics. Using Rodych's work (2008) as a point of departure, we then consider the various contradictory elements in Wittgenstein's intermediate work, and, using insights from Wittgenstein's later work, assess the limitations of Rodych's characterization of Wittgenstein's mature verificationist position and Wittgenstein's developed intermediate period verificationist view itself. In the course of this examination, we see how and why Wittgenstein's use of the verification principle, in relation to his philosophy of mathematics, is restricted, and the consequences of this restriction.

Chapter 4 examines Wittgenstein's views on infinity. This material heavily relies on the verification principle. We shall begin by examining Wittgenstein's opposition to the extensional infinite, first considering his arguments as they relate to the empirical world and then considering his arguments as they relate to the *a priori* discipline of mathematics. We then turn to how the concept of the infinite relates to actuality and possibility before examining, in more detail, the confusions that underlie the extensional conception of the infinite. As we shall see, this serves as an early example of what is to become the 'genetic method'. We turn next to the *Tractatus*' influence on Wittgenstein's thought on the concept of the infinite in 1929, in this case focusing on a few select passages that exemplify a confusion in Wittgenstein's thought that is subsequently identified in 1931 when he re-evaluates this earlier position. We see how the elimination of this (general) confusion is used to reassess specific arguments Wittgenstein gave in the early part of the intermediate period (related to the infinite). The chapter concludes by outlining Wittgenstein's overall view of infinity in 1931 and its limitations.

Chapter 5 focuses on Wittgenstein's comments on inductive proof. This material is heavily dependent on Chapter 4. We begin by reviewing Wittgenstein's views on quantification and verification, and relate this to his views about proof-schemas. The stage will then be set by a brief discussion of Wittgenstein's interest in Skolem and inductive proof. We then present Skolem's proof of the associative law of addition as a particular example of an inductive proof. We then proceed to focus on some of the preliminary clarifications Wittgenstein makes about the proof, before examining his more specific comments about the variables within the proof. From these comments, largely meant to avoid possible confusions that could result from reflection on the proof, we proceed to investigate Wittgenstein's positive characterization of it. As we shall see, he characterizes the inductive proof as the general form of a proof, which captures the form any specific proof will take. In this way, it shows its infinite applicability by showing the possibility of the construction of an endless number of particular proofs. We then proceed to examine some of the additional examples of 'showing' Wittgenstein uses in the context of the philosophy of mathematics partly to shed light on his claims about inductive proof. These examples include the multiplication system, periodicity, and the relation between the general and the particular. On this basis, we then further examine Wittgenstein's contrast between inductive proofs and decision procedures, and explain his notion of 'generality'. We then briefly assess a debate about the extent to which Wittgenstein's work constitutes a refutation of Skolem's. We conclude the chapter by considering the influence of Wittgenstein's thoughts on inductive proof on his philosophy of mathematics and philosophy more generally.

Chapter 6 examines Wittgenstein's views on set theory. After presenting a brief history of set theory and some of its technical elements, we turn to how Cantor thought of his project in terms of its purpose, justification, and application(s). We then review the calculus/prose distinction, as well as general points about Wittgenstein's philosophy of mathematics, emphasizing elements of his position that are particularly pertinent to set theory. Wittgenstein's views about infinite sets (and the categorial divide between the infinite and finite), extensions and intensions, and numbers and the number line are then dealt with. With these topics in mind, and the possible confusions connected with them explained, we proceed to explore the different uses of 'description' Wittgenstein employed in relation to set theory, which helps to bring Wittgenstein's criticism of set theory to the fore. We then discuss Kienzler's and Sebastian Grève's claim about Wittgenstein's diminished use of the calculus/prose distinction in his later work. Their claim is made in the context of a discussion of Gödel's first incompleteness theorem proof, which naturally leads us to a comparison between Wittgenstein's treatment of Gödel's proof and his comments on set theory. We conclude the chapter by weighing in on a debate about the extent of Wittgenstein's criticisms of set theory, using the work of Ryan Dawson (2015) and Rodych (2000) as proponents of the 'descriptivist' and ‘revisionist’ positions, respectively. We conclude that a *proper understanding* of *what is correct in these positions*, in relation to the details of Wittgenstein’s intermediate period comments about set theory, reveals that the two positions emphasize two different, but *complementary*, components of Wittgenstein’s mature thinking on set theory (and the philosophy of mathematics more generally). With this understanding, it is apparent that the two positions are easily reconcilable with each other.

1. Wittgenstein's Philosophy of Mathematics: The *Tractatus*

An understanding of Wittgenstein's work in general, and his intermediate period in particular, is best set against the background of the *Tractatus*. For Wittgenstein's project in 1929, when he returns to philosophy, largely picks up where the former project left off, and, even in cases where Wittgenstein gives new emphasis to specific topics not dealt with in much detail in the *Tractatus*, or where he focuses on new topics, it is against the background of his views in the *Tractatus* that all of these (closely connected) developments are best understood. Of central importance to the evolution of Wittgenstein's intermediate thought was his philosophy of mathematics, which will be a major focus of this thesis. Thus, a brief review of the philosophy of the *Tractatus* sets the stage for a discussion of his early philosophy of mathematics, which includes an examination of operation theory and Wittgenstein's position on the nature of mathematical propositions. This is followed by a critical discussion of Frascolla's work as it relates to his discussion of the reductive nature of operation theory, as well as both Frascolla's work and Wittgenstein's as this relates to the 'superfluousness' of mathematics. The chapter concludes with a discussion of Wittgenstein's views on infinity in the *Tractatus*.

1.1 Operation Theory

In order to properly understand Wittgenstein's philosophy of mathematics in the *Tractatus*, it is necessary briefly to set the stage with a review of his general project in the *Tractatus*.[2] To explain how propositions describe, and thus how representation is possible, Wittgenstein likened a proposition to a picture (*TLP* 2.11). Elementary propositions are essentially bipolar. They are capable of being true or false, and it is their comparison with reality that determines their truth or falsity. Propositions limit the space of reality to two possibilities (which reality either fits or fails to fit). Just as pictures essentially represent by sharing their pictorial form with what they picture, any representation whatsoever represents by sharing its form with what it represents. Elementary propositions are compared with reality by observing whether reality is the way the proposition says it is (i.e., whether the objects are combined as the proposition

[2] I have consulted Baker and Hacker's *Language, Sense and Nonsense* (40-43) for a helpful overview of the views expressed in the *Tractatus*.

says they are). Corresponding to an elementary proposition in language is a state of affairs in the world. Both states of affairs and elementary propositions are logically independent (*TLP* 2.061, 2.062 and 4.211). From the truth or falsity of any elementary proposition or the obtaining or non-obtaining of any state of affairs, the truth (or falsity) of any proposition or obtaining (or non-obtaining) of any state-affairs does not logically follow. Elementary propositions consist of simple names combined. These names denote the simple, sempiternal objects in reality. The meaning of a name is the object it denotes. The combination of these names into elementary propositions is given by rules.[3] These rules, in order for representation to be possible, must reflect the same combinatorial possibilities as the objects in the world. This is a reflection of the shared logical form between language and the world (*TLP* 2.18).

Molecular propositions are formed from truth-functional combinations of elementary propositions. Their sense is identical to their truth possibilities. However, that there are molecular propositions presupposes the existence of elementary propositions containing simple names. For this is what ensures the determinacy of sense (that every well-formed proposition is true or false). Without elementary propositions, one proposition would always depend on the truth or falsity of another, which would lead to an infinite regress (*TLP* 2.0211). Analysis consists in decomposing molecular propositions into their elementary propositions, which will show that meaning has been given to all the signs. It is an important fact that Wittgenstein could not himself give an example of a single elementary proposition or simple name. To avoid conflation of empirical matters with the *a priori* method of the *Tractatus*, Wittgenstein left actual analysis to the 'application of logic' (*TLP* 5.557). Pseudo-propositions are malformed propositions in disguise. Upon analysis, the fact they do not conform to the *a priori* structure required of language will become evident. Hence, there are numerous propositions that are, strictly speaking, nonsense. They do not say anything (because they are malformed). But some[4] of these do *show* themselves to be ineffable truths in our representation. For example, 'red is a colour' is nonsense. For 'colour' is not a name, but rather a formal concept. It is represented by a variable of which any colour

[3] Although these rules are stipulated by us, we often will not be able to state what they are. Thus, we all regularly 'follow' a large number of rules the existence of which plays no role in our normative practices.
[4] Propositions that say nothing, but reveal themselves to be truths in our symbolism, fall into a number of different categories. For a helpful categorization, see Hacker (2001, 146-151).

name is a value. That red is a colour is shown by the combinatorial possibilities of red with other objects (Hacker 2001, 148).

We needn't go into all the details of Wittgenstein's technical developments of arithmetic within operation theory in the *Tractatus*.[5] The sum total of Wittgenstein's *Tractatus* philosophy of mathematics can be understood by citing two propositions, 6.01 and 6.02, and presenting them in relation to the overall philosophy of the *Tractatus*. 6.01 reads:

> Therefore the general form of an operation $\Omega'(\bar{\eta})$ is
>
> $$[\bar{\xi}, N(\bar{\xi})]'(\bar{\eta}) \; (=[\bar{\eta}, \bar{\xi}, N(\bar{\xi})]).$$
>
> This is the most general form of transition from one proposition to another.

6.02 reads:

> And this is how we arrive at numbers. I give the following definitions
>
> $$x = \Omega^{0\prime}x \text{ Def.,}$$
> $$\Omega'\Omega^{\nu\prime}x = \Omega^{\nu+1\prime}x \text{ Def.}$$

A detailed and excellent analysis of this notation can be found in the first chapter of Frascolla's *Wittgenstein's Philosophy of Mathematics*. Here, largely relying on his examination, I shall present an explanation of this notation and its philosophical significance within the context of the *Tractatus* insofar as it is relevant to my overall project.[6]

First it is necessary to explain Wittgenstein's notion of an operation – for which 'Ω' stands as a variable. The notion of a logical operation is conceived of in the following way. 'The operation is what has to be done to the one proposition in order to make the other out of it' (*TLP* 5.23).[7] An operation is a constant procedure for generating expressions from other expressions (often propositions). What the operation is applied to is called 'the base' and what is generated by application of the operation is

[5] I ignore the technical definitions and more complicated proofs that are all connected with Wittgenstein's expositions surrounding the function of the product of two numbers in the language of operation theory. See Frascolla (1994, 14-20).
[6] Therefore, I shall ignore Frascolla's perceptive criticisms of earlier scholars' work and just focus on his conclusions, which I consider largely correct. For further specifics, see Frascolla (1994, 2-8).
[7] 5.231 continues: 'And that will, of course, depend on their formal properties, on the internal similarity of their forms' (*TLP* 5.231). What is meant by this will become clearer through the explanations given in the subsequent paragraph.

called ‘the result’. Operations can be applied to their own results, starting from a given base, and thus a formal series is generated by an operation (Wittgenstein calls repeated applications of an operation ‘successive applications’ [*fortgesetzte Anwendung*]); the internal relation orders the members of the series. A given series would not be the particular series it is if not for the internal relation that constitutes the relationship between its members. A particular formal series or operation can be specified by a ‘variable’, as Wittgenstein explains in 5.2522:

> Accordingly I use the sign ‘[a, x, O′x]’ for the general term of the series of forms a, O′a, O′O′a, … .This bracketed expression is a variable: the first term of the bracketed expression is the beginning of the series of forms, the second is the form of a term x arbitrarily selected from the series, and the third is the form of the term that immediately follows x in the series.

Such a ‘variable’ specifies particular operations or formal series since it gives both the base and the way of generating any given term from an arbitrary term of the series. Hence, it gives everything necessary to produce the formal series and specify the operation.

‘Ω’ is ‘the symbol of the formal concept of an operation’ (Frascolla 1994, 8). That this is the case is shown by Frascolla by replacing ‘($\bar{\eta}$)’ in the formula in 6.01 with ‘($\bar{p}$)’ which denotes the class of elementary propositions. The resulting formula to the right of ‘=’ is the general form of a truth-function (and the general form of a proposition) as presented by Wittgenstein in proposition 6. Hence, as Marion notes, the general form of a proposition is a *particular case* of the general form of an ‘operation’ (Marion 1998, 21). 6.001 explains the general form of a proposition: ‘every proposition is a result of successive applications to elementary propositions of the operation N($\bar{\xi}$)’. This, in turn, means that ‘[$\bar{\xi}$, N($\bar{\xi}$)]′($\bar{\eta}$)’ indicates the procedure of successive applications of the operation of joint negation, in this case, applied to, as indicated by the ‘$\bar{\eta}$’, *any combination* of one or more propositions. At the same time this represents the general form of an operation for, as Frascolla says, ‘such an operation is conceived as a procedure to generate, from one or more given propositions, a proposition which is a truth-function of these; and whatever the procedure may be, the appropriate iteration of Sheffer stroke-operation will produce exactly that truth-function’ (1994, 2). Frascolla, I take it, means the operation of joint negation, which is the variant of the single connective used by Wittgenstein. A procedure or operation on one or more propositions will result in some specific truth-function and, since any truth-function can be produced by the N – joint negation – operator, the complex symbol in fact represents the general form of an operation.

Let's return to 6.02. I believe Frascolla is right to claim that Wittgenstein intends to define the endless expressions of the form '$\Omega^{0+1+1+1+\ldots+1}$'x' using an inductive definition on the number of occurrences of '+1' in the expression '0 + 1 + 1 … + 1'.[8] The definition itself must be viewed as taking place in a metalanguage which has the operation language as its object language (this, of course, makes sense of the variables). '0 + 1 + 1 + 1 + … + 1' is a term of the general theory of operations that corresponds to the ordinary algebraic expression. According to Frascolla, the definition has a reductionist aim: '0 + 1 + 1 … + 1' as an ordinary algebraic expression is to be derived from the corresponding operation term – what will be represented by the same number of '+1's in the exponent of an operation in operation theory. Whether or not this is best viewed as a reduction is discussed below (Section 1.3). The use of the variables has a tendency to mislead.[9] According to Frascolla's interpretation of the variable 'ν', it 'should be regarded as a schematic letter for an expression of the form 0 + 1 + 1 + … + 1' (1994, 6). The prime sign indicates the form of the result of an application of an operation to a given base. Such expressions are then appended to 'Ω' to 'represent a specific formal property common to the elements of a definite, wide class of linguistic (non-mathematical) constructs' (1994, 6). 'x' has the role here of representing the form of an expression that has not been generated by an operation (the 'base term'). Such expressions belong to the metalanguage and thus Wittgenstein requires the distinction between language and metalanguage as well as the notion of numbers included in the metalanguage (which, as Frascolla says, 'weakens the reductionist claim') (Frascolla 1994, 6).

It is possible to go into further technical detail with respect to proofs within operation theory. For our purposes, it is just necessary to highlight some of the main features of Wittgenstein's conception of the philosophy of mathematics within the *Tractatus*. First, as already noted, we can observe that mathematics deals with the

[8] It has been noted that the '+1's in the operation notation can be confusing since '+' is also used for the sum function. Since, in my own exposition, it is quite clear when exponents of an operation or addition are being referred to, I have not bothered adding the notation 'S0, SS0, etc.' as a replacement for the '0 + 1 + 1 … + 1' notation used in operation theory.

[9] This is a specific example of what could have led to misinterpretations by scholars such as Anscombe and Black, as briefly mentioned in footnote 6. See Frascolla (1994, 6).

shared form of a certain class of linguistic expressions. This form is *shown* through the operation language and cannot be, strictly speaking, said. For example, individual terms in operation theory show the shared form of a class of linguistic expressions. Their meaning[10] consists in their form, and forms themselves are not objects (in the *Tractarian* sense). Thus, equations, which consist of forms, are not themselves *sinnvoll* propositions, which picture states of affairs, since only propositions that consist of objects can do so. The *Bedeutung* of an arithmetical term, or, as Wittgenstein defines it, the arithmetical term as the exponent of an operation, is the property that is common to a shared class of linguistic expressions. 'The propositions of mathematics are equations, and therefore pseudo-propositions' (*TLP* 6.2). 'A proposition of mathematics does not express a thought' (*TLP* 6.21). Equations in operation theory, rather, indicate the synonymy of two linguistic expressions having the same form. They indicate the mutual transformability of two linguistic expressions (in terms of the transformation of groupings, as discussed above). Similarly, the equality sign does not actually assert anything. Wittgenstein says:

> It is impossible to *assert* the identity of meaning of two expressions. For in order to be able to assert anything about their meaning, I must know their meaning, and I cannot know their meaning without knowing whether what they mean is the same or different. (*TLP* 6.2322)

Thus, according to Frascolla, the pseudo-proposition equations of mathematics do not say anything, but facilitate the recognition of the synonymy of two forms. This is useful in complex cases where the synonymy of forms is not easily recognizable. In a perfectly regimented language, with an omniscient speaker such as God, there would be no need for the equations of mathematics since identity of forms could be immediately seen from the symbol alone (Frascolla 1994, 31-32).

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[10] It is apparent that the terms of operation theory, on either side of an equation, do indeed have meaning [*Bedeutung*] (*TLP* 6.232). It would appear that Wittgenstein, already at this stage, rejected parts of the word-object theory of meaning, even though he had not worked through the details in his philosophy of mathematics. One would have thought that operation symbols would have been treated the same as connectives, which were said not to denote.

1.2 Mathematical Pseudo-Propositions and Logical Propositions

Wittgenstein says: ‘The logic of the world, which is shown in tautologies by the propositions of logic, is shown in equations by mathematics’ (*TLP* 6.22). We can best understand this claim by contrasting how tautologies and the equations of mathematics show the logic of the world and by bringing to light both the similarities and differences between propositions of logic and equations of mathematics.

Let’s begin with what is meant by a proposition of logic or an equation of mathematics showing ‘the logic of the world’. The logical propositions are the limiting cases of propositions and are tautologies (hence, they receive the value ‘T’ under every interpretation; although Wittgenstein admits contradictions would serve the same purpose (*TLP* 6.1202)). They are molecular propositions formed from truth-functional combinations of elementary propositions that end up being true independently of how the world is (i.e. no matter what). Such propositions can be determined ‘from the symbol alone’, either by the truth-table method or by a process of derivation within the symbolic language. Logical propositions arise with the possibility of representation (with any language). For they are well-formed propositions, but, since they are not bipolar (i.e. capable of being true and false), they have, so-to-speak, zero sense[11]; they are limiting cases, but are still well-formed and not nonsense. Although logical propositions do not say anything (since they are not bipolar and thus can’t picture a state of affairs), logical propositions do *show* the internal relations between propositions. Tautologies show that they are so by their symbol alone and it is inconceivable they should be different (hence they express internal relations between their constituent propositions). Although they do not say anything, they do *show* the logic of the world since this is the shared form common to both language and the world that is necessary for the possibility of representation. ‘The fact that the propositions of logic are tautologies *shows* the formal – logical – properties of language and the world… If propositions are to yield a tautology when they are connected in a certain way, they must have certain structural properties. So their yielding a tautology when combined *in this way* shows that they possess these structural properties’ (*TLP* 6.12). ‘For example, the fact that the propositions ‘p’ and ‘~p’ in the combination ‘~(p·~p)’ yield a tautology shows that they contradict one another. The fact that the propositions ‘p ⊃ q’, ‘p’, and ‘q’, combined with one another in the form ‘((p ⊃ q) · (p):⊃:(q)’, yield

[11] Here I follow Hacker in his use of ‘zero sense’ (2001, 144).

a tautology shows that q follows from p and $p \supset q$... (*TLP* 6.1201). The various relations between propositions show themselves in our symbolism. These relations between propositions in turn reflect the logic of the world.

Arithmetical equations also show the logic of the world. For they too essentially deal with the formal properties of sign construction. Take, for example, the equation '$\Omega^{(3\times2)\prime}x = \Omega^{6\prime}x$'. This true arithmetical equation expressed in the language of operation theory expresses the mutual transformability of the two expressions: '$(\Omega'\Omega'\Omega')'(\Omega'\Omega'\Omega')'x$' and '$\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'x$'. The mutual transformability of these expressions is derived from the purely formal properties of sign construction as exemplified by operation theory. It is *inconceivable* that the triple application of the third iteration of an operation to a given base symbol would not result in the same proposition as five applications of the same operation to the result of the application of the same operation to the same initial symbol. Just as with tautologies, the truth of arithmetical equations can be seen from the symbol alone. Using conventional distinctions, we can say that both are known *a priori*. Wittgenstein says:

> And the possibility of proving the propositions of mathematics means simply that their correctness can be perceived without its being necessary that what they express should itself be compared with the facts in order to determine its correctness. (*TLP* 6.2321)

The truth of the above equation can be seen in the same way that the truth of the tautology $(p \supset q) \equiv (\sim q \supset \sim p)$ is. Both can be seen from the symbol alone, and in both cases it would be inconceivable that they should not be true. This is a feature of the expression of all formal relations for Wittgenstein. A world where these were not truths would be *impossible*. Both the notations for logic and mathematics are designed to show the relevant properties of linguistic expressions which can only be *shown* by language and not *said*. In logic this is facilitated by truth-tables or by the 'semi-mechanical' derivation of tautologies within an axiomatized system. The formulas of logic are constructed in order to display the forms of propositions including metalogical properties of propositions and logical relations between propositions. Similarly, arithmetical notation is developed to clearly show the forms of results of successive applications of operations. Arithmetical calculation, in turn, is similar to logical 'calculation': it is designed to clearly display the relation of synonymy between two expressions (once again facilitating the recognition of certain formal properties). Hence Wittgenstein says: ‘Mathematics is a logical method’ (*TLP* 6.2).[12]

The important difference between logical propositions and arithmetic equations must now be dealt with. Whereas, as explained, logical propositions are *senseless*, arithmetic equations are pseudo-propositions. Logical propositions, as already intimated, necessarily arise with any language since they are the limiting cases of genuine propositions. Arithmetic equations are, in contrast, Frascolla argues, actually superfluous. This use of ‘superfluous’ has *some* basis in Wittgenstein’s philosophy, although it also creates problems for Frascolla’s interpretation and, arguably, Wittgenstein’s own philosophy (discussed in the next section). According to Frascolla, arithmetic equations are necessary for beings like us who can’t always take in the identity of forms between two expressions of operation theory. This claim is further borne out by the symbols themselves employed in the respective domains of formal relations. Unlike ‘≡’, which itself belongs to language, ‘=’ expresses a metalogical relationship. ‘$(p \supset q) \equiv (\sim q \supset \sim p)$’ *shows* its metalogical status as a tautology even though ‘≡’ belongs to the language of logic. In contrast ‘=’ does not actually belong to language; it does seem to try to assert an identity of meaning, although, as we have already seen, this is not actually the case. Arithmetical expressions of operation theory doubtless show their forms, but equations themselves neither show nor state anything. They are used in facilitating recognition of synonymy of forms, but are not themselves expressions of the language. Thus, Frascolla claims, in a perfectly regimented language, employed by an ideal speaker such as God, equations would disappear since the identity of meaning of forms would be immediately evident. In contrast, tautologies are actually formed using ‘≡’; such tautologies are well-formed expressions of the language that also serve to *show* their metalogical properties (that two propositions have the same sense).

It is now useful to explain briefly Wittgenstein’s objection to logicism. Wittgenstein says: ‘The theory of classes is completely superfluous in mathematics. This is connected with the fact that the generality required in mathematics is not *accidental* generality’ (*TLP* 6.031). Wittgenstein did, at least retrospectively, see his account of mathematics as ‘foundational’ in nature, although it is clear, as will be further discussed, that he merely means by this the clarificatory reconstruction of some

[12] The interpretation of this quotation is supported further by the discussion below (Section 1.3).

elementary parts of mathematics using the *Tractatus* concept of an operation. This foundation is in contrast to the traditional logicist reduction where arithmetic would be reduced to logic (which includes set theory – also referred to as the 'theory of classes'). Wittgenstein rejects this possibility instead of rewriting mathematics in terms of propositional logic, he tried to show that both mathematics and propositional logic involve the fundamental concept of an operation. Moreover, he rejected logicism since at least some of the axioms of set theory are importantly *accidentally general*. The mark of logical or mathematical truth is *essential* general validity:

> The general validity of logic might be called essential, in contrast with the accidental general validity of such propositions as 'All men are mortal'. Propositions like Russell's 'axiom of reducibility' are not logical propositions, and this explains our feeling that, even if they were true, their truth could only be the result of fortunate accident. (*TLP* 6.1232)

It is not simply general validity that is the mark of a logical proposition (as Russell thought), but essential general validity – in contrast to accidental. That is, general validity is not enough, for a proposition such as 'all men are mortal' may be generally valid, but only so for a particular world. Logic, for Wittgenstein, importantly demarcates what are the necessary features of any possible world. For what breaks the rules of logic is an impossible world. So, connected with this, as explained in the subsequent proposition of the *Tractatus*, worlds can be envisioned where the axiom of reducibility or the axiom of infinity is not valid, thus making them at most accidentally valid.[13] This can be contrasted with the validity that is essentially general, as found in operation theory. As we have seen, an equation of arithmetic can be translated into the operation language and then transformed according to strict procedures to exhibit the synonymy of expressions on either side of the equality sign. The generality of an equation in operation theory is due to its purely formal nature. These forms are not objects but the shared properties of linguistic expressions and their possibility of symbolic transformation (the repeated application of operations). Hence these forms are independent of the actual configuration of objects in the world (recall *TLP* 6.2321). Thus the equations of mathematics, since they deal with forms that apply to general properties of symbols, are completely general (only dealing with 'bases' and the

[13] Our main focus in this chapter will be understanding the axiom of infinity, which Wittgenstein also doesn't think is a logical proposition. By the lights of the *Tractatus*, it would be a metaphysical truth.

application of general operations). Moreover, since they deal with the purely formal properties of symbolic construction, they are essentially valid since they in no way depend on contingent ways the world is.

Following Frascolla, we can note that the views of the *Tractatus* run contrary to a basic tenet of *both* Platonism and Constructivism. Platonism is the view that mathematical objects exist in their own realm independent of human practices. The job of the mathematician is thus to discover the relationships between mathematical objects as they are revealed through mathematical proofs. Constructivism, on the other hand, at least according to one of its standard definitions, is the view that mathematical objects are constituted by mathematical proof. The common tenet of both is that philosophy deals with *objects*. Wittgenstein's approach, as should already be clear, undermines this view. Mathematics deals with the forms of linguistic expression; 'he regards these forms as mere possibilities of symbolic construction, which cannot be treated as objects' (Frascolla 1994, 34). In this respect Wittgenstein undermines the views of both Platonism and Constructivism.[14]

1.3 Challenges to Frascolla and Wittgenstein: On Reductionism and Superfluousness

Frascolla sees Wittgenstein's project here as a type of reductionism. While, as he admits, Wittgenstein uses certain mathematical terms (e.g. numbers) in the operational definitions, Frascolla seems to maintain that Wittgenstein would have likely viewed these parts of the definition as part of the inescapable ineffable 'knowledge' of forms (Frascolla 1994, 6), which would serve to confirm his general view about forms (it would be understandable that not all parts of the definition would be reducible). Combined with his view regarding what exactly is meant by Wittgenstein thinking mathematical propositions are pseudo-propositions, he concludes that the project is a reductionist one.

However, it should be noted that there are other reasons not to think of this as a reduction. First, as Frascolla would admit, this would not have been considered a reduction by Wittgenstein in the sense of reducing mathematics to logic (including set

[14] This is not to say that any version of Constructivism is committed to this view. Arguably, Wittgenstein's later philosophy of mathematics is a version of Constructivism that avoids any commitment to mathematical objects.

theory). Wittgenstein was already critical of set theory. Moreover, in addition to the fact that it wouldn't actually achieve what is required of it (some arithmetical terms are needed in the language of operation theory), the project of reduction does not seem to fit into the overall project of the *Tractatus*. Wittgenstein's interest in operation theory as an explanation of mathematics is not best understood as an epistemological or metaphysical/ontological project. Unlike a project such as Russell's and Whitehead's, Wittgenstein is not interested in reducing one thing to another in order to ensure/prove its certainty. He is also, unlike Frege, not trying to establish the true meaning of numerals by identifying the objects they denote in logical terms. Instead, Wittgenstein is arguably interested in the *clarification* of mathematical terms. This clarification takes the form of a *reconstruction* of number terms and equations into operation theory. The use of operation theory clarifies the logic of number terms and the equations they appear in. It is by using operation theory that the theory of representation, as it relates to different domains, is unified, and the unique status of mathematics within Wittgenstein's overall early philosophy is made clear.

Wittgenstein's entire project is meant to show what is necessary for representation. The appeal for Wittgenstein of his account of mathematics is that it can explain the features of mathematics all within the symbolic/metaphysical framework already provided by the *Tractatus*. This would suggest that at most Wittgenstein is interested in reconstruction, since this shows precisely in what way mathematics can *already be understood as a kind of logic*.[15] Throughout the *Tractatus*, Wittgenstein advocates for different notations that shed light on the rules we already follow (in order to represent anything about the world). It seems apt then to think of operation theory itself as another (very general) notation that sheds light on the general rules that make possible mathematics and propositional logic. Wittgenstein arguably saw logic as a broader concept than, and one that includes, mathematics. There is a logic of concepts such as 'if… then', 'not', etc., but also one for the number concepts, '+', '=', etc. All of these concepts can be explained using the broad notion of 'logic', as this was explained in the *Tractatus*. And, most tellingly, in both cases this logic could be explicated using the technical notion of an operation. Thus, the idea of an operation serves as a unifying feature for everything from the general form of a proposition (what was at the very

[15] I owe this point, through to the explanation of the idea that logical concepts and mathematical ones can both be explained using a broad notion of 'logic', to Severin Schroeder.

heart of the project of the *Tractatus*) to the propositions of logic and the pseudo-propositions of mathematics. In this case, as with several others in the *Tractatus*, the philosophy of language of the *Tractatus* and/or the metaphysics found therein determines what is meaningful and what is not, and with this determines how/why mathematical propositions are arguably ‘pseudo-propositions’ (*TLP* 6.2).

While the logical operation has the advantage of explaining both logical propositions and mathematical equations within the framework of the *Tractatus*, it has the disadvantage, because of the metaphysical element upon which Wittgenstein’s explanation is dogmatically built, of trivializing mathematics. If all internal relations exist because of fundamental features of the symbolism, there would appear to be little work involved for the mathematician. At least, Wittgenstein does not explain this important question in any detail and, moreover, perhaps connected with this, chooses to only deal with a very small part of mathematics (arguably the area of mathematics where his account would be most intuitive).

There are a number of areas that could be filled out in more detail by Wittgenstein, but aren’t. Frascolla, as we have seen, attempts to explain some of these gaps with, to my mind, varying degrees of success. Wittgenstein, unlike Frascolla, makes little reference to God in the *Tractatus*, although it does make sense that God would be ascribed the ability of recognizing all internal relations (without the use of ‘pseudo-propositions’) and Wittgenstein does suggest that in a properly regimented notation there would be no need for mathematical equations (see below). But there is no explanation at all of how we could still do mathematics, in all of its richness, in this properly regimented notation, nor whether all of mathematics – understood as a plurality of proofs and techniques – can actually be understood within Wittgenstein’s framework.[16] Thus, it is unsurprising to find Wittgenstein reconsidering operation theory (discussed in Chapter 2).[17]

[16] Here is it apt to note, as Frascolla does, that Wittgenstein only attempts to account for a remarkably small part of mathematics: arithmetic equations. Thus, it is unclear exactly how the explanation in the *Tractatus* could serve as an explanation for all of mathematics. In addition to the reconsideration of the nature of mathematical equations in the early intermediate period, as we shall see in more detail in Chapters 3 and 5, Wittgenstein also comes to examine some of the diversity of mathematical practices.

[17] It is strange to what extent Frascolla’s explanation of propositions rely on God (something that plays little role in the *Tractatus* itself). It is unclear if Wittgenstein himself precisely thought that formal relations would be evident to God (in an extensional form, or any form), although, when considered from a historical philosophical perspective, it would make sense that a philosopher would give the role of that which sees all that shows itself to God (God is typically employed to represent an omniscient perspective and thus serves as an explanation in such cases). However, whether Wittgenstein thought precisely this is unclear. Moreover, I see no reason to think that God (or the ‘metaphysical subject’) must be understood

Frascolla's use of 'superfluous' is interesting for the additional reason that recent work on the *Tractatus* by Engelmann supports this choice of wording in particular. Engelmann emphasizes the possible use of this term when Wittgenstein explains the role of the nonsense [*unsinnig*] propositions of the *Tractatus*. These are the ones that make explicit (thus 'elucidating') the implicit rules we already follow in language (since they are necessary to all representation). The notations Wittgenstein employs in the *Tractatus*, Engelmann convincingly argues, are *not* nonsense (e.g. the general form of a proposition or the truth-table method). And, while the propositions that are used to explicate these rules can be seen as nonsense, Wittgenstein, even before writing the *Tractatus*, was instead inclined at times (*NB*, 28. 05.1915; *NB*, Appendix II: p. 110) to consider such 'propositions' as 'tautologous' or 'superfluous' – or as Engelmann also calls them: 'trivialit[ies]' (Engelmann 2013, 134). There are reasons to emphasize their 'nonsensicality', but also ones to emphasize their 'superfluousness'. This tension is again explored in the intermediate period by Wittgenstein – and expertly explained, with a likely resolution proposed, by Engelmann (Engelmann 2013, 131-139).

It seems that there is a tension in Frascolla's account of Wittgenstein's work (which doubtless arises from Wittgenstein's own work). In one sense, all of the pseudo-propositions in the *Tractatus* are 'superfluous' in the technical way already explained. On the other hand, understood in the ordinary way, mathematical propositions are anything but 'superfluous'. Indeed, under Frascolla's interpretation of Wittgenstein (which *is* charitable to Wittgenstein's views in the *Tractatus*), mathematical propositions would be quite essential for humans.[18] By any *normal sense* of 'superfluous', logical propositions should be considered superfluous (especially if

as grasping all internal relations *extensionally*. Wittgenstein's entire account, as Frascolla expertly outlines it, involves explaining arithmetic on the basis of an intensional characterization (this is arguably one of the most important aspects of the book). Granted this, I see no reason why Wittgenstein would backstep in this regard, even if it involves trying to give an explanation of how anyone/anything could understand all internal relations. This is especially the case considering that the details of how God would have a complete extensional knowledge of logical forms is just as opaque as (indeed logically impossible, and therefore possibly *more opaque than*) other possible explanations. Just to give one example: one could just as easily argue that God is able to instantaneously 'calculate' the identity of 'forms' (based on intensions) rather than that he sees their infinite extensions (which are logically impossible).

[18] Although not dealt with by Wittgenstein or Frascolla in any detail, it is not clear that mathematical pseudo-propositions and the pseudo-propositions of the *Tractatus* are in an identical situation. Part of this problem is based on Wittgenstein's lack of detail on the subject, and Frascolla's explanation does not wholly answer this point either (since he makes no distinction between the two). Thus, this label of 'superfluous' may not apply in any way; in the above, making sense of a way in which it could was attempted.

mathematical ones are). Indeed, according to the ordinary use, logical propositions should be paradigmatic examples of superfluous propositions (more so than mathematical ones). Frascolla's account of what makes a proposition 'superfluous' depends entirely on the (technical) notation in which it is expressed (and how it is interpreted), but it is unclear why arithmetic, even if not part of normal language, would not be implied by the analysis of number words (and ordinary language generally), and it is equally unclear why a tautology, expressed in ordinary language (or in a logical notation), would not be equally superfluous (e.g. 'A table is a table'). It is only reflection on language from the perspective of a dogmatic *a priori* logic that leads to the position that mathematical equations are 'superfluous'. The reconsideration of these themes in the intermediate period leads to a reorientation away from abstract operational definitions to the consideration of the signs (e.g. numerals), rules, proofs, and practices generally, as they are actually used. This is a reorientation that continues to develop throughout the intermediate period.

There is at least one reason to think Wittgenstein did not think mathematical pseudo-propositions were 'superfluous' in the ordinary way; and, even if he did hold this view, Severin Schroeder has convincingly argued that it must be mistaken. Regardless of whether Wittgenstein thought mathematical propositions were superfluous in the ordinary way (whether or not Frascolla's interpretation is correct), he arguably came to think of them as different from logical ones in the intermediate period, and likely did so by reflecting on their differences rather than their similarities. Wittgenstein does use 'superfluous' to describe the 'theory of classes' (*TLP* 6.031). This suggests everything in his notation (as an account of mathematics) is *not* superfluous. But even if he did think of mathematical propositions as superfluous, he arguably did come to stop comparing them to tautologies and instead compared them to meaningful propositions in the intermediate period, and likely did so on the basis of an awareness of their differences. As Schroeder has pointed out, in logic, proofs are not strictly required. In the propositional calculus the truth-table method can show a tautology to be one, and seemingly Wittgenstein thought that similar methods would be applicable to the predicate calculus. However, we could not get by without line-by-line proofs or calculations in mathematics (e.g. solving a quadratic equation). Moreover, as claimed in the *Tractatus*, logical propositions are not needed at all, but we could not get by without mathematical propositions. In order to work with quantities, we require arithmetical calculations, but logic is already implicit in language and does not need to be taught at all for people to be able to reason. Finally, whereas it can seem plausible that a tautology could be recognized from the symbol alone, there is less temptation to think this with respect to mathematical propositions. Conjectures in mathematics serve as an obvious example of something that isn't obviously true or false (Schroeder 2021, 36-37).

1.4 Infinity in the *Tractatus*

As a last important part of Wittgenstein's views in the *Tractatus*, we must examine his position on infinity. His view on the infinite in the philosophy of mathematics will be discussed before proceeding to consider his view on the axiom of infinity. Then, using the work of Anderson Luis Nakano, we will consider these views in relation to Wittgenstein's philosophy of language and metaphysics in the *Tractatus* generally.

Wittgenstein says: 'The concept of successive applications of an operation is equivalent to the concept "and so on"' (*TLP* 5.2523).[19] 'When the general form of operations is found we have also found the general form of the occurrence of the concept "and so on" (*NB*, 24.11.16). The concept of infinity for Wittgenstein is essentially a general procedure for generating relevant propositions belonging to a set. As explained above, it is given by a 'variable that represents the general term of the series of forms' (e.g. '[0, ξ, ξ+1]'). In this way it is a recursive definition that allows one to immediately 'take in' the entire infinite totality of the series. Such a definition will provide the base and the procedure for generating a term from an arbitrary term of the series. As Frascolla notes, and as should be obvious from the discussion up until now, such a rule is not a method for generating mathematical objects unlimitedly, since there are no such things in the *Tractatus* ontology. Rather, it is the theoretical possibility of generating an unlimited number of meaningful linguistic expressions from a given (base) linguistic expression.

In the *Tractatus* Wittgenstein briefly discusses the axiom of infinity. There he suggests that it is not a meaningful proposition at all and, instead, what it tries to assert is shown – in the technical sense – by a notation that contains an infinite number of names (*TLP* 5.535). This can be made sense of when one considers Wittgenstein's other

[19] In the *Notebooks* it is said: 'The concept "and so on" and the concept of the operation are equivalent' (5.25.23).

views in the *Tractatus*. Just prior to the above passages he calls into question the legitimacy of employing the identity sign (this has already been discussed in the context of mathematical equations). The problem can be roughly stated as follows: ‘to say of *two* things that they are identical is nonsense, and to say of *one* thing that it is identical with itself is to say nothing at all’ (*TLP* 5.5303). Connected with this, as already discussed, the identity sign can’t be used to *assert* anything at all. Instead, whatever the identity sign is used to try to correctly say (about objects) is actually shown in a proper notation by each object having a unique name (*TLP* 5.535). Various propositions, among them the axiom of infinity, require the identity sign to express and thus are called into question. Moreover, Wittgenstein argues that no meaningful proposition can express the number of objects there are; since ‘object’ is a formal concept and represented by a variable, a statement purporting to express the number of objects must treat the concept as a ‘proper’ one and thus produce nonsense (*TLP* 4.1272); the axiom of infinity, in this case, would just be an example of the general prohibition against doing this.[20]

A virtually identical claim about the number of objects being shown in the ‘language’ is made in the *Notebooks* (28.10.14). Prior to this, Wittgenstein suggests that ‘(∃x) x=x’ can be ‘investigated’ in place of the axiom of infinity (*NB*, 13.10.14); indeed, he even claims that all of the problems that are part and parcel with the axiom of infinity already occur in this other proposition (*NB*, 9. 10. 14). Shortly thereafter, on 13.10.14, Wittgenstein instead advocates investigating the logical formula that translates ‘there is a class with only one member’.[21] This was difficult to interpret, but the idea seems to be that the first formula could plausibly be read as a tautology (Wittgenstein will also subsequently challenge this),[22] while the second is clearly contingent (and thus a better comparison with the axiom of infinity).[23]

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[20] Of course, Wittgenstein even gives the example of ‘there are $\aleph_0$ objects’ in the aforementioned section of the *Tractatus*. It is clear that Wittgenstein did not always think of infinity as a number. However, given his own testimony in the intermediate period that he did *at least at times* do so in the early intermediate period (*PR* 305-306 – and what reason have we not to think a similar confusion did not occur in the *Tractatus*?), as well as how he describes the axiom of infinity in the relevant passages (including the aforementioned passage of the *Tractatus* and his discussion of the axiom specifically in a letter to Russell (*NB*, p. 128) – both passages which include the noteworthy symbol ‘$\aleph_0$’), it seems clear that he did in this context.

[21] The exact formula is ‘$(\exists\phi):.(\exists x):\phi x:\phi y.\phi z. \supset_{y,z}.y = z$’.

[22] Wittgenstein subsequently notes in the *Tractatus* (5.5352) that the negation of this proposition would equally be true if ‘nothing that existed were identical to itself’. Applied to the above proposition, this would immediately remove the appearance of its tautological character.

[23] I owe this point to Severin Schroeder.

In the *Notebooks*, although he does not discuss the idea in much detail, he argues that propositions about ‘infinite numbers’ are given by means of ‘finite signs’ (*NB*, 11.10.14). He even suggests that ‘infinite numbers’ are somehow ‘got’ by calculating the ‘signs’ – seemingly finite ones – themselves. So, while he does not go into much detail, it seems he is already hinting at the idea that discussion of the infinite is somehow a qualification of what is done with the finite. This discussion culminates in the claim that ‘It would be necessary to investigate the definitions of the cardinal numbers more exactly in order to understand the real sense of propositions like the Axiom of Infinity’ (*NB*, 12.10.14). Although far from a clear statement of what his view was to become in the intermediate period, it does make sense that one would look to how an infinite series is generated (and with that, the ‘finite signs’ that are used) in order to make sense of the infinite.

In a letter to Russell (in 1913 – clearly this is an earlier position), Wittgenstein took a different position by suggesting that the axiom is indeed a ‘proposition of physics’ to be determined by experience. Once again in this case, he likens the axiom to the proposition ‘(∃x) x=x’, which he now also argues is a ‘proposition of physics’ and not tautological because it is always a matter of experience whether something actually exists (he contrasts this with the ‘tautology’: ‘(x):x = x.⊃.(∃y).y = y’). Even though he argues that the axiom of infinity is a proposition of physics, he, in this case, seems aware that there is a problem with saying this since he adds: ‘and experience can’t decide it’ (*NB*, p. 128). Seemingly Wittgenstein’s problem, *even at this stage*, is with actually ‘verifying’ the proposition. As will be shown in Chapter 4, this position will be investigated by Wittgenstein in much more detail in the intermediate period and serve as an important basis for much of his other work in that period.

My claims here are supported by the work of Anderson Luis Nakano, who gives one of the best and most detailed accounts of infinity in the *Tractatus*. Given the technical nature of the discussion, a detailed examination of Nakano’s findings is beyond the scope of this chapter. However, a brief summary of Nakano’s position, insofar as it supports my findings in this chapter, is worth mentioning.

Nakano distinguishes between potential, actual, and possible infinities, and concludes that it appears that in general Wittgenstein was not *committed* to actual infinities in the *Tractatus*.[24] That is, while the application of logic could uncover that there are indeed actual infinities, few passages of the *Tractatus* would suggest that Wittgenstein was committed to such entities based on the *a priori* logic of the *Tractatus* (Nakano 2017, 168-172). Indeed, using Nakano's own distinctions, it is more apt to say that Wittgenstein thought that there were at most a few areas of his philosophy that admitted of being possible infinites (i.e., they are capable of being actually infinite, although they aren't necessarily so). As we have seen, this is exemplified by Wittgenstein's claims about the axiom of infinity and the possibility of the world being infinite (*TLP* 5.535 and 4.2211, respectively).

While Wittgenstein develops and maintains a consistent intensional characterization of the infinite for his philosophy of mathematics, a couple of passages suggest the necessity of an actual infinite in his philosophy of language. That is, these passages would suggest that an actual infinite is implied by the *a priori* logic of the *Tractatus*: 4.463 and 2.0131 (Nakano 2017, 171). However, these passages create tensions in Wittgenstein's thought. We needn't examine these passages in detail. The first involves the claim that 'logical space' is infinite, while the second claims that space is. The first suggests the need for an infinite number of propositions and, thus, following Nakano's reasoning, an infinite number of elementary propositions (both entail the other) (Nakano 2017, 166-167 and 171). The second suggests that space is infinite which, as Nakano can best make sense of it, requires an actual infinity of elementary propositions (Nakano 2017, 171). Even if these passages could be considered on the model of a possible infinity, this would require, based on Wittgenstein's technical explanations of 'world' and 'language', and the harmony between the two, a possible infinity of elementary propositions. And Nakano convincingly argues that there is no way to understand elementary propositions in this way (Nakano 2017, 172). Finally, it should be noted that Wittgenstein suggests the possibility of a potential infinite in certain domains of 'reality', namely with respect to the visual field and 'life without end' (*TLP* 2.0131 and 6.4311, respectively). Both examples indicate a lack of a limit and not an actual infinite (Nakano 2017, 171).

[24] As will be explained further in Chapter 4, a potential infinite is one in which members of a series can be continually constructed according to a rule. An actual infinity is one in which an infinity of objects/members exists as a totality. To these common distinctions, Nakano adds the idea of a 'possible infinite'. This refers to areas that could be an actual infinite (but also needn't be). I think this idea nicely captures an important element of Wittgenstein's philosophy at this time, although, as we will see, he will come to reject it (implicitly) in his intermediate period work.

However, once again, there is no way to make sense of this potential infinite in the terms of the *Tractatus*. Nonetheless, it is important to note that, while not developed in any detail, the idea that the potential infinite has applications to reality arises in Wittgenstein's early philosophy.

From all of this we can see that Wittgenstein's position in the philosophy of mathematics involves the potential (or intensional) infinite. In contrast, his position on language and the world allows for the possibility of actual infinities, although, except for a couple of comments, he does not appear to be committed to the idea that the world and language are actually infinite. Using Nakano's terminology, except for two comments that suggest otherwise, Wittgenstein is committed to the possible infinite when it comes to language and the world. And, as we have seen, those two comments are not definitive in any way, and create tensions with other parts of Wittgenstein's thought. Thus, as already suggested, in terms of Wittgenstein's positive account of the infinite, he seems to characterize it accurately as a potential infinite. However, in terms of his philosophy of language and metaphysics he does appear to allow for the possibility of the actual infinite. That is, he was not, in general, committed to this being the case, but allowed that it *could* be. Precisely to avoid having to give an account of the actual infinite, perhaps even because he already anticipated problems with doing so, Wittgenstein seemed to both be very tentative about whether the world was actually infinite (*TLP* 4.2211), and was even inclined to put the question off to the analysis of language (as suggested by the axiom of infinity). Assessing the truth of the axiom of infinity was left to the application of logic, and possible problems with assessing its truth were arguably brushed under the carpet by appealing to the saying/showing distinction. In this way, he put off having to think through all of the implications of this commitment to the intelligibility of the actual infinite.

We have examined Wittgenstein's use of operation theory as it is used in his philosophy of mathematics. We have seen that he can reconstruct arithmetic proofs in the language of operation theory thereby, as he likely saw it, giving arithmetic a foundation. We have also examined the similarities and differences between mathematical propositions and logical propositions. We were able to ascertain his general views about the philosophy of mathematics which, already at this stage, denied the idea that mathematics is descriptive or even that it is the study of objects. We pointed out some of the possible limitations of both Frascolla's interpretation of Wittgenstein's work and the work itself, as this related to whether operation theory involves reduction (in the case of Frascolla) and what could be meant by, and the limitations of, the idea that arithmetic equations are ‘superfluous’. Finally, in Wittgenstein’s account of infinity in the *Tractatus*, we saw that while Wittgenstein limited himself to an account of the potential infinite in mathematics, he does leave open the possibility of an actual infinite when it comes to the world and language, but at this time is clearly already somewhat aware of limitations of the idea of an actual infinite (at least in the world).

2. Transitioning from the *Tractatus*: Number, Arithmetic, Its Autonomy, and the Phenomenological Language and Its Demise

After having thought he solved all of the major problems of philosophy with the *Tractatus*, Wittgenstein returned to doing philosophy in 1929. Likely prompted by meetings with other philosophers, especially Ramsey and the Vienna Circle, Wittgenstein became aware that there were problems with his book.[25] Thus, his return to philosophy either involves trying to account for these problems (e.g. the development of the phenomenological language) or developing relatively new lines of thought (e.g. his discussion of number and arithmetic). In contrast to his work in the *Tractatus*, the intermediate period involves a much greater focus on the philosophy of mathematics. This is discussed in the introduction to the mathematical sections below. Discussion of Wittgenstein's philosophy of mathematics then continues with his new analysis of number, his comparison of arithmetic to geometry, a comparison of his views of number with the *Tractatus*, and his rejection of, and explanation for the uselessness of, giving a foundation to arithmetic. In the course of dealing with this, the idea of a grammatical rule, the normative nature of mathematics, the autonomy of mathematics, and Wittgenstein's new focus on the numerals (and rules) we actually use will be explained. The chapter concludes by explaining Wittgenstein's new phenomenological project and what caused its demise. It should be noted here that this is not a rigorous chronological account of Wittgenstein's work. Wittgenstein's mathematical insights discussed here begin at the time of his return to philosophy, but continue on into 1930.[26]

[25] Wittgenstein's meetings with members of the Vienna Circle began in 1927, although we only have documentation of their meetings from 1929 (Wrigley 1989, 270). Wittgenstein met with Ramsey long before this to discuss the *Tractatus*, and Wittgenstein would have been aware of Ramsey's criticisms because Ramsey wrote a critical notice about it in 1923. In the context of discussing the extensional infinite, Wittgenstein mentions, in early 1929, that, when speaking with Ramsey, he 'once [*einmal*] said', suggesting he is thinking of a discussion they had considerably earlier (likely in 1923 or 1924) (Marion 2019).

[26] Arguably, the development of Wittgenstein's thought is roughly the following. Wittgenstein first thinks of mathematics as being essentially rules (i.e. syntax). Geometry played some role in this realization and he already referred to this syntactical component as 'grammar' (MS 105, 53), although this is not his very first reference to the idea of 'grammar'; the idea is expressed even before the intermediate period (e.g. MS 104, 54) and he uses it also in relation to the phenomenological project (MS 105, 5). It then appears he realized that he needed to investigate the signs that we actually employ (and/or the rules that govern them). This is hinted at early in 1929 (MS 105, 115), but is made clearer later in 1929 (*WVC* 34-35). It is around September of 1929 that Wittgenstein comes to realize that a law of which no one is aware is not a law (MS 107, 117). This would make sense given it would be closely related to, or even involved in, the decision to investigate the rules that actually govern the use of the numerals we actually use. Around this same time (i.e., later in 1929), Wittgenstein thinks of any calculation in arithmetic as being an 'application of itself' (MS 107, 180), and thinks this more generally about mathematics (*WVC* 34). At that time, he also comes to question the idea of giving arithmetic a foundation (MS 107, 180). Then, early in 1930, he comes to think of arithmetic as being (explicitly) the 'grammar of number' and its application

In contrast to his mathematical ideas, the phenomenological language is given up by Wittgenstein by October 1929. The purpose of this chapter is not to focus on the details of these developments in Wittgenstein's philosophy, but rather to lay out some of the most important changes in Wittgenstein's thought, especially as these relate to what was discussed in Chapter 1 and what is necessary to the rest of the thesis.

2.1 Wittgenstein's Interest in the Philosophy of Mathematics

The intermediate period in Wittgenstein's work contains a large number of notes on the philosophy of mathematics. The amount of space devoted to this topic stands in stark contrast to the *Tractatus* for, as we have seen, there the entire topic is explicated through a few propositions and definitions. It is unsurprising, of course, that when re-examining central theses of the linguistic theory of the *Tractatus*, Wittgenstein would also come to question theses in his philosophy of mathematics. However, in addition, Frascolla notes, the particularly large body of writings in the philosophy of mathematics in the intermediate period has both a possible 'external' and 'internal' explanation. The 'external' explanation consists in all the philosophically relevant mathematical work, particularly in the area of the foundations of mathematics, which was reaching its most developed form in the 1920's. So, for example, Brouwer's and Weyl's intuitionism and Hilbert's formalism (as well as Ramsey's modified version of logicism) presented competing accounts of the foundations of mathematics. Although not explicitly mentioned by Frascolla, potentially relevant to an understanding of the development of Wittgenstein's work in this respect was a lecture given by Brouwer in 1928 that Wittgenstein attended and in all likelihood was influenced by.[27] In addition to his frequent mention of theses and work by Brouwer, Weyl, Hilbert, and Skolem, typical topics of discussion that belong to an 'external' explanation include: the interpretation of generalized propositions, the problem of the consistency of an axiomatic system, the

as 'taking care of itself' (MS 108, 115-116), and labels arithmetic, like geometry, 'autonomous' (MS 108, 119). Of course, the developments of these ideas are interrelated, making it at times difficult to properly differentiate every thread in his thought. And a detailed investigation of these developments is beyond the scope of this chapter.

[27] Hacker is pessimistic about any (positive) influence Brouwer had on Wittgenstein (1986, 122). But other more recent scholarship has shown some definite points of convergence between their respective philosophies, including likely influences Brouwer had on Wittgenstein (including an influence on his mathematical verificationism). See, for example, Schroeder (2021, 38: n. 3) and Marion (2008).

status of a proof by complete induction, and the clarification of a real number.[28] To this list, Frascolla notes that Wittgenstein was influenced by the work of G.H. Hardy, an eminent Cambridge mathematician, who, although not directly involved in foundational research, expressed various philosophical positions.[29] Perhaps because Frascolla deems it to be too obvious to require mention, he omits that both Wittgenstein's clarification of the concept of infinity, and the very important role conversations with Ramsey had for Wittgenstein, can also be added to the list.[30]

For Frascolla, to the 'internal' explanation belongs Wittgenstein's linking of the concept of grammar and mathematics. It is this important connection that essentially explains Wittgenstein's interest in the philosophy of mathematics. According to Frascolla, the writings on the philosophy of mathematics are of special importance for three additional reasons. First, there are important themes that are examined in the intermediate period that are not examined again in comparable depth. Second, the clear antecedent to one of the principal theses of Wittgenstein's mature writings (i.e., 'the view of necessity derived from his rule-following considerations') is clearly anticipated by Wittgenstein's description of the relationship between the general and the particular within mathematics. Lastly, it is in the intermediate phase that Wittgenstein still *tries* to preserve the sharp distinction between a linguistic rule and that of a meaningful proposition (as developed in the *Tractatus*). For, just as in the *Tractatus*, where arithmetic shows relations between the common forms of linguistic expressions (the rules we implicitly follow), Wittgenstein now claims that arithmetic 'produces rules of grammar of the factual language which describes the outcomes of manipulations of arithmetical symbols, as well as the outcomes of certain operations – union, partition etc. – on classes of empirical objects (in the case of cardinal arithmetic)' (Frascolla 1994, 43). Despite his preservation attempts, it is in the intermediate phase when a new idea foreign to the *Tractatus* explanation emerges: that, in certain cases, the content of rules of grammar can be expressed in *meaningful propositions* (this, of course, in

[28] One may add to this list (what Frascolla also deals with in his own chapter) the topics of recursive algebra and set theory (Chapters 5 and 6 of this thesis, respectively).

[29] Wittgenstein's respect for Hardy and desire to correspond with him regarding philosophical matters is evidenced by, for example, the references to him in letters (e.g. *WC* 194) and occasional mention of Hardy in Wittgenstein's notes.

[30] These two things are related. For Wittgenstein mentions the importance of conversations with Ramsey in some of the earliest notes from this time and is troubled by questions about the concept of infinity which are posed by Ramsey (see, for example, MS 105, 4 and 23-25 and Kienzler (1997, 160-174), who analyzes the import of some of the earliest parts of the debate). Of course, Wittgenstein famously mentions the influence of discussions with Ramsey in the preface to the *Philosophical Investigations*.

contrast to the *Tractatus* conception, where any attempt to express such a thing results in nonsense). This idea is further discussed in Chapter 3.

2.2 The Analysis of Number

Wittgenstein, in the intermediate period, says that a sequence of strokes (e.g. '|||') 'carries out the function of a numeral' by serving as 'a paradigmatic representation of the class property of having three elements' (Frascolla 1994, 46). '|||' used as a numeral acquires the aforementioned property by definition, which, as Frascolla points out, corresponds to the *Tractatus* use of 'shows'.[31] The laying down of such definitions (i.e., the establishing of paradigms) is a necessary step in establishing what can and cannot be meaningfully said about a class to which the paradigmatic property belongs. This can be compared to establishing a colour grammar, where establishing the definition of the individual colours also establishes what can be meaningfully said about the colours. With such paradigms and the grammar that is established by them, basic elementary processes can be described as operations on symbolism. Such processes are not concerned with the physical properties of the sign constructions, nor with establishing inductive evidence for universal empirical hypotheses concerning the corresponding transformations on classes possessing the relevant numerical property. In addition, we can note that the adoption of paradigms and the laying down of a grammar for numerical terms leads to the adoption of rules for non-mathematical language. This can be explained in the following way.

When established as a paradigm, the sign (e.g. '|||') is defined as representing a certain property and thus becomes a symbol – in Wittgenstein's technical use of that term (i.e., a sign with meaning).[32] The meaning of the sign is established by all the rules for the sign given by the calculus to which it belongs. Hence, Wittgenstein says:

> The system of rules determining a calculus thereby determines the 'meaning' of its signs too. Put more strictly: The form and the rules of syntax are equivalent. So if I change the rules – seemingly supplement them, say – then I change the form, the meaning. (*PR* 178)

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[31] This is one example of several that are given in the intermediate period that illustrate the saying/showing distinction. In addition to what else is presented in this chapter, see Section 5.7.
[32] The contrast between a sign and symbol is examined repeatedly in Wittgenstein's work – e.g. *LL* 26-28.

The meaning of a mathematical sign is its position within the calculus. With the various other arithmetic properties established by the paradigm (e.g. that '||||' consists of two groups of '||') comes a grammatical rule such that any description employing these paradigms in factual language and not agreeing with the aforementioned arithmetical property will be senseless. So, for example, any empirical statement describing the union of two classes each containing two objects must result in the conclusion that the result is a class consisting of four objects. Any other result is senseless. This serves as an introduction to Wittgenstein's notion that arithmetic is autonomous; further explanation is provided by Wittgenstein's comparison of arithmetic to geometry, which will be explained next. As will become clear, one of the reasons for the comparison is to emphasize the autonomy of arithmetic by pointing out important similarities with geometry which itself is autonomous.[33] Whereas geometry is the grammar of space, arithmetic is the grammar of number. Wittgenstein does not go through great lengths to investigate geometry for its own sake, and, when commenting on geometry, he comes to conclusions that are applicable to other areas of mathematics (or mathematics generally) (e.g. *PR* 216-218; *WVC* 62). It is the comparison to arithmetic, or utility to the phenomenological project (see Engelmann 2013, 54-56), that he sees as most fruitful. The other specific investigation of geometry relates to contrasting it with the rules of the visual field (see, for example, *WVC* 55-59 and 61-63) or with actual figures/shapes/drawings in the world (*AL* 50-51). Once again, this is largely for the sake of making an important comparison, or, as in the latter case, is simply too brief to draw any additional conclusions. Nonetheless, arguments for thinking geometry is autonomous are clearly given (and mentioned in this chapter).

2.3 Arithmetic: A Comparison to Geometry

It is important to note that Wittgenstein chooses to speak of the arithmetic of strokes and not simply to speak of arithmetic as we normally understand it. I believe, in part, this serves to emphasize Wittgenstein's conception of arithmetic at this time generally, and to emphasize the similarities between arithmetic and geometry. Moreover, the development of the stroke notation is a natural progression from operation theory.

[33] Engelmann also first emphasizes the important idea of geometry as a grammatical system and then goes on to speak of specific properties of arithmetic (see Engelmann 2013, 54-60).

Stroke notation preserves the idea of a sign being manipulated in a mechanical way. In what follows, we will examine the stroke notation and Wittgenstein's views about it in more detail, and, with this, what is meant by the autonomy of arithmetic will also become clear.

Wittgenstein continually emphasizes that arithmetic (and mathematics more generally) is not the study of signs (of any kind). The stroke notation, what could equally be served by any particular signs, does not provide signs to be described or studied. Wittgenstein says:

> Let's remember that in mathematics, the signs themselves *do* mathematics, they don't describe it. The mathematical signs *are* like the beads of an abacus. And the beads are in space, and the investigation of the abacus is an investigation of space. (*PR* 186)

And in *Philosophical Grammar* he says:

> What arithmetic is concerned with is the schema ||||. – But does arithmetic talk about the lines I draw with pencil on paper? – Arithmetic doesn't talk about the lines, it *operates* with them. (*PG* 333)

Insofar as signs are used as paradigms and conventions are employed for those signs used as paradigms (e.g. the manipulation of signs to serve as examples of unions, partitions, etc. of sets), one does mathematics *with* the signs and one isn't describing anything (neither abstract objects conceived as meanings for the signs nor any properties of the signs themselves).[34] Wittgenstein repudiates this descriptive conception. As he says: calculation, the process of sign construction, 'is only a study of logical forms, of structures' (*PR* 133). This is further explained by an examination of Wittgenstein's conception of geometry and how it can be usefully compared to arithmetic.

Wittgenstein says:

> The point of the remark that arithmetic is a kind of geometry is simply that arithmetical constructions are autonomous like geometrical ones, and hence so to speak themselves guarantee their applicability.

[34] This is further supported by the following quotations from Wittgenstein's conversations with Waismann. 'Mathematics is always a machine, a calculus… A calculus is an abacus, a calculator, a calculating machine; it works by means of strokes, numerals, etc.' (*WVC* 106). 'What we find in books of mathematics is not a *description of something* but the thing itself. We *make* mathematics. Just as one speaks of "writing history" and "making history", mathematics can in a certain sense only be made' (*WVC* 34: n.1).

> For it must be possible to say of geometry, too, that it is its own application. (*PR* 132)[35]

With this general comment in mind, it is worthwhile to look some more at specific examples Wittgenstein gives. He says:

> 'the sum of the angles of a triangle is 180 degrees' means that if it doesn't appear to be 180 degrees when they are measured, I will assume there has been a mistake in the measurement. So the proposition is a postulate about the method of describing facts, and therefore a proposition of syntax. (*PG* 320)

Working with figures in geometry is not a study of the particular figure one is using; the point is not to describe features of any individual figures or figures in general. Rather, geometrical proofs serve a normative function: to stipulate what must be the case when certain determinate processes, correctly performed, are performed on a figure correctly described. So, using an example Frascolla gives, one can say: 'Necessarily, if the inner angles of a figure correctly identified as a triangle are correctly measured, then their sum is 180°' (Frascolla 1994, 51-52). This statement serves the function of a definition. It states that for any figure, if it is correctly described as a triangle and if its angles are correctly measured, then the sum of its angles must add up to 180°. This definition serves to exclude certain descriptions (of factual language) that claim a *triangle* has been measured and its interior angles have not added up to 180°. Such a statement is nonsense (i.e., prohibited by the definition already given). Any deviation from this definition shows that the figure identified is not a triangle or that there has been a mistake of measurement (cf. Frascolla 1994, 52). Hence, such a statement is clearly a rule of grammar, one belonging to geometry. It helps *stipulate* what is to count as a triangle. Thus, a particular geometrical proof does not serve as an empirical generalization about appropriately similar figures. Wittgenstein says:

> It wouldn't be possible for a doctor to examine *one* man and then conclude that what he had found in his case must also be true of every other. And if I now measure the angles of the triangle and add them, I can't in fact conclude that the sum of the angles in every other triangle will be the same. It is clear that the Euclidean proof can say nothing about the totality of triangles. A proof can't go beyond itself. (*PR* 152)

[35] Comments emphasizing the comparison are plentiful. One additional example is: 'You could say arithmetic is a kind of geometry; that is to say, what are constructions on paper in geometry are calculations (on paper) in arithmetic. – You could say it is a more general kind of geometry' (*BT* 550-551).

The reason for choosing the stroke notation should be apparent. It helps establish the fundamental similarity between geometry and arithmetic when it comes to the figures employed as paradigms and the manipulations of such figures.[36] In addition, Wittgenstein's likening of arithmetic to geometry should now be clear. The arithmetic of strokes doesn't deal with general properties of signs, empirical generalizations extrapolated from the examination of particular signs and their manipulation(s). Rather, it is from the strokes and their manipulations that a rule of grammar is established. The rule of grammar stipulates that any grouping of signs correctly identified as strokes, when certain manipulations are applied to them, will result in another specific grouping of strokes. It establishes, *by definition*, what must be the case when such-and-such strokes are transformed in such-and-such ways. The result of a specific sign manipulation is determined to be an essential property of that process, such that that result is given by definition. As Wittgenstein says, 'In the calculus process and result are equivalent to each other' (*PG* 457). Such a rule of grammar serves to rule out descriptive statements that do not accord with this (either what has been observed is not a stroke, or transformations on the strokes have not been properly done).

2.4 The Analysis of Number: A Comparison with the *Tractatus*

Wittgenstein's new approach to explaining numbers can be usefully contrasted with his approach in the *Tractatus*. This is importantly connected to more general developments in his philosophy in this period. Wittgenstein no longer seeks absolute definitions which will give the essence of what is defined through a set of necessary and sufficient conditions (see also footnote 26). Recall, it was impossible to *state* the essence of number or even 'three is a number' according to the lights of the *Tractatus*. Since 'number' is a formal concept and not a material one it is not represented by a predicate and thus it is impossible to state that three is a number, but the fact that something[37] is a

[36] Although I think this to be an accurate account of the development of Wittgenstein's thought, it should be noted that this particular comparison, which is one articulation of Wittgenstein's idea that there is a similarity between arithmetic and geometry, seems superficial and does not seem to carry on in Wittgenstein's thought. It has its origin in the idea that 'figure' is used in both constructions (geometry) and calculations (arithmetic).

[37] Of course, here 'something' is merely a placeholder for that which can be said to be 'represented' by numerals. That is, what is represented need not be *some thing*. That numbers are not objects of any kind is already a point emphasized by Wittgenstein in the *Tractatus*. It is how he argues for this, as his thought evolves, that changes.

number and what this entails are *shown* by facts of the symbolism (that ‘number’ is correctly represented by a variable and all the combinatorial rules that this entails). More specifically, a number is the exponent of an operation and ‘number’ is the shared formal relation that holds between all numbers and is represented by a ‘variable’. As already explained, it only arises because of the existence of the general operation (even more generally, the general form of a proposition). Now, in contrast, Wittgenstein intends to investigate numbers through investigating the grammar of the terms that represent them. Hence, he says:

> I mean: numbers are what I represent in my language by number schemata.
>
> That is to say, I take (so to speak) the number schemata of the language as what I know, and say numbers are what these represent.
>
> This is what I once meant when I said, it is with the calculus [system of calculation] that numbers enter into logic. (*PR* 129)

A marginal note to this comment states: ‘Instead of a question of the definition of number, it’s only a question of the grammar of numerals’ (*PR* 129: n. 2). It is through investigating numerals and the grammatical rules that apply to them, that the correct understanding of numbers will become apparent. The same point is made in the *Philosophical Grammar* where Wittgenstein, in answer to the question ‘what is number?’ says ‘what numerals signify’. In the next comment he says: ‘What we are looking for is not a definition of the concept of number, but an exposition of the grammar of the word “number” and of the numerals’ (*PG* 321). Similarly, Wittgenstein emphasized to Waismann in conversation that ‘numbers are not represented by proxies; numbers are there. Only objects are represented by proxies’ (*WVC* 34). By this Wittgenstein is indicating that there isn’t anything outside the calculus to which the number terms must refer to have meaning. Numbers aren’t *merely* numerals, but there is nothing aside from their place in the calculus that determines their meaning. And their place is determined by the rules in which they belong. Moreover, mathematics does not talk about these signs, but instead operates with them. It is *by means of* these signs that we *do* mathematics. By emphasizing that we are not ‘talking about’ the signs themselves, nor anything to which the signs refer (they don’t refer), this part of Wittgenstein’s philosophy goes hand in hand with his more general view that mathematics is invented, not discovered. It is by means of the signs that we do mathematics. The idea that numbers aren't objects is an idea that is maintained into his later work.

2.5 The Autonomy of Arithmetic and the Uselessness of Giving it a Foundation

We are now in a position to properly understand Wittgenstein's views on the autonomy of arithmetic (and mathematics more generally – clearly, as already seen, geometry would also be considered to be autonomous). With this, we can understand the uselessness of giving it a foundation (albeit a non-reductive one), such as that proposed in the *Tractatus*.

One insight of the *Tractatus* was to claim that number terms have meaning, but only arise with the general operation. The general operation itself is necessary for molecular propositions and is thus at the very heart of the *Tractatus* with respect to the workings of language and the possibility of representation. It is the general operation applied to a 'base term' and its subsequent applications that give rise to numbers. Hence, any individual number is the exponent of an operation and the concept of number itself arises with the constant relationship generated by repeated applications of the general operation. Thus, numbers are *similar* to the logical connectives (cf. Engelmann 2013, 57), but not defined in terms of logical connectives. They are not found in elementary propositions,[38] but are the result of applications of the general operation to elementary propositions. It is the constant internal relationship generated by repeated applications of the general operation that is *the* essential element in the *a priori* definition that characterizes the concept of number. Moreover, thinks Engelmann, this insight makes some sense of the difficult remark in the *Tractatus*: 'Mathematics is a logical method' (*TLP* 6.2).[39] Both mathematics and logic arise from the general operation. The general operation makes possible the use of numbers in propositions. It is the theory of operations that makes possible inferences with (genuine)

[38] It is worth noting that numbers also could not be part of elementary propositions. For the appearance of number as a constituent in the proposition would entail other propositions (e.g. 'The rod is 3 meters' entails 'the rod is not 4 meters' etc.). That is, in most cases, if not all, where numbers appear in a genuine proposition, Wittgenstein would refer to it as a 'statement of degree'.

[39] This is an alternative reading to Frascolla's (and my own – discussed in Sections 1.2 and 1.3). This interpretation sees the actual reasoning with mathematical propositions as essential to the interpretation. Here it can merely be noted that these two interpretations are not difficult to reconcile. It would seem that if my elaboration of Frascolla's interpretation is correct, it allows for the possibility of making logical inferences on the basis of numbers occurring in empirical statements. Engelmann's interpretation is essentially given in an explanatory proposition (*TLP* 6.211) to that proposition.

propositions containing numbers. It is, argues Engelmann, in this sense that mathematics is a logical method, because, just like with the emergence of the connectives, genuine inferences can be made on the basis of numbers in empirical propositions. Moreover, it is the general form of the proposition that allows for the possibility of the application of arithmetic. Here 'application' means numbers can be used in propositions and this is facilitated by the general operation – what ultimately arises with the general form of a proposition (Engelmann 2013, 57). So, in the *Tractatus*, it is the *a priori* definition arising with the investigation of the essence of the proposition that itself entails that arithmetic can be applied.

In the intermediate period Wittgenstein argues against these views. As already mentioned, the search for *a priori* essential definitions is dispensed with. It is now necessary to make clear the rules of our language as they are actually employed; Wittgenstein's essential point becomes that mathematics itself consists of rules of syntax – equations in essence are rules that fix the meaning of the terms contained in them. This insight is essential to understanding the autonomy of arithmetic which depends on two important points: that the meanings of numerical terms are given by their place in the mathematical system and that this alone, in turn, allows for the possibility of arithmetic's application (Engelmann 2013, 56).[40] In the intermediate period, it is the place of numerical terms in the system of rules that fixes the meaning of those terms. It is the rules applied to the terms in contrast to other rules of the system that fix the specific meaning of any individual arithmetical term. Every numerical calculation, Wittgenstein tells us, can be considered an 'application of itself'. And this applies to mathematics more generally (*WVC* 34: n. 1). This emphasizes that the meaning of a calculus, much like the statements and terms making up the calculus itself, are entirely self-contained. Just as terms don't need to refer to have meaning, mathematical statements don't need to describe or correspond to anything, a point emphasized by thinking of these statements as grammatical rules. It is doubtful whether an 'application of itself' is actually best viewed as an application at all, but it is clear what Wittgenstein's *intention* is: the calculus endows meaning (upon itself) *exclusively* through its rules. That is, Wittgenstein, at this stage in his development, emphasizes the

[40] One additional quotation serves to help confirm the point in this context: 'One always has an aversion to giving arithmetic a foundation by saying something about its application. It appears firmly enough grounded in itself. And that of course derives from the fact that arithmetic is its own application' (*PR* 130).

purely intrasystemic, syntactical nature of mathematics. It is the primitive propositions together with the rules of the system that *wholly* define what is a meaningful proposition of the calculus (Rodych 1997, 199). This is in contrast to his *Tractatus* view, where an application outside of mathematics was not only the primary purpose of mathematics (*TLP* 6.211), but was guaranteed by the very metaphysical framework developed therein. By emphasizing that mathematics is an 'application of itself' Wittgenstein suggests that *all* that is important to meaning is the intrasystemic determination of meaning that is characteristic of mathematical rules. As we shall see (especially in Chapter 6), Wittgenstein, at least partly based upon his reflections on set theory, shall come to reassess this position later in the intermediate period.

Even though his own explanations in the *Tractatus* weren't definitions proper, as so characterized by Frascolla, his rejection of the possibility of a foundation for arithmetic in the intermediate period is meant to apply to any possibility of a foundation.[41] Wittgenstein says:

> Every mathematical calculation is an application of itself and only as such does it have a sense. *That* is why it isn't necessary to speak about the general form of logical operation when giving a foundation to arithmetic. (*PR* 130)
>
> On the one hand it seems to me that you can develop arithmetic completely autonomously and its application takes care of itself, since wherever it's applicable we may also apply it. On the other hand a nebulous introduction of the concept of number by means of the general form of operation – such as I gave – can't be what's needed. (*PR* 130-131)

There are several important points to note in these quotations. I believe Frascolla is right to claim that Wittgenstein is rejecting any possibility of a foundation for mathematics in the sense he attempted this in the *Tractatus*. This may not seem like the case with the first quotation where Wittgenstein seems to be maintaining that it is possible to give a foundation to arithmetic, just not with the theory of operations. Nonetheless, it is clear that Wittgenstein (ultimately) wishes to emphasize the autonomous nature of mathematics, something Frascolla doesn't mention in this context. Insofar as 'foundation' has any use in Wittgenstein's intermediate work

[41] Frascolla considers it not a proper mathematical definition because of its inductive character. A small amendment – for example, by using the lambda calculus – can avoid the inductive component and thus make it a legitimate mathematical definition. I am grateful to Frascolla for explaining his interpretation in greater detail to me.

onwards, it merely refers to clarification, but such clarification, in contrast to what Wittgenstein undertook in the *Tractatus*, is never dependent on another technical notation that, based on metaphysical insights, is meant to capture rules of which we are unaware. Thus, in his later work he never speaks of *giving* a foundation to mathematics, but merely uses ‘foundation of mathematics’ to refer to the uniquely conceptual questions related to mathematics, which are the province of philosophy. Wittgenstein moves away from the idea of giving a foundation to mathematics, where this means, through a special notation, making explicit the rules we follow, to the idea of describing or bringing out a foundation, where ‘foundation’ merely indicates our conceptual scheme.

With all of this mind, not only is the specific definition of number (i.e., the reconstruction of arithmetic) in the *Tractatus* pointless, but any definition is useless. Moreover, although these terms may be fixed, it is impossible to determine in advance whether any (additional) application of arithmetic (as Wittgenstein understands ‘application’ at this time) is possible or not. Arithmetic is simply applicable where it is applicable and no *a priori* justification can make it so. The meanings of its signs originate with their places in the calculus and it is unnecessary to explain them in terms of anything else (e.g. the general operation). This accords well with the development of Wittgenstein’s views in general. For it is already becoming clear to Wittgenstein that ‘definitions’ – or more generally, rules – that are not used to direct our normative practices can’t serve as real definitions. It would appear that it is on the basis of the view that mathematics consists of grammatical rules that Wittgenstein ultimately[42] realizes that there is no such thing as a law of which nobody is aware (*PR* 176). Viewing mathematics as a collection of practices and techniques essentially constituted by rules connects it with the idea of invention. Thus, there can be no such thing as *discovering* a rule in mathematics. And this makes it even more clear why the explanation of mathematics in the *Tractatus*, grounded in the general operation, can’t serve any normative function in our actual mathematical practices. Thus, Wittgenstein’s definitions that employ the general operation can’t serve as definitions of *the mathematics we actually employ*. Similar to problems about discovering the ‘real’ meanings of our ordinary words (as this is connected with philosophical analysis in the *Tractatus*), here we have the problem of discovering the ‘real’ meaning of numerals or

[42] This possibly relates to other intermediary steps in his thought. See footnote 26 for further details.

numbers. In contrast to this, in the intermediate period, Wittgenstein realizes that 'we understand and apply the propositions of arithmetic perfectly well without adding anything whatever to them' (*PR* 128). So, the definition of number given in the *Tractatus* is completely superfluous along with any foundation for arithmetic.

Moreover, all numbers, by the lights of the *Tractatus*, had to have something in common. It was a very general internal relation that characterizes the concept of number. Now Wittgenstein rejects this. There are many different things that all go under the name 'number' and they may not have any one thing in common. Instead, if numbers can be grouped together (e.g. natural numbers, integers, etc.) it is insofar as they comprise a system and they are defined by the rules that stipulate their meaning within that system. '[T]here are no gaps in mathematics' (*PR* 187). Since mathematics does not describe anything, it can't do so more or less accurately. If a mathematical system does not correspond to anything, there is no point of comparison for its completeness. That is, there is no way by which to evaluate whether it is complete or not. Thus, no system can be more or less complete. To be more precise: 'complete', insofar as it has meaning in relation to mathematical systems, can't be viewed on the empirical model.[43] Based on this, together with the aforementioned idea that mathematics essentially involves invention, it is wrong to think there could be gaps in mathematics. Instead, there are various systems more or less related to each other, and it is partly up to us whether we view the concepts in the diverse systems as importantly similar (enough to call them by the same name). Given this view, together with Wittgenstein's new focus on numerals and how they are actually used, it is natural for Wittgenstein to conclude that no *a priori* definition along the lines of that given in the *Tractatus* can capture the essence of number *in advance* of how we actually choose to use the concept (with the multiplicity of systems we actually use). And this contributes to the development of the view that mathematics consists of different systems as these are constituted by different rules.

Whether arithmetic has additional applications to those that determine the meaning of its signs can't be determined *a priori*; this is based upon the notion of the autonomy of arithmetic. The idea of the autonomy of a system will develop (in the

[43] Engelmann suggests that every system is 'complete' in the sense of 'closed'; that is, entirely constituted by its rules and never incomplete (as Engelmann says: 'the system of naturals is not completed by the system of negative numbers') (Engelmann 2013, 59-60). But, of course, this must actually mean that there is no meaning for 'incomplete' as this applies to mathematical systems.

philosophy of language) in relation to Wittgenstein's investigation of the phenomenological language (and its demise – both examined below). The idea of an autonomous system as it applies to mathematics develops into the idea that grammar itself, conceived of as a comprehensive discipline, is autonomous;[44] this is an important development in his intermediate period, then, for the autonomy (and arbitrariness) of – the more general – grammar is an important development in Wittgenstein's intermediate work.

2.6 The Phenomenological Project and Its Demise

Upon his return to philosophy, Wittgenstein's initial philosophical reflections largely[45] focused on Ramsey's criticism of this passage in the *Tractatus*:[46]

> For example, the simultaneous presence of two colours at the same place in the visual field is impossible, in fact logically impossible, since it is ruled out by the logical structure of colour.
>
> Let us think how this contradiction appears in physics: more or less as follows – a particle cannot have two velocities at the same time; that is to say, it cannot be in two places at the same time; that is to say, particles that are in different places at the same time cannot be identical.
>
> (It is clear that the logical product of two elementary propositions can neither be a tautology nor a contradiction. The statement that a point in the visual field has two different colors at the same time is a contradiction.) (*TLP* 6.3751)

The problem, which Ramsey already pointed out in his 'Critical Notice' of the *Tractatus*, is that two propositions asserting a certain point in the visual field has two different colours at the same time is a logical contradiction (e.g. 'A is blue and A is red') and, yet, this is not easily accounted for in the terms of the *Tractatus*.[47] It is

[44] Arguments already suggesting at least the arbitrariness, if not autonomy, of grammar – referring to language – are evident in the *Philosophical Remarks*. However, the exact term 'autonomy' is first used in relation to mathematics – beginning with laws (MS 107, 62), and then geometry and arithmetic (and the comparison between the two) (MS 108, 119). Hence, the calculus conception of language originates with the investigation of mathematical systems.

[45] I say 'largely' because several different problems exercised him at this time. As we have already partly seen, in the earliest notes from that time period he is already mentioning problems in the philosophy of mathematics, logic, and the philosophy of language.

[46] This part of the chapter all the way through to the explanation of the phenomenological language owes a great deal to Engelmann (*Wittgenstein's Philosophical Development* – Chapter 1). It would have been impossible to bring this kind of order to Wittgenstein's diverse and dispersed comments without his numerous insights.

[47] With respect to Wittgenstein's doctrine that all 'genuine' propositions assert something possible but not necessary, that the only necessity is expressed by tautologies and the only impossibility by contradictions, Ramsey actually says in the 'Critical Notice': 'There is great difficulty in holding this; for

implied by the above quotation that ‘A is blue’ and ‘A is red’ are not elementary propositions. For Wittgenstein states that their conjunction is a contradiction, yet no two elementary propositions can contradict each other since they are, by the lights of the *Tractatus*, logically independent. It is a mark of an elementary proposition, Wittgenstein says, that there can be no other elementary proposition that contradicts it (*TLP* 4.211). So, Wittgenstein, at the time of writing the *Tractatus*, did indeed believe that these propositions were complex. Although a proper analysis would reveal why their conjunction forms a contradiction, such an analysis did not concern Wittgenstein at the time of writing the *Tractatus*. The ‘method of logic’ (i.e., delineating the necessary *a priori* features that are required for representation), which was Wittgenstein’s concern in the *Tractatus*, need not concern itself with its own application,[48] so the matter of the actual analysis of these propositions could be left for the ‘application of logic’.[49]

Ramsey’s challenge to Wittgenstein amounted to the following: if the aforementioned propositions are complex, show the analysis of them. And if they aren’t, the threat to Wittgenstein’s account of logic and language given in the *Tractatus* would be obvious. For ‘statements of degree’,[50] as Wittgenstein calls them, seem not to be logically independent. Propositions such as ‘A is blue’ entail a number of other propositions including ‘A is not red’, ‘A is not green’, etc. The existence and necessity of elementary propositions for the possibility of representation generally was a linchpin of the *Tractatus*. In 1929, Wittgenstein did indeed try to meet Ramsey’s demand by trying to analyse colour statements by the methods available in the *Tractatus*. We needn’t concern ourselves with the details of his analysis here.[51] It is sufficient to say

Mr. Wittgenstein admits that a point in the visual field cannot be red and blue; and, indeed, otherwise, since he thinks induction has no logical basis, we should have no reason for thinking that we may not come upon a visual point which is both red and blue. Hence, he says that ‘This is both red and blue’ is a contradiction. This implies that the apparently simple concepts red, blue (supposing us to mean by those words absolutely specific shades) are really complex and formally incompatible’ (Ramsey 1923, 473).

[48] Wittgenstein did not concern himself with determining actual elementary propositions or simple names. Famously, he couldn’t give an example of an elementary proposition or simple name at all. It was enough that his *a priori* arguments showed there had to be such things and it was not of much concern to be able to provide examples. Such a matter would be empirical and thus not a part of his *a priori* method.

[49] Ramsey’s concerns were not limited to statements of colour. Statements involving space are also a concern.

[50] Statements of degree aren’t limited to colour statements. As Wittgenstein explains in ‘Some Remarks on Logical Form’: ‘[W]e are dealing with properties which admit of gradation, i.e. properties as a length of an interval, the pitch of a tone, the brightness or redness of a shade of colour etc.’ (166 -167). So, statements about length, pitch, tone, temperature (as Wittgenstein mentions immediately after this quotation) and colour can be seen as statements of degree.

[51] By way of a brief example: Wittgenstein initially, in ‘Some Remarks on Logical Form’, thought that statements of degree could be analysed as the logical product of ‘simple propositions of quantity’ with an

that early in 1929 he realized that any analysis of statements of degree – considered as complex statements – in terms of truth-functional analysis[52] was bound to fail.[53] With the failure of this analysis, Wittgenstein was indeed forced to say that there are elementary propositions that are not logically independent.

Accepting elementary propositions as not logically independent immediately led to additional problems. For it was evident that the notation of the *Tractatus* was insufficient for representing colour exclusion (and thus other statement of degree incompatibilities). The *Tractatus* would have to analyse the molecular statement ‘RPT and BPT’ (where ‘R’ and ‘B’ are different colours had by the point P at the time T) as follows:

RPT	BPT	RPT and BPT
T	T	T
T	F	F
F	T	F
F	F	F

additional clause stating ‘and nothing else’. However, this fails. ‘Even in the case that one could analyse a statement of degree in such a way that its entailments are shown to be implicitly contained within it, even the basic analysis of ‘A is blue’ is unsatisfactory (Engelmann 2013, 11). Wittgenstein considers the statement that the entity E has the brightness b, represented as E(b). Then if an entity were to have two degrees of brightness this would be represented as E(2b). But this should be analysable into E(b) ∧ E(b) which is simply E(b). If one tries to distinguish between the units of brightness and write E(2b)= E(b’) ∧ E(b’’), the notation for an entity with one degree of brightness would have to decide between the two (which is ‘absurd’ (SRLF 167-168)). Similar problems arise with explaining colour mixture and complementary colours, which also do not conform to truth-functional analysis.

52 It should be noted that Wittgenstein considers analysing statements of degree by the form of function and argument. Function and argument also aren’t sufficient (see *PR* 106-107). The contradiction does not show itself in the symbols (as used in the *Tractatus*).

53 The overall movement of Wittgenstein’s thought to this conclusion is clearly and concisely explained: ‘One’s first thought is that it’s incompatible for two colours to be in *one* place at the same time. The next is that two colours in one place simply combine to make another. But third comes the objection: how about the complementary colours? What do red and green make? Black perhaps? But do I then see green in the black colour? – But even apart from that: how about the mixed colours, e.g. mixtures of red and blue? These contain a greater or lesser element of red: what does that mean? It’s clear what it means to say that something *is red*: but that it *contains* more or less red? – And different degrees of red are incompatible with one another. Someone might perhaps imagine this being explained by supposing that certain small quantities of red added together would yield a specified degree of red. But in that case what does it mean if we say, for example, that five of these quantities of red are present? It cannot, of course, be a logical product of quantity no. 1 being present, and quantity no. 2 etc., up to 5; for how would these be distinguished from one another? Thus the proposition that 5 degrees of red are present can’t be analysed like this. Neither can I have a concluding proposition that this is all the red that is present in the colour: for there is no sense in saying that no more red is needed, since I can’t add quantities of red with the “and” of logic’ (*PR* 105).

Clearly the first line should not result in a 'T'. The alternative of putting an 'F' is no better, for this deviates from the rule of conjunction. Moreover, this does not show the *impossibility* of the first line as dictated by the rules governing space and colour.[54] Because the notation of the *Tractatus* fails to capture the 'logical multiplicity' of the phenomena, Wittgenstein searches out a complementary notation.

This complementary notation, called a 'phenomenological' or 'primary language', is a brief focus of Wittgenstein's in 1929.[55] Its purpose is to show perspicuously the rules of the phenomena in visual space. It is in the visual field that the 'forms' of space and colour meet and so this is the natural place to look for the answer to Ramsey's concerns.[56] The phenomenological notation is not concerned with truth or falsity, but rather with what can sensibly be said about the visual field (MS 105, 3).[57] A proper notation would accurately capture the 'forms' of colour and space. It would be 'the explicit notational presentation of the rules of the forms space, time, and colour that should be grasped in "the ultimate analysis of the phenomena" (SRLF 171)' (Engelmann 2013, 15). Such a notation is to contain no 'hypothetical addition'. It is the forms of space and colour as they relate in the visual field that is under investigation; it is the visual phenomena of colour that are of interest. Colours as physical or chemical phenomena are not of interest (since they would presuppose what is actually seen). Moreover, 'hypothetical addition' also refers to the possible forms elementary propositions might have. In the *Tractatus*, it was admitted that one could not say in advance what the specific form elementary propositions would take (other than that they consist of function and argument) (*TLP* 4.24). In 1929 Wittgenstein takes this further. Even the claim that, for example, elementary propositions must, as part of their *a priori* structure, contain functions and arguments generally is a distortion. As already noted, function and argument could not accurately represent the phenomena

[54] 'It is, of course, a deficiency of our notation that it does not prevent the formation of such nonsensical constructions, and a perfect notation will have to exclude such sentences by definite rules of syntax' (SRLF 170-171).

[55] He eventually decides (22 Oct. 1929) the phenomenological language is not possible and that even its purported goal that 'it would first say what we have to/want to express in philosophy is… absurd' (MS 107, 176; Engelmann's Translation).

[56] Ramsey was also concerned about showing the tautological nature of certain spatial rules. For example, the transitivity of 'between' also could not be accounted for in terms of the truth-functional operations of the *Tractatus*.

[57] Although the notation itself is meant to show the form of the visual field, what is actually necessary *to develop* the notation is at least partly an *a posteriori* investigation. See pp. 52-53 for further details.

(Engelmann 2013, 16). They too, at this point, are thus taken to be ‘hypothetical’ and are to be avoided.

Such a complementary notation should show the form of colour and space. It is noteworthy that Wittgenstein still uses the word ‘show’. For this indicates the new notation’s connection with the project of the *Tractatus*. For it *should* show these relationships in the same way logical propositions show their status by their representation in the truth-table or in the way a picture shows spatial relationships. It is also noteworthy – and something Engelmann does not spend time on – that there is now little (if any) talk of the ineffable truths of metaphysics. Wittgenstein now takes his investigation to be that of the form of the phenomena. Such a notation would perspicuously show the necessary colour exclusions, mixtures, incompatibilities, as well as the necessary spatial relationship (e.g. the transitivity of ‘between’). This involves looking to the phenomena themselves. In ‘Some Remarks on Logical Form’, Wittgenstein says:

> This task is very difficult [determining what the atomic propositions are], and Philosophy has hardly yet begun to tackle it at some points. What method have we for tackling it? The idea is to express in an appropriate symbolism what in ordinary language leads to endless misunderstandings. That is to say, where ordinary language disguises logical structure, where it allows the formation of pseudopropositions, where it uses one term in an infinity of different meanings, we must replace it by a symbolism which gives a clear picture of the logical structure, excludes pseudopropositions, and uses its terms unambiguously. Now we can only substitute a clear symbolism for the unprecise one by inspecting the phenomena which we want to describe, thus trying to understand their logical multiplicity. That is to say, we can only arrive at a correct analysis by, what might be called, the logical investigation of the phenomena themselves, i.e., in a certain sense *a posteriori*, and not by conjecturing about *a priori* possibilities. One is often tempted to ask from an *a priori* standpoint: What, after all, can be the only forms of atomic propositions, and to answer, e.g., subject-predicate and relational propositions with two or more terms further, perhaps, propositions relating predicates and relations to one another, and so on. But this, I believe, is mere playing with words. An atomic form cannot be foreseen. And it would be surprising if the actual phenomena had nothing more to teach us about their structure. (163-164)

It is by examining the phenomena themselves that we come to understand their form. For unlike in the *Tractatus*, where the substance of the world was the sempiternal objects, here the phenomena could be otherwise (or even not exist).[58] This characterizes the investigation as in some way *a posteriori*. For *a priori* forms given in the *Tractatus* are not adequate for their own application. Where the *Tractatus* had demarcated the *a*

[58] Hence, *TLP* 5.634: ‘This is connected with the fact that no part of our experience is at the same time *a priori*. Whatever we see could be other than it is’.

priori structure necessary for the possibility of representation, Wittgenstein now turns to the phenomena to see what can sensibly be said about the visual field.[59] The phenomenological notation can, in combination with the truth-functional analysis of the *Tractatus*, be used as a tool of analysis (in the spirit of the *Tractatus*). It can also function as a philosophical tool of clarification, for it can be used, as in the *Tractatus*, to show the real logical form of statements that are distorted by the clothing of language. As in the *Tractatus*, the meaning of words will ultimately derive from the connection the 'names',[60] in the fully analysed statements of the phenomenological language, have to the world.

We needn't go through all of the arguments concerning what can be sensibly said about the visual field here. It will suffice to briefly illustrate what Wittgenstein meant by the phenomenological notation. In 'Some Remarks on Logical Form' he gives an example of the phenomenological notation with a graph. The graph uses coordinates[61] to identify individual patches. These patches can have letters affixed to them to indicate their colour. As Engelmann goes on to explain, such a notation could be used to show the correct multiplicity by indicating that any given patch can't have two letters affixed to it (thus representing colour exclusion) (Engelmann 2013, 33). Although this is not pursued by Wittgenstein in detail, he also indicates that numbers could be affixed to the coordinates, thus indicating colour exclusion (MS 105, 70). Conceivably this notation could be extended to also represent colour mixture, complementary colours, and incompatible colours. Instead, Wittgenstein does introduce the colour octahedron which is designed to perspicuously show the internal relationships between colours (see *PR* 278 for the visual aid). This latter notation is itself not a phenomenological notation (which Wittgenstein thinks of as the graph and coordinates which presupposes the concept of direction and distance as forms of the visual field), but it is a perspicuous representation, for one can easily read the form of colour off of the octahedron. Of course, the octahedron does not include the form of space and thus could not constitute the form of the visual field or a method of analysis.

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[59] It should be noted in passing that this was not meant to be some sort of private language or sense-datum language. It was *not* to be the description of the form of *my* visual field, but of the *visual field* in general. Thus, neither private objects nor solipsism are concerns for Wittgenstein (Engelmann 2013, 19-20).

[60] In this case, 'names' could be represented by coordinates.

[61] With the restriction removed on elementary propositions having to be logically independent, it was no longer necessary to restrict numbers as being part of elementary propositions. For propositions including numbers would have also been statements of degree and, therefore, not elementary by the lights of the *Tractatus*.

Wittgenstein's project to construct a phenomenological language is short-lived. As with the *Tractatus*, the notation can't properly represent the phenomena. It is unnecessary to spend much time on the specific reasons for its failure, but sufficient to note the following.[62] In the visual field there is no way to represent the distinction between appearance and reality. For, by assumption, the notation is to represent how the phenomena are immediately perceived (with *no* hypothetical additions). Hence, there is no distinction in this notation between how something 'appears' and how it 'is'. But it is an obvious fact that something can appear other than it is. A line may appear to be one length, but upon inspection turn out to be another. Can such a distinction apply to the visual field? Engelmann chooses an important quotation:

> Should I now, nonetheless, say that in the visual field something can *appear* differently than it *is*? Certainly not! Or that a distance in the visual [*sic*] that once turns out to be n and once turns out to be n+1 turn out to be the same? Just as little that. (MS 107, 29; Engelmann's Translation, 36)[63]

n=n+1 cannot be a rule of the visual field (it is a contradiction). However, one equally can't apply the distinction of appearance and reality to the visual field. This would necessitate the 'hypothetical element' that is disallowed by the notation. For in order to say how something 'appears' one requires the 'hypothetical element' (an external standard) to determine how it actually is (its 'being'). We can fill out this argument a bit with the following.[64] Wittgenstein investigates the relationships between parts of a line and the lines themselves. Imagining *a* and *b* to be two lines that appear to be the same and *c* and *d* as parts of those lines that appear to be the same, Wittgenstein says that it is perfectly conceivable that one counts 24 units of *c* and 25 units of *d*, yet one would not be inclined to say anything other than *a*=*b* and *c*=*d*. By assumption Wittgenstein limits his example, for much of the discussion, to a number of parts that can't be taken in at a glance (MS 107, 29). If one can't see the difference in the *number of parts in a line* (for the line looks the same regardless), is one justified in speaking of 'number of parts of

[62] This part of the argument is particularly difficult, so I am indebted to Engelmann here.
[63] The manuscripts indicate that the omission is *Gesichtsraum*. For the sake of consistency (with Engelmann's translation), this can also be translated as 'visual *field*'.
[64] It would appear that Wittgenstein's argument begins by considering lines and the parts of lines. Thus, the above quotation already happens in this context. While Engelmann immediately starts talking about lengths of individual lines, I think this is justified given that he can consider the parts of the line to be the units used to measure the line. However, Engelmann's conclusion about not being able to speak of 'the same' seems premature given the arguments that follow. Otherwise, Engelmann's argument seems perfectly faithful to Wittgenstein's presentation.

the line' – in the ordinary sense – at all (MS 107, 29)? Wittgenstein tentatively suggests that there is no such thing as being wrong about the number one has counted and, more importantly, he suggests that not being able to see the number of parts *in the line* (*a* and *b*) means that there is no justification for concluding that *c* and *d* are the same number (MS 107, 29). Even never counting a different number of parts would not provide a justification for such a claim, for seeing a different number could still result in seeing the lines themselves as the same (MS 107, 29-30). Without a justification for being able to use the concept 'same', which Wittgenstein suggests undermines the idea of a criterion of identity for 'number of parts', he still wonders whether, if one is justified in thinking there are numbers of parts, one could be certain that what one is counting is really what one sees (MS 107, 30). From the fact that *a* can look the same regardless of whether it is made up of 24 or 25 parts (and, thus, he can't notice more or less parts), and that this change could even happen in an instance without his noticing, he tentatively concludes that what he is counting is not necessarily the number that he sees (MS 107, 30). Since what is seen is supposed to be represented by coordinates in the phenomenological language, this is an example of the indeterminacy of the sense data that will spell the demise of the phenomenological project. Much of this entire discussion is, as suggested, probing and open-minded, but the direction of his thought seems clear: describing the visual field in this way is difficult (MS 107, 30), if not impossible. His further reflections, and attempts at working around these problems, supports the latter conclusion.

The problem for representing the phenomena, which manifests itself in a variety of ways, is most generally: when describing the visual field one cannot appeal to the distinction between appearance and reality. But at the same time the phenomenological language requires this distinction already found in ordinary language. The words used to describe the immediate experience in the visual field are dependent on this distinction for their meaning. Wittgenstein says: 'We would need new concepts and we take always again the ones from the physical language' (MS 107, 163; Engelmann's Translation). When trying to describe the visual field, we naturally return to our everyday concepts, but denying the aforementioned distinction implies a change of the meaning of these concepts (e.g. 'same'), leading to paradoxical results. Wittgenstein tries other means of representing the 'inexactness of sense data' in the visual field. It is not just that he must accurately represent the rules of the visual field (using new concepts), but these rules must be translatable into the coordinate system at his disposal;

only then would it be a proper complementary notation (to the logic of the *Tractatus*). This too is impossible.[65] So, the phenomenological language, like the *Tractatus*, fails to properly represent the phenomena, and is given up.

With the demise of the phenomenological language, Wittgenstein gives up on the idea that notations have a fundamental role in philosophy (he still makes a place for some notations to represent rules perspicuously – e.g. the colour octahedron). Replacing the phenomenological notation is Wittgenstein's idea of 'grammar'. Grammar is syntax; all of the forms investigated with the phenomenological notation, together with all other theoretical philosophical investigations, should now be undertaken by examining the rules for the use of the words we actually follow. Moreover, mathematics too, as we have seen, is part of grammar. This unifying, comprehensive grammar preserves the goal of the phenomenological language (going back to the *Tractatus*): to delimit sense from nonsense.

2.7 Continuities with the *Tractatus*

As we have seen, Wittgenstein's intermediate project almost entirely picks up where the *Tractatus* leaves off. Both his philosophy of mathematics and his phenomenological language project largely involve reassessing his earlier views. Thus, most generally, there is continuity between Wittgenstein's early and intermediate work. Nonetheless, some points of continuity are short-lived and there are already, even in the early intermediate period, a few points of divergence with his earlier thought. In what follows, I shall examine some of the most important ideas that serve as continuities between Wittgenstein's early and intermediate work.

(i) Mathematics. Mathematics remains an *a priori* normative discipline. Mathematics does not describe, but rather stipulates rules which, when applicable, license the transformation of empirical statements. The elaboration of the idea of the general operation in the *Tractatus* was meant to ensure all of these elements. In the early intermediate period this is taken over by his idea that mathematics is syntax, and that it is autonomous.

[65] The inexactness that he is able to represent using the coordinate system is not the inexactness that is actually seen. See Engelmann (2013, 38-41).

(ii) Equations. Connected with this, Wittgenstein continues to think of equations (and proofs containing and employing them) as paradigm cases of mathematics (at least in relation to the areas of algebra and arithmetic – geometry is now, unlike in the *Tractatus*, given some discussion).[66] As we shall see in Chapter 5, it is against this assumption that his investigation of the philosophy of mathematics develops.

(iii) Saying/Showing Distinction. As we have already seen, Wittgenstein continues to maintain the saying/showing distinction with the phenomenological language. It is with his movement to the idea of grammar that this idea will lose importance for him in the philosophy of language. In the philosophy of mathematics, the idea of 'showing' is connected to any internal relation or form.

(iv) Logical Form and Logical Concept. Connected with the last distinction, Wittgenstein continues to speak of 'form' in the intermediate period. This has already been clearly shown with the phenomenological project. Applications of this term to mathematics are dealt with in more detail in Chapter 6. The 'variable' of the *Tractatus* reappears as a 'logical concept' that serves as a rule for the construction of a (potentially) infinite series. The axiom of infinity is also referred to as a 'logical concept'.

(v) Potential Infinite. Wittgenstein continues to make reference to the potential infinite in the philosophy of mathematics and now elaborates this notion so that it takes on the role of the only legitimate form of the infinite. This is extensively discussed in Chapter 4.

2.8 Discontinuities with the *Tractatus*

While overall Wittgenstein's early and intermediate work exemplifies considerable continuity, there are several ideas that serve as new directions in his thought. These are ideas that either occupied his thought right upon his return to philosophy (likely because of discussions he already had about them earlier – e.g. infinity) or ones that emerged as a consequence of his rapidly developing philosophy (e.g. giving up trying to represent the form of the phenomena). In what follows I shall outline some of the biggest, or most pertinent (for our purposes), discontinuities between Wittgenstein's early and intermediate work. I shall begin with those that have already been suggested

[66] Wrigley (1993, 76-77) also notes this continuity.

with the work in this chapter, and finish with three discontinuities that either play a central role in the development of his philosophy generally, or are important for understanding the developments that will be subsequently discussed in later chapters.

(i) Numerals and Stroke Notation. In the philosophy of mathematics, Wittgenstein gives up investigating the general operation as a means of understanding numbers. As we have seen, he develops a ‘stroke notation’ that is meant to emphasize certain properties of numerals (while, arguably, maintaining some continuity with the general operation notation and creating a point of comparison with geometry). It would appear that his focus on numerals is a consequence of his new focus on rules – specifically, as they are now conceived, the rules we actually follow when we do mathematics. This is, of course, connected to his more detailed views about mathematics generally: that it is essentially invention and characterized by rules. Once his interest in the phenomenological language ends, Wittgenstein correspondingly begins to investigate rules for the use of words. Moreover, Wittgenstein applies ideas and elements from his analysis of the philosophy of mathematics to the philosophy of language (e.g. ‘calculus’, ‘system’, and ‘autonomy’).

(ii) Demise of the Phenomenological Language. In the philosophy of language, Wittgenstein, upon his return to philosophy, tries to answer Ramsey’s criticisms and realizes he can’t. Relatively early into the intermediate period, then, Wittgenstein comes to give up the idea of representing the form of the phenomena. With this, his project in the philosophy of language will change to conceive of language as a calculus; unlike in the *Tractatus*, the rules that make up this calculus can be expressed in ordinary language and Wittgenstein conceives of the perspicuous representation of the rules as the central task of philosophy.

(iii) Application of Mathematics. Wittgenstein’s ideas surrounding application change in the early intermediate period in relation to his philosophy of mathematics. In the *Tractatus*, it was Wittgenstein’s entire philosophy of language and metaphysics that guaranteed the application of language to the world (explaining the possibility of representation was at the essence of the project of the *Tractatus*). It is apparent that the general operation importantly contributed to this in the philosophy of mathematics. Moreover, Wittgenstein insists that it is in the use of mathematical propositions to make inferences among empirical propositions where the true value of mathematics lies (*TLP* 6.211). Thus, this extra-mathematical application (what Rodych (2000) refers to as an ‘extrasystemic application’) is already essential in the *Tractatus*. As we have seen, in

the early intermediate period, Wittgenstein insists on a mathematical system's autonomy (e.g. arithmetic and geometry). The rules for these disciplines are self-contained and not logically dependent on the world. They are rules of syntax. At least in terms of his philosophy of mathematics, he will come to think that mathematics does not require an application to be meaningful. However, as we shall see in more detail in Chapter 3 and Chapter 6, after further reflection, Wittgenstein will return to consider once again the possible application of a mathematical calculus as an essential component of a paradigmatic mathematical calculus.

(iv) Meaningful Mathematical Propositions. Schroeder has rightly pointed out that Wittgenstein, in the early intermediate period, now compares mathematical propositions (equations) to meaningful propositions (instead of tautologies) (Schroeder 2021, 37). It is on the basis of this (*PR* 142) that arguably Wittgenstein also investigates the verification principle in relation to mathematical propositions (discussed in much more detail in Chapter 3). This point thus has important ramifications for the development of his philosophy.[67]

(v) Verification. Although the concept of 'verification' expresses an idea that can be traced back to the *Tractatus* (i.e., the comparison of a proposition with reality), the actual use of the term, together with the doctrine of verificationism itself (which develops alongside other parts of Wittgenstein's intermediate period thought), is only employed in the intermediate period. The evolution and use of the verification principle is a central theme in Wittgenstein's intermediate period. This is extensively discussed in Chapter 3.

(vi) Actual Infinite. Wittgenstein immediately starts investigating the intelligibility of the actual infinite. This makes up some of his earliest material in 1929, and is relatively self-contained (prompted, at least in part, by discussions he had with Ramsey even (likely) earlier). He will investigate this again later in the intermediate period, slightly adjusting earlier comments he made on this subject. All of this is extensively examined in Chapter 4. The result of his investigation is a complete

[67] As we have seen (Section 2.1), Frascolla also notes that Wittgenstein now thinks that some grammatical rules can be expressed in meaningful propositions. However, this arguably happens not only as a result of the application of the verification principle to mathematics (as Frascolla suggests), but actually anticipates Wittgenstein's application of the verification principle to mathematics. This is supported by how early in the intermediate period Wittgenstein suggests mathematical equations should be compared to meaningful propositions (MS 106, 172-174). Thus, it was likely this realization that actually *encouraged* his use of the verification principle to the philosophy of mathematics (there would be no point in applying the verification principle to the philosophy of mathematics if he still viewed mathematical propositions as meaningless).

repudiation of the actual infinite. Wittgenstein identifies this as one of the important changes in his thought when discussing the first four propositions of the *Tractatus* with his student Desmond Lee. *TLP* 1.12 states: 'For the totality of facts determines both what is the case, and also all that is not the case'. When discussing this proposition Wittgenstein is reported as saying:

> This is connected with the idea that there are elementary propositions, each describing an atomic fact, into which all propositions can be analysed. This is an erroneous idea. It arises from two sources. (1) Treating infinite as a number, and supposing there can be an infinite number of propositions. (*LL* 119)

In line with the interpretation I provided in Chapter 1, this suggests that Wittgenstein thought that there *could* be an infinite number of propositions, but not that there necessarily was. This is an especially clear example of Wittgenstein thinking that the infinite could be understood on the model of the finite (as a quantity). That is, he merely thought that infinite could be understood this way and assumed the details of exactly how could be worked out (upon logical analysis). Wittgenstein's examination of this assumption in his early intermediate period, and his development and application of these ideas to other areas in his thought, constitutes a major discontinuity between his early and intermediate thought. The applications of this development in Wittgenstein's thought as they developed in relation to inductive proof and set theory are examined in Chapters 5 and 6, respectively.

(vii) Family Resemblance. The family resemblance concept seems to be first mentioned later in the intermediate period in relation to Spengler's work (although it does not appear to originate from Spengler's work)[68] (MS, 111, 119). However, we see ideas in the direction of the family resemblance concept early upon Wittgenstein's return. For example, already in 1930, he notices that different things are referred to by the concepts 'question', 'problem', 'investigation', 'discovery', 'inference', 'proposition', and 'proof' (*PR* 190). This is a ways off from the actual family resemblance concept, but we can begin to see, through its further development in *The Big Typescript* and *Philosophical Grammar*, how this notion takes shape in Wittgenstein's thought. With the inclusion of 'inference' and 'proof' in the

[68] See William J. DeAngelis (2007, 21) for a particularly clear statement of this point. For an overview of his position, see his first chapter from that same work, which is entitled 'Spengler's Influence on Wittgenstein: A First Approximation'.

aforementioned passage, it makes sense to think that Wittgenstein’s investigation of inductive proof played a decisive role in this. This is more extensively discussed in Chapter 5.

In this chapter we have examined Wittgenstein’s new more extensive interest in the philosophy of mathematics, his new analysis of number and his comparison of arithmetic and geometry, and compared his analysis of number to that given in the *Tractatus*. We concluded the sections on mathematics by considering his idea that arithmetic is autonomous and the uselessness of giving it a foundation. In the course of doing this, his view that mathematics is a normative activity, the idea of a grammatical rule, the idea of an autonomous system of mathematics, and Wittgenstein’s new focus on the numerals and rules we actually use and follow were discussed. We concluded by investigating Wittgenstein’s phenomenological language project and what led to its demise. An understanding of Wittgenstein’s views in the philosophy of mathematics will be especially important to Chapters 5 and 6. An understanding of the phenomenological project will be central to Chapter 3.

3. The Principle of Verification

Although short-lived in terms of its central influence on Wittgenstein's thought, the verification principle occupies such a position in the early part of the intermediate period. We shall begin by examining how early ideas that anticipated the principle were used in the philosophy of mathematics shortly after Wittgenstein returned to doing philosophy in early 1929. This is enough to reveal an important continuity with the *Tractatus*, which is then investigated further. A brief historical digression, in which the Vienna Circle's relationship to the principle is examined, will follow. We then look into whether verificationism was 'implicit' in the *Tractatus*, using the work of Michael Wrigley as our point of departure. We shall then examine how the principle developed (in relation to empirical propositions) against the background of the phenomenological language and its demise, together with the newly employed comprehensive grammar. We then see how the verification principle continued to be used to make logical distinctions in the philosophy of mathematics (delimiting meaningful propositions, conjectures, problems, and questions) later in the intermediate period. Finally, using Rodych's work (2008), we shall explore the various contradictory elements in Wittgenstein's intermediate work and, using insights from his later work, assess the limitations of Rodych's characterization of Wittgenstein's mature verificationist position and Wittgenstein's developed intermediate period verificationist view itself.

3.1.1 Early Development of the Verification Principle

The very term 'verification', as it relates to Wittgenstein's views about meaning at this time (i.e., an early version of what will develop into verificationism[69] itself), is first used in the manuscripts (likely in the summer or September of 1929). This predates the *written record* of Wittgenstein's discussion of verificationism with members of the Vienna Circle in December 1929 (*WVC* 47). The earliest mention of the idea involves

[69] Here I take 'verificationism' to be the clear statement of the principle of verification; that is, something along the lines of 'the meaning of a statement is its method of verification'. Prior to its explicit statement, it is difficult to ascertain exactly how committed to verificationism Wittgenstein actually was, since there are clear uses of the term 'verification' that are not appeals to verificationism and others that, while related, are not a *clear* application of the formulated principle (e.g. with the use of 'construction' in the quoted passage below – MS 105, 8-10). It is also, in the earlier stages, not apparent whether Wittgenstein would formulate the principle to apply to any proposition, since the practical applications of the principle, in the early stages, are almost exclusively focused on the philosophy of mathematics.

contrasting empirical generality with mathematical generality (i.e., the different uses of 'all') (MS 105, 8). More specifically, Wittgenstein indicates that one could mistake the type of generality being employed if one does not realize that this type of generality consists of a verification 'in the internal way' [*auf dem internen Weg*]; the verification takes place with a calculation (although seemingly for Wittgenstein a proof would also count – at the very least, this is how his position will develop).

Three comments later, in the context of discussing Fermat's sentence [*Satz*] in his last theorem and what is required to prove certain general claims that involve the infinite domain of the natural numbers, Wittgenstein says (MS 105, 8-10):

> Es ist klar daß ich die Konstruktion der mathematischen Satzzeichen dadurch erklären muß, daß ich angebe wie die so gebildeten Sätze verifiziert warden sollen. Denn jedes Zeichen deutet eine Methode einen Weg (?) der Verifikation an.
>
> It is clear that I have to explain the construction of mathematical propositional signs by indicating/spelling out how the propositions thus formed are to be verified. For every sign indicates/hints at a method, a way/path (?) of verification.[70]

Although Wittgenstein uses 'construction', it is evident that what he is explaining is actually a difference of meaning (which would be reflected in how *different meaningful* propositions are constructed). This is, at least, certainly borne out by later passages: his principal concern is to discuss differences of meaning between meaningful propositions, which are established by differences between how the propositions are verified. The earliest use of what is essentially an early version of the verification principle is thus a tool for making logical distinctions. More precisely, Wittgenstein emphasizes that different (propositional) signs or propositions have their own verification, but does not talk much specifically, or confidently, about actual 'methods' or 'ways' of verification, and does not yet elegantly express the principle (as, for example, a catchy one-liner). Wittgenstein clearly thinks that one can employ the idea of different verifications in order to make logical distinctions between types of propositions (both between empirical and mathematical propositions and between mathematical propositions themselves) (MS 105, 42).[71] And he emphasizes the difference between verifying existential and universal arithmetical propositions (MS

[70] I am grateful to Severin Schroeder for his assistance with the translation of this passage.
[71] Thus, early on, the reference to 'verification'/the verification principle is used to justify/explain the distinction Wittgenstein already maintained in his early thought between empirical and mathematical propositions: the former is 'verified' by comparing it with the world while the latter is not.

105, 42). This clearly shows the mathematical origins of verificationism and illustrates the point that Wittgenstein now takes mathematical propositions to be meaningful, for otherwise there would be no reason to apply these verificationist ideas. This is supported by his use of ‘sense’ [*Sinn*] here. This is also one of the earliest references to what will become a much greater part of Wittgenstein’s philosophy when he re-examines his previous (at least implicit) assumptions about the quantifiers and how these relate to his extensive clarifications with respect to the concept of the infinite (extensively discussed in Chapter 4).

Taking ‘the method of verification is the sense of a proposition’ (or some close variant to this) as the clearest statement of verificationism, it is apparent that Wittgenstein’s position develops towards it later in 1929. While there are occasional passing references to ‘method’, ‘way’, and ‘sense’ in the first half of 1929, this is made more explicit *later* in 1929. There are few passages that use ‘verification’ or ‘verify’ when discussing the phenomenological language at all (or outside the philosophy of mathematics prior to Wittgenstein giving up the phenomenological language). Those that do (e.g. MS 105, 120 and 107, 5) use ‘verification’ or ‘verify’ on its own, and the term is used to mean the positive comparison of language with the world (i.e., that a proposition correctly describes how the world is). This, along with other textual evidence,[72] shows the continuity between the idea of comparing language with the world (in the *Tractatus*) and the idea of verification, which continues to develop into verificationism when Wittgenstein gives up the phenomenological language (see Section 3.1.5 below). While Wittgenstein is sometimes already practically making use of the idea expressed by the verification principle in order to make important logical distinctions in the philosophy of mathematics in the early intermediate period, he does not express this itself as a principle until later, when he considers empirical propositions in more detail. Understood as the explicit statement of a principle, verificationism takes shape when Wittgenstein considers empirical propositions in greater detail and it develops against the background of the philosophy of the *Tractatus* and the phenomenological notation. At the very least, it is the earlier practical use of the idea expressed in the principle that then led him to formulate the principle itself (which subsequently had a wider application to empirical propositions); and, regardless of

[72] Here I have in mind (what is discussed below) the fact that *WVC* 244 and *PR* 170 develop from *Tractatus* 4.024.

whether the principle is formulated because of his consideration of empirical propositions specifically (or whether coincidentally it just happens at this time), it is apparent that the principle, because of the focus of Wittgenstein's philosophy at this time, occupies a more central place in his philosophy.

3.1.2 The Verification Principle and its Relationship to the *Tractatus*

The clear continuity between the philosophy of the *Tractatus* and the development of the verification principle is further substantiated by some of the (later) published remarks of Wittgenstein, as well as the notes taken on his work or the material published on the basis of his work. In fact, as we shall see below, there are intermediate steps between the philosophy of the *Tractatus* and the verification principle: the phenomenological notation, its demise, and the (roughly co-occurrent) emergence of the idea of grammar. However, given that the phenomenological notation preserves many elements of the philosophy of the *Tractatus,* and since many of the people commenting on Wittgenstein's work (including those examined here) did not view his comments about verification in light of the phenomenological project, it is apt to note some of the clear connections between the *Tractatus* and verificationism (and what other scholars made of them).

In December 1929 we already find Wittgenstein talking about his principle of verification with Waismann and Schlick. On the 22nd of December part of their meeting concerns what Waismann records as 'The Sense of a Proposition is its Verification'. That the sense of a proposition is determined by a method for determining its truth or falsity is given in several other quotations from the period.[73] For example:

> To understand the sense of a proposition means to know how the issue of its truth or falsity is to be decided. *(PR* 77)
>
> How a proposition is verified is what it says. (*PR* 200)

[73] Again, I am not attempting, *in this section*, to explain all of these quotations in their right chronological order or context. How Wittgenstein's thought evolved specifically against the background of the phenomenological language is discussed in a later section. Instead, these are the types of things Wittgenstein would have been saying (beginning around the summer of 1929) to the people that he discussed verificationism with. Just how much the phenomenological language was at play was rarely, if ever, understood by the people Wittgenstein was discussing this with, nor the other scholars (dealt with also in the next two sections) who attempted to place verificationism in its right historical context. Thus, I am justified in limiting this section to the spirit of his thought, which was grasped by the aforementioned scholars, insofar as it is necessary to weigh in on the *relevant concerns* in these sections.

> The verification is not *one* token of the truth, it is *the* sense of the proposition. (*PR* 200)
>
> For a statement gets its sense from its verification. (*AL* 17)[74]

In all likelihood[75] the earliest mention of the verification principle *in print* is to be found in Waismann's 'A Logical Analysis of the Concept of Probability' (1977, 5). Here he openly says that he is 'using Wittgenstein's ideas' (1977, 4). This work seems to have developed from Waismann's 'Theses' (Hacker 1986, 136), a philosophical undertaking meant to explicate the ideas of the *Tractatus as carefully as possible* in conjunction with a few new ideas of Wittgenstein's from 1929 (which were communicated to Waismann and sometimes Schlick, in conversation). A slightly later version of this is printed as Appendix B to *Wittgenstein and the Vienna Circle*. This work is pivotal to understanding Wittgenstein's verification principle, for it not only provides one of the earliest and lengthiest statements of it, but it does so against the background of Wittgenstein's earlier work, thus providing us with the clearest possible explanation of Wittgenstein's verification principle in relation to the *Tractatus*. Although presented by Waismann, these are clearly Wittgenstein's ideas. So, they are the best possible source for Wittgenstein's thought next to Wittgenstein's own writings and notes.

There are many passages expounding Wittgenstein's verificationism in Appendix B of *Wittgenstein and the Vienna Circle*. It should suffice to quote two:

> A person who utters a proposition must know under what conditions the proposition is to be called true or false; if he is not able to specify that, he also does not know what he has said.

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[74] Additional quotations can be found in other sources. For example: 'The meaning of a proposition is the mode of its verification; two propositions cannot have the same verification' (*LL* 66). In the notes by G.E. Moore, we find the following especially pertinent quotation: 'A proposition can be verified or falsified, & is equivalent to a method of verifying of falsifying' (*NM* 44). And in *The Voices of Wittgenstein*, there is an entire section devoted to 'Verification' (*VW* 117-121), although even by this time (likely sometime after March of 1931), Wittgenstein was already clearly qualifying his position greatly from what appears in Appendix B of *Wittgenstein and the Vienna Circle*.

[75] This is Hacker's educated surmise, which uses the support of J. Passmore's *A Hundred Years of Philosophy* (1957, 371). The work by Wrigley gives no evidence to suggest otherwise, even though, at least for the purposes of his paper, he seriously entertains the possibility that the Circle could have influenced Wittgenstein's thought in this regard (i.e., the principle of verification either came from them, or meetings with the Circle somehow inspired Wittgenstein's thought). Of course, I would readily agree that conversations with the Circle could have 'inspired' Wittgenstein, in the sense that these were occasions to further think through his ideas and respond to questions about his thought. However, there is certainly no textual evidence to suggest that any members of the Circle had any hand in coming up with the principle. And it seems to me unprofitable, without further evidence, to contemplate some of the other logically possible options.

> To understand a proposition means to know how things stand if the proposition is true.
> One can understand it without knowing *whether* it is true.
> In order to get an idea of the sense of a proposition, it is necessary to become clear about the procedure leading to the determination of its truth. If one does not know that procedure, one cannot understand the proposition either.
> A proposition cannot say more than is established by means of the method of its verification…
> *The sense of a proposition is the way it is verified*. (*WVC* 243-244)
>
> A proposition that cannot be verified in any way has no sense. (*WVC* 245)

There are many similar passages in this part of Waismann's work. The important point in the first passage is that it clearly begins with outlining a proposition in the *Tractatus*. Taking 'how things stand' as equivalent to 'what is the case' means that the third line of the above passage is almost a perfect restatement of 4.024: 'To understand a proposition means to know what is the case if it is true'.[76] Moreover, one needn't, either by the lights of the *Tractatus* or Wittgenstein's ideas at this stage, actually know whether any given proposition is true or not. For, in both cases, the truth of the proposition is independent of its sense. In the *Tractatus*, it was possible to simply compare an elementary proposition with reality (i.e., make sure the simple objects are in the combination indicated by the proposition – that is, the way the proposition says they are). It is *the additional fact* that one must now 'become clear about the procedure leading to the determination of its truth' that takes one beyond proposition 4.024 and into talking about verificationism.

3.1.3 A Historical Digression: The Verification Principle and the Vienna Circle

It is undeniable that verificationism played a decisive role in the thought of the Vienna Circle. In their hands it became a dogma which stood at the very centre of their worldview and condemnation of metaphysics. It is clear it also played an important role in the thought of Waismann who, clearly influenced by Wittgenstein, first speaks of the verification principle and transmits this idea more widely to the Vienna Circle. As proof of its influence on the Circle in general, the following should suffice.[77] In 'The

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[76] This is not the only case where Wittgenstein makes reference to related *Tractatus* propositions. In *PR* 170 the reference to *TLP* 4.022 is made explicit with quotation marks.
[77] There is also 'Meaning and Verification', by Schlick, which contains an important passage (Schlick 1936, 341). Moreover, there are numerous passages where members of the Circle, including Schlick, but also Carnap, Juhos, and Kraft, unequivocally credit Wittgenstein as being the originator of the

Elimination of Metaphysics through Logical Analysis of Language', published in *Erkenntnis* II (1931-1932), Carnap wrote:

> the meaning of a word is determined by its criterion of application (in other words: by the relations of deducibility entered into by its elementary sentence-form, by its truth-conditions, by the method of its verification)… (Carnap 1931, 63)

And Schlick wrote a year later in his 'Positivism and Realism', also published in *Erkenntnis*:

> If I am *unable*, in principle, to verify a proposition, that is, if I am absolutely ignorant of how to proceed, of what I must do in order to ascertain its truth or falsity, then obviously I do not know what the proposition actually states… in so far as I am able to do this I am also able in the same way to state at least in principle the method of verification… The statement of the conditions under which a proposition is true is *the same* as the statement of its meaning, and not something different. (Schlick 1932, 87)

The earliest references to the verification principle are made by Waismann. In the decisive material referenced in 3.1.2, he clearly acknowledges his debt to Wittgenstein. And he was not the only member of the Circle to do so (Hacker 1986, 135 and 137). Hence, it is most likely through Wittgenstein's extensive discussions with Waismann about his own work, which primarily included explanations of the *Tractatus* together with some of his newly emerging views (discussed below), that the verification principle developed. Even in his later conversations with Waismann, Wittgenstein is already carefully qualifying his position. This much more careful position was to live on in slightly different forms in *The Big Typescript* (§60) and *Philosophical Investigations* (§353). In a more dogmatic and unqualified form, the verification principle, through Waismann, entered the general parlance of the Vienna Circle.[78]

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verification principle. See Hacker (1986, 135) for these details. Kraft very loosely links it to the *Tractatus* itself, but it should be noted that, given the difficulty of Wittgenstein's work and his tendency towards not carefully explicating his own views, there very easily could have arisen confusions surrounding exactly where the principle was to be found.

[78] Contrary to this position, it has been claimed by Michael Hymers that 'the emphasis on verification was already present in the Vienna Circle's discussions of the *Tractatus* in 1926-27, as the mathematician Karl Menger testifies' (Hymers 2005, 213). If this were true, it would clearly undermine my claim that talk of the verification principle originated with Wittgenstein and was disseminated to the Circle through Waismann in 1929, and *only then*, in the hands of the Circle, developed into having a central and unique role in their own philosophy. However, Hymers presents insufficient evidence to support his claim. While Menger does indeed talk about tautologies and atomic propositions in the area of text Hymers cites, there is no mention specifically of 'verification' there or anywhere else in his discussion of that time (Menger 1982, 86-87). Moreover, rather confusingly, Hymers otherwise clearly holds my position, since he expresses his view in the preceding paragraph that there is little reason to think Wittgenstein held any verification theory of meaning in the *Tractatus* and that it is only the Circle's preoccupation with

3.1.4 Wrigley and McGuinness on Wittgenstein's Verificationism and the *Tractatus*

Given certain quotations from the intermediate period (e.g. *WVC* 244 and *PR* 170), it may be tempting to trace Wittgenstein's verificationism back to the *Tractatus* itself.[79] Both quotations, after all, seem to make reference to pertinent passages about the nature of propositions. Was the principle not already present at that time? This would initially seem, as Michael Wrigley has argued without making reference to the same quotations, to best explain the origin of the principle. Wrigely's starting point is that Wittgenstein was clearly a proponent of verificationism upon returning to philosophy in 1929, so the key to understanding where this idea came from must be found in the intervening years when he finished the *Tractatus* to when he returned to doing philosophy. By a lengthy, convoluted, process of elimination, Wrigley concludes that the principle of verification must have been 'implicit' in the *Tractatus*. He reaches this conclusion by eliminating other equally implausible explanations of the origin of the principle.

Wrigley does not consider carefully Waismann's work or comments. He does not attempt to trace how or when the principle first came into circulation. He does not make mention of Wittgenstein's project with Waismann when Wittgenstein first returns to philosophy (in the second part of 1929, if not before). He uses the word 'implicitly', but does not say in this context, relative to the ideas of the *Tractatus*, what this actually means. It is also clear that 'implicitly' can't refer to the same interpretation I have just given, for Wrigley does not acknowledge the possibility that the verification principle was first *developed* by Wittgenstein early upon his *return* to doing philosophy. Wrigley seems to *assume* that there had to be some clear evidence of a progression to this view and, therefore, without any (based on Feigl's testimony), this option is to be rejected. For Wrigley readily accepts Feigl's testimony (Wrigley 1989, 270) that Wittgenstein still thought he had solved all the major problems of philosophy with the *Tractatus* and

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their own problems and ideas that could have led them to see a doctrine of verificationism in (a few select) passages of the *Tractatus* (Hymers 2005, 213). Thus, he clearly doesn't think that talk of tautologies or atomic propositions is itself talk of verificationism. But, in that case, there is also no reason to think verificationism was already being discussed in 1926-27 by the Circle. Instead, it is still likely that verificationism derived from Wittgenstein and spread through Waismann, and, if this is true, while the *Tractatus* may, at that time, have been 'being read through the lens of their [the Circle's] own philosophical preoccupations' (Hymers 2005, 213), these 'preoccupations' were certainly not yet themselves any sort of worked out verificationist position.

[79] Michael Wrigley, although he does so for very different reasons, is not the only one to think that Wittgenstein's verificationism can be found in the *Tractatus*. This idea was also held by members of the Vienna Circle (Hacker 1986, 135). Passmore confirms this (1957, 371).

therefore did not have the inclination to do original philosophy in the two years before 1929[80] (and he does not make any reference to the conversations between Wittgenstein, Waismann and Schlick that show otherwise). He makes no reference to proposition 4.024 and does not contrast this proposition or any parts of the philosophy of the *Tractatus* with Wittgenstein's views in the intermediate period.

Other commentators have made similar claims. Ignoring some of Brian McGuinness's broader views about the *Tractatus*, at least some of which I remain unconvinced about,[81] McGuinness, at the end of his paper 'Language and Reality', makes reference to Wittgenstein's 'implicit verificationism' in the *Tractatus* (McGuinness 2002, 102). However, it should be noted that he does not mean by 'verification' what is meant by using the term to refer to Wittgenstein's position in the intermediate period. Instead, he means that one establishes the truth (or perhaps even falsity) of a proposition in the way explained in the *Tractatus*.[82] To be sure, his interpretation of how exactly this is done in the *Tractatus* is unique, but, regardless of the specifics, it is clear he does not think propositions get meaning in the *Tractatus*

[80] Without evidence that the Circle published views about verificationism before 1929, there is no reason not to use Wittgenstein's meetings with Waismann in 1929 as the source of the original and influential principle of verification. The most important point is that Feigl's testimony regarding Wittgenstein's lack of enthusiasm with respect to doing original philosophy – which Wrigley cites (1989, 270) – aptly applies to the larger meetings with members of the Circle which clearly did not last long (because of tensions with Wittgenstein), but not to the meetings with Waismann and Schlick (*VW* xviii). Specifically, Waismann's project of writing *Logik, Sprache, Philosophie* – what was meant to clearly explicate the philosophy of the *Tractatus* – got underway at least by 1928 (*VW* xxiii) and already by that time Schlick and Waismann would have been at least seeking additional meetings with Wittgenstein (*VW* xviii and xxiii). Admitting Feigl's testimony does not amount to anything with respect to Waismann's project with Wittgenstein, there should be no problem claiming Wittgenstein did start developing new ideas when rethinking his *Tractatus* views (which would have been encouraged by Waismann) and there is still no reason to think these came from the Vienna Circle.

[81] For example, McGuinness insists that it is important to understand the propositions of the *Tractatus* as propounding a myth and that commentators should 'not fail to make the necessary leap to the destruction of that myth by its own absurdity' (McGuinness 2002, 95). This seems to me to be unnecessarily pessimistic, reductionist, and absolute. The *Tractatus* has paradoxical or contradictory conclusions, but it is certainly an exaggeration to suggest that (virtually) everything is myth just on the basis of choice propositions or claims in that work. Good commentators would still (rightly) point out what are genuine insights, either because they were important contributions to the history of philosophy, or because they were parts of his philosophy that Wittgenstein either preserved or reinterpreted in the light of his later philosophy.

[82] It is clear that he is aware of the difference between these views in another work: 'However, the process of explaining the truth-value of a proposition cannot be broken down into any simpler operations than that of grasping or expressing the proposition. All these operations possess the same multiplicity. Wittgenstein was to urge this point as a justification of his picture theory in the early 1930's, and it also obviously underlies the development from the *Tractatus* to the dictum that the sense of a proposition is the method of its verification' (2002, 93). Unfortunately, even in this case, where the difference is properly articulated, it is unclear to me *exactly what he means* when claiming the 'operations' listed possessing the same multiplicity 'obviously underl[ie] the development from the *Tractatus* to the dictum that the sense of a proposition is the method of its verification'.

from their ‘method of verification’ – as required in the intermediate period. One may claim that the comparing of a proposition with reality to establish whether or not it is picturing a state of affairs is a type of ‘verification’. This is an intuitively correct use of the word (even more so when the propositions can indeed be ascertained to be true), but even if this is accepted, there is clearly a great difference between the scope of what is considered ‘verification’ in the intermediate period and in the *Tractatus*. For example, ‘verification’ in the *Tractatus* would really only contain *a* method and thus would also not be a way of making logical distinctions (as it is clearly employed in the early intermediate period). As explained further below, ‘methods’, for Wittgenstein, are individuated by logical ‘spaces’ and, in the *Tractatus*, he is only committed to one logical space. Given the difference between what ‘verification’ would mean in the two cases, we can only understand McGuinness’s claim that ‘verificationism’ is ‘implicit’ in the *Tractatus* as really meaning that ‘verification’ in the first sense occurs despite the term ‘verification’ not being used in the *Tractatus*. As should be clear, there is more possibility for confusion with the use of this terminology than what is clarified by invoking it.

In contrast, while the principle itself clearly develops out of the *Tractatus* passages (in relation to Wittgenstein’s other emerging ideas), it is also clear that it is not present there in the exact form it takes in the intermediate period. So, while Wittgenstein does say that ‘To understand a proposition means to know what is the case if it is true’ (*TLP* 4.024) this must be read in conjunction with the more general ideas propounded in the *Tractatus*. The meaning of a proposition, in the *Tractatus*, was what was pictured by a genuine combination of simple names or a truth-functional combination of such propositions (i.e., elementary ones). One could understand a proposition by reading off from it what must be the case for it to be true. We shall see below how the verification principle developed not only out of these ideas but also out of ones involving the phenomenological language.

3.1.5 The Development of the Verification Principle Continued: Its Relationship to the Demise of the Phenomenological Language

The purpose of this section is to explain further the development of verificationism as it relates to the demise of the phenomenological notation (beginning around September 1929). As we have seen, verificationism (like the phenomenological notation) contains seeds that can be found in the *Tractatus*. An examination of the details of Wittgenstein's thinking around the autumn of 1929 will show how verificationism developed in relation to other parts of his thought.

It is worth noting, given how quickly some of these changes are happening, as well as limitations inherent in the presentation of Wittgenstein's ideas, that it can be difficult to ascertain *exactly* how Wittgenstein's thought developed. For example, it is difficult to assess whether the verification principle influenced Wittgenstein's conception of a comprehensive grammar, or *vice versa*. Regardless, it is apparent that the problems with the phenomenological language inspire the development of grammar *together* with a more elaborate version of, and use for, the verification principle. There is an important interrelationship here that has not been noted in the literature.[83] In this way, at least until Wittgenstein develops the idea of grammar further, the verification principle occupies a central position in Wittgenstein's philosophy. Similarly, it is difficult to assess whether the verification principle is meant to account for the problem of the epistemological status of the phenomenological language, or whether reflection on the verification principle (or grammar more generally) inspires this development. Finally, the problem of exactly when Wittgenstein gives up the shared form between language and the world, and how exactly this fits into the overall evolution of Wittgenstein's thought, is difficult decisively to determine. I have tried my best to reconstruct these developments on the basis of the evidence available.

It is noteworthy that discussion of the 'the sense of the proposition' occurs in September 1929, but still isn't a very clear statement of the principle ('method of verification' still doesn't occur here) (MS 107, 143). Immediately following the passage Engelmann gives as indicative of Wittgenstein's giving up the phenomenological project (MS 107, 176), Wittgenstein gives an early elaboration of the idea of grammar and, in relation to this, employs the verification principle. Indeed, the verification principle has a very important role in this context: to preserve the idea that language isn't merely a game, but what makes a language applicable to reality is the fact that

[83] In one of his papers, Engelmann (2018, 66-67) does correctly note that the development of Wittgenstein's technical idea of 'sentence -hypothesis' in contrast to 'phenomenological description' occupied a central place in his thought in 1930. In particular, the sentence-hypothesis, understood as a law for generating phenomenological descriptions, was capable of being verified over a variety of 'spaces' and thus served as a unifying feature for verification (since the phenomenological notation would no longer be sufficient for this purpose). I agree with Engelmann's interpretation here, although, as will be argued, Engelmann's account of the development of Wittgenstein's use of the verification principle is at best unclear and at worst incorrect.

there is a method [*Art*] of verification indicated by the propositions that make up a language. And this application, in turn, is necessary for a language to count as a language.

Wittgenstein says:

25. [10.1929]

> Jeder Satz ist ein leeres Spiel von Strichen oder Lauten ohne die Beziehung zur Wirklichkeit und die /seine/ einzige Beziehung zur Wirklichkeit ist die Art seiner Verifikation.

26. [10.1929]

> Alles wesentliche ist, daß die Zeichen sich in wie immer complizierter Weise am Schluß doch auf die unmittelbare Erfahrung beziehen und nicht auf ein Mittelglied (ein Ding an sich).

25. [10. 1929]

> Each sentence is an empty game of strokes or sounds without the relation to reality and the /its/ only relationship is the mode of its verification.

26. [10. 1929]

> All that is essential is that the signs – it does not matter in which complicated way – at the end relate to the immediate experience, and not to an intermediate link (a thing in itself). (MS 107, 177)

13. [12. 1929]

> Unter Anwendung meine ich das was die Lautverbindungen oder Striche auf dem Papier etc. überhaupt zu einer Sprache macht. In dem Sinn in dem es die Anwendung ist die den Stab mit Strichen zu einem Maßstab machen. Das Anlegen der Sprache an die Wirklichkeit.
> Und dieses Anlegen der Sprache ist die Verification der Sätze.
>
> By application I mean what makes the sound combinations or signs on the paper into a language. In the sense in which the application makes the bar/stick with marks into a ruler. The application of language to reality.
> And this application of language is the verification of the propositions.[84] (MS 108, 1)

As Engelmann also notes (2013, 45), signs, for Wittgenstein at this stage, still ultimately have meaning because they refer (in a ‘complicated manner’) to experienced phenomena (i.e., immediate experience – e.g. patches of light, sounds etc.) (MS 107: 177, 223, and 255). This indicates that Wittgenstein is still (perhaps dogmatically) committed to the connection between language and the world, although, at this point, with the failure of the phenomenological notation, he is aware that this connection can’t

[84] I am grateful to Severin Schroeder for his assistance with the translation of this block of quotations.

be easily specified (or specified at all). Moreover, given that all of these developments originate directly from the philosophy of the *Tractatus*, it is reasonable to assume that Wittgenstein is still committed to the shared form between language and the world, although he is aware that this form, with the demise of the phenomenological language, can't be specified either. While parts of the phenomenological project live on ('phenomenology' remains a part of grammar), and parts of the form can be represented with a notation (e.g. the colour octahedron), the goal of representing the form in a single notation *that is fundamental* is now thought to be mistaken (Engelmann 2013, 43-44). Within the framework of the *Tractatus*, the connection between language and the world would seem only to make sense together with the idea of the shared form between language and the world.[85] Thus, it is reasonable to assume that Wittgenstein was still committed to the idea of the shared form of language and the world. Indeed, it is reasonable to assume that this shared form is only decisively rejected with the autonomy of grammar arguments that arise in the second half of 1930 (i.e., primarily in MS 109, beginning around 59-60 (August 1930), but also MS 110). In contrast, Engelmann thinks the shared form between language and the world is given up with the phenomenological language (Engelmann 2013, 151). We needn't settle this here, but rather simply note that even if Engelmann is correct, my interpretation requires only the slightest change to fit with what Engelmann thinks (explained below).

With the acceptance of different 'spaces', but the demise of the phenomenological language, *how* the proposition is verified becomes an important part of language; the phenomenological language was meant to be *the* notation in which any proposition can be verified, so giving this up means, at least at this stage, that a proposition has meaning insofar as it has a method of verification. Different 'spaces' will require different methods of verification, since the different 'spaces' will contain different rules as to what does and does not make sense. The truth of certain statements will imply the truth or falsity of other statements, and nonsense can't be verified at all (and thus won't have a method of verification). 'Method of verification' has a technical meaning. Normally, one would not consider the rules of sense that characterize a logical 'space' to be related at all to a method of verification. A method of verification would instead be a way of determining the truth (or falsity) of some (empirical) statement (e.g. the methods used in a criminal investigation).

[85] I am grateful to William Crooks for emphasizing this point.

It is reasonable to assume that because of the problems with the status of the phenomenological language, how it is verified is happily conceived as an *a priori* feature of the proposition itself. The verification principle, as applied to empirical propositions, only gains prominence around this time, and so it is unlikely that Wittgenstein thinks of the verification principle on a par (in terms of its epistemological status) with the phenomenological notation he has already abandoned. Wittgenstein's commitment to the idea that language and the world must be connected leads naturally to a use of the verification principle as a means of guaranteeing the application of language to the world. If it is impossible to engage in an analysis of language as Wittgenstein conceived of it in the *Tractatus* and with the phenomenological language, then, at least when it comes to empirical propositions, the verifiability of the proposition becomes of central importance. And this possible verification is best articulated by saying that the proposition has 'a method of verification'; this eliminates the need for the proposition to be actually verified, and allows that propositions are verified relative to different 'spaces'. Since it becomes impossible to specify the specific form that language and the world will share, then something must replace the specification of this shared form. And given that the logic employed in the *Tractatus*[86] is inadequate for capturing the multiplicity of the phenomena, and the phenomenological notation too is incapable of this task, it makes sense that Wittgenstein appeals to the method of verification as a way of bypassing the problem of specifying the shared logical form. The 'method of verification' is a linguistic rule/instruction (i.e., it is given within language) that will make sure that an actual verification could take place for any (meaningful) proposition (i.e., one is logically possible), even if it isn't done in practice. And the different 'methods' essentially account for the different rules of sense that apply to the different 'spaces'. In this way, even without the ability to specify the form, one can be certain that a (meaningful empirical) proposition pictures a state of affairs that *could* be verified, even before any verification actually takes place. And, it should be noted, even if Engelmann is correct,

[86] I find Marion's claim (1998, 115) that Wittgenstein was, in the *Tractatus*, already aware of the different 'spaces', not convincing. Of course, he would have been aware that the colour exclusion problem would be a problem for his interpretation, *if not for* the all-encompassing logic of the *Tractatus*, which would give the analysis of the colour exclusion problem. But seemingly his views about the *Tractatus* meant that he was confident the problem would be so analyzed and thus any additional 'space' was just an illusion. This would parallel other problems that would seemingly arise with the *Tractatus* (e.g. not being able to give an example of a simple object). As far as I can tell, there is nothing in the *Tractatus* to suggest otherwise, and Marion does not provide any reference for his claim.

and Wittgenstein did give up on the idea of the shared form between language and the world with the demise of the phenomenological language, then the point of the verification principle would be to *replace* the shared form between language and the world (for a discussion of they ways in which Engelmann (and Hacker) is wrong, see the appendix).

With the introduction of Wittgenstein's notion of a comprehensive grammar at the same time, as well as his recent rejection of the phenomenological notation (and, seemingly, the problems it embodied), it is reasonable to assume that Wittgenstein conceives of the verification principle itself as a rule of grammar and/or that consideration of the verification principle helped to develop his views about rules of grammar (or grammar more generally – that is, conceived as a comprehensive discipline). That is, both the verification principle conceived *as a principle*, as well as specifications of the methods of verification that are given by any meaningful proposition (which would, at the very least, show what does and doesn't make sense), would be considered by Wittgenstein to be *a priori* features of the propositions being considered. This seems supported by everything Wittgenstein says about the principle, and it is noteworthy that arguments for the arbitrariness and, later, the autonomy of grammar are being developed soon after (Wittgenstein gives arguments both against the idea that grammar can be justified and that language and the world are connected – e.g. MS 108, 98 and104, and MS 109, 98-99, respectively).[87] The arguments against the idea that grammar can be justified, as Engelmann notes (2013, 60), are meant to eliminate the problem of the epistemological status of the phenomenological language. These arguments are presented in early 1930, but, given everything Wittgenstein says, as well as how soon after giving the verification principle a more important place in his philosophy he presents them, it is likely that the introduction of the verification

[87] Arguments for the arbitrariness of grammar are developed already in early 1930. These arguments, however, do not contradict the idea that language and the world are connected (cf. Engelmann 2013, 63). Thus, it is reasonable also to assume that, at this point, Wittgenstein still holds to the shared form between language and the world. Arguably, it is with the subsequent autonomy of grammar arguments, which develop out of his consideration of the causal theory of meaning and, related to this, intentionality, that Wittgenstein will reject the shared form between language and the world. It makes sense that the shared form between language and the world would be given up when any 'relation' is considered 'internal' and the connection between language and the world is given up. However, as I already suggested, my interpretation of the role of the verification principle needs only to change slightly to accommodate the idea that the shared form between language and the world was given up with the demise of the phenomenological language. See Englemann (2013) for excellent discussions of the arbitrariness arguments (pp. 60-64) and both the autonomy arguments (pp. 90-99) and what led to them (Chapter 2 – especially pp. 65-93).

principle is meant to represent a decisive break with the problematic epistemological status of the phenomenological language (even though the arguments that actually serve to establish this are only provided later).

With the development of the idea of grammar, specifically the development of the idea of a grammatical rule, Wittgenstein will see the verification principle as no longer the only/primary way of establishing the meaning of a proposition; there are a variety of propositions that count as grammatical rules and thus what counts as a verification is just one way of specifying the meaning of a proposition. Moreover, with arguments for the autonomy of grammar that develop further, Wittgenstein will no longer see language and the world as connected. Arguments for the autonomy of grammar will also serve to decisively refute the idea that language and the world must share a form. Wittgenstein can continue to hold that a proposition is compared with the world without thinking there is a connection between language and the world in the form of some basic part of reality that stands as the meaning of the signs in our language. With this development, the verification principle will no longer occupy the central role it did around the time of the failure of the phenomenological notation.

Since its initial use was a tool for making logical distinctions, it makes sense that the verification principle is employed by Wittgenstein in order to make other distinctions during the intermediate period: between ‘hypotheses’ and ‘phenomenological descriptions’ (MS 107, 252-254) and between first-person avowals and ordinary descriptions (the earliest mention of what will become an important part of the private language argument) (MS 113, 52-53). The former distinction develops as an immediate consequence of the central use of the verification principle (as this relates to empirical propositions) together with the failure of the phenomenological project. The phenomenological language would make verification a simple affair: ordinary propositions would be truth-functions of elementary propositions and analysis within the notation would allow for determinate verification. Together with the abandonment of the phenomenological language, Wittgenstein now reconsiders the determinate verification of, and reconceives, ordinary propositions. Ordinary propositions are better understood as having the logical structure of a ‘hypothesis’.[88] Thus, we see the

[88] Understanding ordinary propositions as ‘hypotheses’ is to think of ordinary propositions as laws that generate an indefinite number of phenomenological propositions. This means that no number of confirmed phenomenological propositions will *entail* the truth of the ordinary statement. For this synopsis I am indebted to Engelmann (2018, 65-66).

development of the idea of ‘hypotheses’ (as a way of understanding ‘ordinary statements’), as this is explained using the verification principle, early in 1930, shortly after the verification principle first takes on a central role in Wittgenstein’s philosophy. And it makes sense that Wittgenstein would use the principle (most broadly) as a tool to test the intelligibility of an idea, to help understand the nature of expectation (MS 107, 235), and to clarify the concept of the infinite (extensively discussed in Chapter 4) (these aren’t mutually exclusive).

3.1.6 The Restriction of the Verification Principle

Already in 1930 Wittgenstein is aware of the limitations of the verification principle. First, he indicates that simply using the term ‘verification’ for what are categorially distinct domains is misleading at best (MS 108, 295): ‘Nothing is more fatal to philosophical understanding than the notion of proof and experience as two different but comparable methods of verification’ (translated in *PG* 361). As we shall see, Wittgenstein does explore the idea that the concept of ‘verification’ can be profitably employed in both the empirical and mathematical domains, but it is already apparent that he does not think that simply using the term ‘verification’ indicates an important similarity between the two domains. It is worth considering whether Wittgenstein was already aware of this before writing this qualifying comment. I think it is clear that he was, since he already argued for the categorial divide between the two domains in the *Tractatus*, and already noted how a method of verification can happen in the ‘internal way’. And, as already suggested (footnote 71), in line with its use as a tool for making logical distinctions, he appeals to the principle as a way of explaining/justifying the distinction between *a priori* and *a posteriori* domains.

In addition, Wittgenstein articulates an objection that is commonly made against verificationism. He cites poetry as an example of a discipline that is clearly meaningful (which contains meaningful sentences) but does not have a verification (MS 109, 26). Wittgenstein thus qualifies his position as we move into his later work. The method of verification is *one* indication of a proposition’s meaning, but not the only one. Insofar as a proposition has a verification (first-person avowals don’t), the method of verification serves as one way of specifying its meaning. This, together with other parts of Wittgenstein’s evolving philosophy (e.g. his emphasis on language-games instead of the calculus conception), means the centrality of verificationism, and with this his appeal to 'method of verification' (particularly as part of slogan), decreases in his later work; instead, he will now often simply note whether a proposition is verifiable or not (often to make logical distinctions). Mention, and use, of 'method of verification' is made in the *Whewell Lectures* (e.g. pp. 10-11) where the 'sameness' of colours, 'impressions', and pains are contrasted, and passing use of the idea of a statement not having a 'verification', in order to make distinctions between propositions, occurs in *On Certainty* (§510).[89] In this way, a qualified version of verificationism can be found running through Wittgenstein's later philosophy (cf. Hacker 1986, 144-145).

3.2.1 Verificationism in the Philosophy of Mathematics

As noted already (Section 2.8), Wittgenstein, in the early intermediate period, came to think of mathematical propositions as importantly similar to empirical ones (*PR* 142), likely on the basis of a realization of the differences of the former from tautologies. As we have seen, that mathematical propositions have meaning is suggested by Wittgenstein's initial use of early verificationist ideas in order to make logical distinctions, and is even made explicit with his use of 'sense' [*Sinn*] in this context (MS 105, 42).

It is against this background that, together with ideas that still live on from the *Tractatus*,[90] Wittgenstein's mathematical verificationism develops. The truth-conditional semantics view held in the *Tractatus* leads, when applied in the philosophy of mathematics, to a particularly stringent form of verificationism. A mathematical proposition's sense can't be a description of a state of affairs, as is the case with an empirical proposition. Since there is *some* similarity between the role of a state of affairs in relation to an empirical proposition and a proof in relation to a mathematical proposition, this leads Wittgenstein to the view that the sense of a mathematical proposition must be 'the way in which it is to be proved' (*PR* 170). It should be noted that this doesn't mean that the proof must actually be given, but rather that a method for

[89] See also Malcolm (2001, 55).

[90] In addition to the idea mentioned immediately below, there are also the ideas of determinacy of sense (*TLP* 3.23) and the complete analysability of a proposition (*TLP* 3.25), which leads to the idea that the method of proof is the mathematical proposition's verification (Schroeder 2021, 41-42). A proposition's proof can be seen as its complete analysis.

constructing the proof must be available. In contrast to this case, while knowledge of what evidence would determine an empirical proposition is necessary to understanding an empirical proposition, it is unnecessary to actually be able to provide such evidence. It is in this way that empirical verification is less strict. In the case of mathematics, to be able to give a description of how one would verify the proposition is to be able actually to give the verification (cf. Schroeder 2021, 37-38). And a description of what would prove the proposition is the proof itself.

Prompted by obvious problems that arise from this position (discussed below), Wittgenstein went on to investigate further the relationship between mathematical concatenation(s) of signs (henceforth 'formula(s)'[91]) and mathematical propositions, mathematical sense, proof, and decision procedures. Thus, verificationism remains an important part of Wittgenstein's work later in the intermediate period (especially *Philosophical Grammar*), even after he has already greatly reduced its role in examining empirical propositions (it still remains a way of testing an idea for its intelligibility and is *one way* of getting clear about a proposition's meaning) and around the time that he has already insisted one must not confuse what verification means in the empirical case with the mathematical. And it continues to be a way of making logical distinctions.

To start with, we shall examine how his verificationist ideas apply to mathematical propositions and check procedures, and then proceed to see how these ideas can be applied to mathematical questions, problems, and conjectures. We shall then, through the work of Rodych, examine some of the inadequacies with this position, together with the tensions, and conflicting ideas, in Wittgenstein's thought that arise as a consequence of applications of the verification principle to the philosophy of mathematics. Thus, in the following two sections I am not endorsing the position I am outlining; the evaluation of this position will occur when we examine Rodych's work. We shall conclude with an examination of two inadequacies of Rodych's position

[91] Concatenation(s) of signs, the abbreviation for which is 'Csign(s)', is used by Rodych in his paper 'Mathematical Sense: Wittgenstein's Syntactical Structuralism'. In this context, it is worth noting that Rodych seems to express skepticism about referring to Csigns/formulas as 'well-formed' at all (Rodych 2008, 91). I'm not sure what the reason for this is. Even within the framework he establishes, it would seem to make perfect sense to distinguish between a 'proposition' that can be calculated/proven using a decision procedure and one that we can immediately see isn't well-formed (e.g. the signs aren't in the right order, such as '2 + =2 4'). One would think, for instance, that he would want to insist *at a minimum* that the 'propositions' in WV_1S would need to be 'well-formed' to count as propositions at all, although obviously this would not be sufficient for something to be a mathematical proposition. I have opted to use 'formula' in place of his 'Csign' and consider its being 'well-formed' to be part of its meaning.

regarding Wittgenstein's developed position in the intermediate period (into his later work).

3.2.2 The Verification Principle Applied to the Philosophy of Mathematics: Meaningful Propositions, General Predicates, and Check Procedures

The principle of verification, in the intermediate period, as applied to the philosophy of mathematics, allows for the possibility of mathematical propositions that are meaningful independently from a proof.[92] These specific propositions are ones that relate to a system of propositions, where what any proposition expresses can be determined to be correct or not by a general 'check' procedure. This is because the basic operations or predicates that are found in the proposition have a meaning that is determined by established rules of the system. To illustrate this, we can use the example of multiplication in decimal notation (as Frascolla does). The meaning of the variable '$\xi \times \eta$' is given by a general method of calculation which applies to all algebraic terms of this form (here 'ξ' and 'η' are variables for numerical expressions in decimal notation). As Frascolla nicely sums it up: 'Any definite description obtained by replacement of "ξ" and "η" in the schema "the product of ξ times η" with two numerical expressions means, indeed, the outcome of the correct application of *that* method to the two numbers, where the method is referred to by means of a general formulation of its operational rules' (Frascolla 1994, 56). This general method is further explained by Wittgenstein in his lectures from 1932-1933:

> Compare this with being taught to multiply. Were we taught all the results, or weren't we? We may not have been taught to do 61 × 175, but we do it according to the rule which we have been taught. Once the rule is known, a new instance is worked out easily. We are not given all the multiplications in the enumerative sense, but we are given *all* in one sense: any multiplication can be carried out according to a rule. Given the law for multiplying, any multiplication can be done. (*AL* 8)

Meanings of some general mathematical expressions such as the schematic descriptions already given and some predicates (to be discussed further below) are given by general rules that apply in an infinite number of cases (to any substitution of a numerical expression in decimal notation for the variables). As Wittgenstein indicates, such a

[92] This section and the next one owe a great deal to Frascolla's *Wittgenstein's Philosophy of Mathematics*, Chapter 2, the section entitled 'Mathematical Propositions'.

general calculation procedure does not give us every possible answer in an enumerative sense. Rather, the rules, easily learnable and finite in number, give a general procedure for calculating the result of each new instance of propositions involving multiplication;[93] they do give us a decisive technique for calculating every possible instance. So, Wittgenstein says:

> the proposition 26 × 13 = 149 is essentially one of a system of propositions (the system given in the formula a × b = c), and the corresponding question one of a system of questions. The question whether 26 × 13 equals 419 is bound up with one particular *general method* by means of which it is answered… [The fundamental law of algebra] seems to get its sense from the proof, while the propositions stating what the product in a multiplication is do not… In the case of the question about the product of 26 and 13, there is something about it which makes it look like an empirical question. Suppose I ask whether there is a man in the garden. I could describe beforehand a complicated way of finding out whether there is or not. There is a resemblance of the multiplication question to this one, in that before you find out I could tell you how to find out.
> (*AL* 197-198)

The similarity between the empirical case and this mathematical case is clear in this quotation. Since the mathematical propositions in question belong to a system of propositions where there is a general method for determining whether they are correct or not, it is possible to say that 'before you find out I could tell you how to find out' whether the proposition is indeed correct or not. That is, in simply articulating the proposition, it is possible to see how one would go about 'verifying' it, as is clearly the case with meaningful empirical propositions. So, in this specific case, mathematical propositions can be said to have a sense precisely because they belong to a system of propositions where rules are stipulated that allow for a 'check procedure' which can clearly check (or 'verify') its truth. If the method of proof is fixed independently of a specific proof, then it is possible to talk about the sense of a mathematical proposition and of 'checking' its truth. Moreover, in this case, the proof doesn't serve to fix the meanings of the terms appearing in the proved proposition, but merely to show the proposition is indeed 'correct'. As mentioned earlier, similar propositions involve predicates such as 'prime', 'divisible by 5', 'not divisible by 5', 'even', etc. (although whether a number is even is clearly easily recognizable once the rules are learned). All of these Wittgenstein calls 'arithmetical predicates' (*PR* 251).

[93] Of course, this anticipates/is supported by Wittgenstein's notion that infinity is not a property of an extension, but of a rule, which is dealt with extensively in the next chapter.

In the case of many arithmetical predicates, their meaning is determined by more general predicates. So, for example, it is clear that ‘prime’ can be analysed as ‘only divisible by 1 and itself without remainder’, which itself employs a similar general check procedure. So, these more general predicates provide, in the cases Wittgenstein is considering, a general method for ascertaining whether any given proposition containing the general arithmetical predicate is correct or not. This means that their meaning, similar to the set of propositions belonging to the system of multiplication already examined, transcends the set of sign figures already established to be results of the correct application of the general method. In the case where a certain proposition is established to be true by the correct application of the general procedure, a rule is adopted to the effect that any empirical result not in accordance with this result is to be deemed senseless (as extensively explained in the last chapter). So, to take an example: granted 11,003 is prime, a new criterion of correctness concerning the results of divisions of 11,003 is then established (Frascolla 1994, 57).

Specific problems about meaningful propositions arise in the context of the philosophy of mathematics. The principle of verification forces two additional requirements: (1) that the negation of every meaningful proposition must itself be a meaningful proposition and (2) that a proposition must act in accordance with the calculus of truth-functions, and so the law of excluded middle must be capable of being applied (Frascolla 1994, 62). In reference to (1), if the meaningfulness of a proposition is tied to its proof, then one of a proposition or its negation in the philosophy of mathematics would always be meaningless. For mathematics consists of necessary truths and thus prohibits any such thing as even imagining $2 \times 2 = 5$ to be true and this is clearly shown to be senseless by the proof of the inequality of $2 \times 2 \neq 5$. Wittgenstein says, in the context of discussing the search for the trisection of the angle:

> But the same paradox would arise if we asked ‘is $25 \times 25 = 620$?’; for after all it’s *logically* impossible that that equation should be correct; I certainly can’t describe what it would be like if… – Well a doubt whether $25 \times 25 = 620$ (or whether it = 625) has no more and no less sense than the method of checking gives it. It is quite correct that we don’t here imagine, or describe, what it is like for 25×25 to be 620. (*PG* 392)

The *application of Wittgenstein’s interpretation* of the principle of verification to the philosophy of mathematics means it is possible to entertain the possibility of truth or falsity of any assertion or its negation that belongs to a system with a decision procedure. It is the general decision procedure that allows either an assertion or its negation, whatever its actual truth value might be, to be meaningful. For it is always possible that a mistake can be made when any particular decision procedure is being undertaken, thus allowing for the possibility that the negation of a necessary proposition could be found to be provable. It is this possibility of failure that indicates the meaningfulness of any assertion or question involving such a mathematical statement. Moreover, the law of excluded middle can hold simply because any formal procedure clearly will prove or disprove the assertion and, at the very same time, correspondingly prove/disprove (depending on what opposing predicate is proven) its negation (Frascolla 1994, 62-63).

This process, as described, has the additional implication that in this period of Wittgenstein's thought it is obvious that the general rules have the ability to determine the adoption of a particular rule. That is, the rules employed in the check method have the ability to normatively establish the adoption of a particular rule in accordance with the results obtained. As Frascolla says:

> In other words, the meaning that the predicate has, *before* any given application of the procedure of calculation definitionally associated with it, would be able to impose rigid constraints on the decision by which a given sign construction is ratified as the sort of construction that *must* be obtained whenever the method is correctly applied (and so, on the decision of adopting a definite grammar rule and not a different one). For this reason, when a certain result is obtained, the acceptance of the rule corresponding to it seems to be forced by the recognition of *that* necessary connection whose existence could be conjectured, in the general terms in which the decision procedure is framed, before the latter was applied. (Frascolla 1994, 57)

So, unacknowledged necessary connections are possible where there is a general procedure to determine whether such a connection obtains in any given instance. We understand, for instance, the meaning of the expression '11,003 is prime' without the fact of its being prime being known to us. Alternatively, we understand the term 'prime' without knowing every instance that falls under that description. It is to these ideas applied to conjectures, problems, and questions that we now turn.

3.2.3 The Verification Principle Applied to Conjectures, Problems, and Questions in Mathematics

Wittgenstein's analyses of conjectures, problems, and questions in mathematics all rely on his principle of verification. The verification principle applied to the philosophy of mathematics determines exactly what counts as a meaningful proposition. With this it also determines what counts as a meaningful question. For a genuine question exists, for Wittgenstein adhering to the verification principle, only where there is a general method for answering it.[94] Wittgenstein says:

> We may only put a question in mathematics (or make a conjecture) where the answer runs 'I must work it out'…
> …
> What 'mathematical questions' share with genuine questions is simply that they can be answered (*PR* 175)

> Tell me *how* you seek and I will tell you *what* you are seeking. (*PG* 370)

> Where you can ask you can look for an answer, and where you cannot look for an answer you cannot ask either. Nor can you find an answer. (*PG* 377)

> 'the equation yields S' means: if I transform the equation in accordance with certain rules, I get S. Just as the equation 25 × 25 = 620 says that I get 620 if I apply the rules for multiplication to 25 × 25. But in this case these rules must already be given to me before the word 'yields' has a meaning, and before the question whether the equation yields S has a sense. (*PG* 378)

A meaningful proposition exists independently of a proof only where its truth can be determined by a general check procedure (where this is readily ascertainable by the signs of the proposition). The case is similar with questions, conjectures, problems, and assertions in mathematics. All of these mean something distinct when the verification principle does not apply to them.

> One could lay down: 'whatever one can tackle [*anfassen*] is a problem. Only where there can be a problem, can something be asserted'.
> Wouldn't all this lead to the paradox that there are no difficult problems in mathematics, since if anything is difficult it isn't a problem? What follows is, that the 'difficult mathematical problems', i.e. the problems for mathematical research, aren't in the same relationship to the problem of '25 × 25 = ?' as a feat of acrobatics is to a simple somersault. They aren't related, that is, just as very easy to very difficult; they are 'problems' in different meanings of the words. (*PG* 379-380)

Even without a general check procedure, one may still call mathematical conjectures, questions, and proofs by those names. But it must be remembered that they mean something different in this case. This nicely illustrates the continued use of verificationism to make logical distinctions. Using the principle of verification,

[94] This is anticipated in the *Tractatus* (6.5): If a question can be framed at all, it is also *possible* to answer it.

Wittgenstein establishes the different things that can be meant by the use of these terms. Although beyond the scope of our examination here, it is worth noting that based on the same idea of mathematical meaning, Wittgenstein analyses ‘looking for’ and ‘searching for’ in a similar way. This serves as an example of one line of thought pursued in the intermediate period.

3.2.4 Rodych on Wittgenstein, Verificationism, and the Philosophy of Mathematics

Rodych (2008) has given a detailed account of the role verificationism plays in Wittgenstein’s philosophy of mathematics. Although it is meant to be an account of how Wittgenstein’s philosophy of mathematics developed into his later work, since this development largely occurred in the intermediate period, this paper also serves to outline the development of Wittgenstein’s philosophy of mathematics, insofar as it involves verificationism, in that period. Rodych emphasizes four different positions found in Wittgenstein’s thought, and argues for what he takes to be the most likely way in which Wittgenstein combined these positions to form his developed view in the philosophy of mathematics. While he admits there are tensions in, and even contrary elements of, these positions in the development of Wittgenstein’s thought, Rodych thinks he can, using the positions he outlines, construct what is arguably the most defensible position Wittgenstein could take.

Rodych elaborates four positions: Strong Verificationism, Weak Verificationsm$_1$, Weak Verificationism$_2$, and Structuralism. Strong verificationism claims that the sense of a mathematical proposition *is* its proof. Weak verificationism$_1$ claims that the sense of a proposition is determined by its proof; and that this proof gives a new meaning to the proposition and thereby creates a new calculus. Weak verificationism$_2$ claims a formula constitutes a mathematical proposition (which has mathematical sense) – if and only if it is algorithmically decidable in an existent mathematical calculus and we know this to be the case. The sense of a mathematical proposition is determined by a decision procedure. This suggests that even the ‘opposite’ of a proved mathematical proposition would have sense (as, we have already seen, Wittgenstein at least at times thinks is the case). Weak verificationism$_2$ is essentially the position we examined above. Finally, there is Wittgenstein’s structuralism. This position identifies the sense of a mathematical proposition with its location in a calculus together with its syntactical connections within that calculus (Rodych 2008, 84-85). This emphasizes Wittgenstein's position that mathematics is syntax (essentially grammatical rules). To be a proposition with sense just means to have a position in the calculus. Mathematical propositions are 'true' insofar as they have sense, which is established by their position within a calculus. Rodych does not think Wittgenstein maintains Strong Verificationism for very long. At least, Wittgenstein's views develop greater nuance and intricacy than just exemplifying Strong Verificationism. Thus, the real tension in Wittgenstein's thought Rodych identifies as being between Weak Verificationism 1 and 2. Rodych says:

> The merit of this position – Wittgenstein's Weak Verificationism$_2$ – is that it clearly defines a mathematical proposition as a Csign that is algorithmically decidable in an existent calculus. The problem, however, is that it does not say *what* the sense of a mathematical proposition *is*. In particular, Weak Verificationism$_2$ says that an undecided mathematical proposition, which is algorithmically decidable, *has sense* (i.e., since it is a *meaningful* or genuine mathematical proposition), whereas Weak Verificationism$_1$ precludes *undecided* (or perhaps *unproved*) mathematical propositions from having sense. (Rodych 2008, 85)

He also notes one of the best ways to resolve this tension: it makes sense to reject the claim that an algorithmically decidable formula has sense before it is decided. It should be noted that there were problems with this position that Wittgenstein also identified. For example, Wittgenstein notes that under this interpretation there might be no real mathematical problems (*PR* 170), but only calculations (i.e., mere 'homework' – *PR* 187).[95] This is clearly implausible, since the paradigm cases of mathematical problems are indeed the ones that occupy mathematicians for years (and are not mere 'homework'). Similarly, this position, at least as it is developed in one way by Wittgenstein, would seem to require the idea that mathematical propositions are bipolar, which is absurd. The decision procedure opens up the possibility that it is intelligible to think that a mathematical proposition, based on a mistake in calculation, could have a different mathematical status (i.e. 'truth value') from what it has. This is to confuse the possibility of our making a mistake when doing mathematics with the necessity of mathematics itself.

[95] Here I have in mind the discussion of *PG* 392 above, where we noted that the decision procedure, together with the possibility of making an error, allowed for the possibility that a mathematical sentence could be imagined to be true or false. As we also saw, Wittgenstein was (rightly) at least aware of the potential objection to this position (also *PG* 392), and, as we shall see (below), develops a convincing argument against this in his later work.

Based on these findings, Rodych is right to conclude that the most plausible way to resolve the tension in Wittgenstein's thought, indeed one that Wittgenstein himself likely would endorse, is to give up the idea that an algorithmically decidable formula has sense before it is decided. Thus, Rodych opts for this position:

> **Weak Verificationist$_1$ Structuralism (WV$_1$S)**
>
> (A1): Mathematical Proposition: A Csign is a mathematical proposition *of* calculus Γ *iff* it is algorithmically decidable *in* calculus Γ and we know this to be the case.
> (B1): Having Mathematical Sense: Only primitive propositions (e.g., axioms) and proved propositions of calculus Γ *have* mathematical sense in calculus Γ.
> (C1): The Sense of a Mathematical Proposition of Calculus Γ: *is* its syntactical position in the syntactical structure that is calculus Γ (Rodych 2008, 86)

The problem is, however, that WV$_1$S falls into a confusion from a similar source. Rodych implausibly believes that Wittgenstein remained committed to the central role of a decision procedure into his later work. This leads Rodych to try to account for Wittgenstein's position in the first clause of WV$_1$S. That is, Rodych ultimately concludes that Wittgenstein held to the position that 'a Csign is a mathematical proposition of calculus Γ iff it is algorithmically decidable in calculus Γ and we know this is the case' (Rodych 2008, 86). There are several interrelated problems with this position. First, it suggests that a mathematical proposition that doesn't have sense is perfectly intelligible. But, arguably, Wittgenstein's whole reason for employing the verification principle in the philosophy of mathematics was because he realized that there are such things as mathematical propositions; based on his reflections in the early intermediate period, he came to see that mathematical sentences can be viewed as having meaning. To separate the concept of meaning from the concept of a proposition would seem to undermine this very development in Wittgenstein's thought. Second, there is little, if any, textual evidence to suggest that Wittgenstein did indeed make this distinction. Instead, as Rodych suggests with Weak Verificationism$_2$, Wittgenstein seems to identify the sense of all propositions that relate to a decision procedure to be given by the decision procedure itself.[96] And it is doubtful that he held to this position in his later work (see below). Rodych provides no textual evidence to suggest that

[96] Of course, Rodych realizes this can't be consistently maintained and thus rejects this claim already when combining his positions to form WV$_2$S. Nonetheless, this is the position according to Weak Verificationism$_2$.

Wittgenstein separated the possibility of something being a proposition from its having sense. Third, even aside from the specifics of Wittgenstein's philosophy, it is strange to think, without considerable argument, that there is any benefit in distinguishing between propositions and the sense of a mathematical sentence. Propositions are typically *understood as* the sense of a sentence. That is, to talk about propositions at all is to think of them as having meaning. Finally, in making the distinctions Rodych does, he essentially defines the idea of a decision procedure into Wittgenstein's mature thinking. This comes at the cost that there are propositions that gain sense, and formulas that become propositions by gaining sense (e.g. any problem or conjecture that is proved and isn't mere 'homework'). That is, there are essentially two ways in which something can be a 'proposition': by being algorithmically decidable or not being algorithmically decidable but having been proven.[97] According to this explanation it is not even entirely clear what is precisely meant by a 'proposition'. In any case, given Rodych's rejection of his WV_2 claim that the decision procedure gives *sense* to the proposition before it is decided, there seems to be no (other) textual evidence to suggest Wittgenstein would call anything a 'proposition' on the basis of the existence of a decision procedure.

This brings us to a separate, but related, problem. As Rodych gladly admits, a conjecture that has not been proved does not have meaning. Thus, he accepts that all conjectures are not propositions (until they are proven). Indeed, he considers this the revisionist consequence of Wittgenstein's philosophy (but one that does follow from Wittgenstein's work). It should be noted that there is textual evidence for this reading (*PR* 175-176). However, there is also textual evidence that points in a different direction, and one that makes Wittgenstein's conclusion regarding mathematical problems and conjectures less revisionist, and therefore more palpable, than Rodych suggests. And this alternative escapes Rodych's notice.[98]

As the problem is set up by Rodych, on the basis of some textual evidence from Wittgenstein, it makes sense to think that a mathematical formula only has sense once it is proved. This, however, leads to the implausible conclusion, as Wittgenstein suggests

[97] Although Rodych doesn't talk about a formula becoming a proposition, this is clearly implied by his idea that the formula acquires sense by being proven. And, from how he discusses conjectures later in his paper (2008, 93), it would appear that they do become propositions by acquiring sense.
[98] It is debatable whether Wittgenstein was even convinced of their actual meaninglessness in the early intermediate period (Schroeder 2021, 42-43). Nonetheless, this is a position Wittgenstein considers in the passages cited above.

(at times), that some of the most famous mathematical 'propositions', such as Goldbach's conjecture, do not have sense at all (until being proven) (*PR* 175-176). Moreover, it raises obvious problems regarding exactly how these meaningless 'propositions' are used by mathematicians. How is it that a meaningless 'proposition' can be the basis of extensive mathematical research? Wittgenstein did indeed struggle with this problem, although we needn't concern ourselves with all of the details here.[99] The important point, for our purposes, is the fact that, already in the intermediate period (e.g. *PG* 374), Wittgenstein suggests that the solution is the fact that a mathematical conjecture may have an empirical sense before being incorporated (with a proof) into our mathematics and thereby being given a proper *mathematical sense*. As an example, we can use (again) Goldbach's conjecture, which states that every even integer greater than two can be expressed as the sum of two primes. All of the concepts employed in Goldbach's conjecture are perfectly understandable (as we have already seen in relation to 'prime') and we even have a way of checking many instances of the conjecture up to a certain even integer (which gives empirical credence to our conjecture) (*BT* 616-618). Indeed, we can easily do this using check procedures; this is perhaps what Wittgenstein means when he refers to this empirical sense as a 'rough pattern of that sense in the word-language' (*PG* 374). In this way, the conjecture has a high degree of probability (cf. Schroeder 2021, 49-50). Nonetheless, this is not to give it a mathematical sense, since it does not have a proof that makes it a part of our mathematical systems (*RFM* 280-281). It is the proof that gives it normative legitimacy (i.e., makes it into a grammatical rule) and gives mathematical meaning to the proposition. As we will see in Chapter 4, it is only a rule that can establish a result for an infinite number of cases, which is given by the proof. By overlooking this possible solution, Rodych makes Wittgenstein's position even more revisionist than it needs to be. We can think of mathematical conjectures and problems as having an empirical sense, although they only obtain a proper mathematical sense once they are proved.

[99] Wittgenstein considers, among other things, the idea that these are problems for which we do not yet have a written system to solve, and that mathematicians come up with solutions in psychic symbolism 'in the head' that they then endeavour to get onto paper (*PR* 176). He also considers the idea that it is only ordinary language that makes one think the proposition has a sense before being proved (*PR* 189), but that this would indicate that the proposition, or problem, does not really have sense before being proved. In addition, he suggests that even though these 'propositions' or 'conjectures' have no proper mathematical meaning, they might have sense insofar as they have heuristic value. Although all of these are lines of thought Wittgenstein considers, they all also contain problems, which Schroeder outlines (2021, 45-47). See also Schroeder and Tomany (2019, 83-86).

Arguably, the *importance* of a decision procedure, as this arises as a result of Wittgenstein's appeal to the verification principle, declines in Wittgenstein's later work. This would not be surprising since Wittgenstein, as even Rodych suggests, is aware of the tensions and limitations related to its central role in his intermediate philosophy. Moreover, Wittgenstein develops an argument to account for the fact that there are formulas that can be checked by a decision procedure and still be rightly seen as not propositions of mathematics. Although Wittgenstein was tempted by the idea that both a proposition and its negation can be seen as meaningful (relative to a system of mathematics where a decision procedure is possible), he ultimately was also aware of the fact that mathematics is not bipolar and that only a formula or its negation can be properly viewed as a proposition of mathematics. Schroeder, on the basis of Wittgenstein's later work (*RFM* 76-79), explains how it is best to describe incorrect equations. He rightly points out that Wittgenstein suggests, in his later work, that an incorrect equation, such as $16 \times 16 = 169$, is better thought of not as a mathematical proposition, but rather, insofar as we think it is correct, we *believe* it to be a mathematical proposition (until we correctly calculate it) (Schroeder 2021, 172-173). This avoids the problem in Rodych's account that requires some mathematical propositions (incorrect equations) not to have any sense (yet they would still be considered to be propositions, according to Rodych's account). According to Wittgenstein's later work, we are perfectly justified in saying that an incorrect multiplication lacks sense. Moreover, under his account, we are not prohibited from still saying that such a formula has a use, insofar as it can be applied outside mathematics (most likely leading to empirical error) and insofar as such formulas always admit of a check procedure. Nonetheless, insofar as such formulas are incorrect they lack genuine mathematical content. And, on the basis of this, they are best viewed as not being *mathematical propositions* at all. Rodych's account has the odd consequence that some formulas are considered propositions before they are proved. This is connected with the development in Wittgenstein's thinking regarding what constitutes the meaning of a mathematical proposition.

Paralleling Wittgenstein's change from the calculus conception of language to a language-game view and, connected with this, the restriction of the verification principle to certain uses of language, Wittgenstein similarly restricts the role the verification principle plays in his philosophy of mathematics. This leads him to place greater emphasis on the use of a mathematical proposition as a source of its meaning.

The view that the meaning of mathematical propositions is given by their proofs is part of the more general verificationist view that states the meaning of a proposition is its method of verification. Of course, there are important differences, as we have seen, between the empirical and mathematical case, but Wittgenstein was still prepared to give the idea that proof gives meaning to a proposition central importance. With Wittgenstein's investigation of language games, he turns to placing emphasis also on the applications of mathematical concepts. This relates to his new focus on the use of a proposition (cf. Schroeder 2021, 183). Proof, it is true, is a source of meaning. A person who knows a true equation based on authority and is unable to work it out himself can't be said to have a complete understanding of it. And, being able to work out a true equation means one must have some understanding of the equation (Schroeder 2021, 184). However, this is not to say that proof is also always sufficient for meaning. For, as Wittgenstein suggests, one may be able to follow a proof step-by-step, but still not understand what it proves (*RFM* 282f). This is because a proof is only one instance of part of intricate, interrelated, systems of calculation that 'stand behind' the proposition, but are not made explicit in the proof (*RFM* 313d). And even someone who has mastered an entire system of calculation may still only use it mechanically and not understand what the signs being employed mean (*RFM* 257-258; cf. Schroeder 2021, 184; Rodych 2000, 301-302). Moreover, Wittgenstein provides a thought experiment to show that the application of mathematical propositions is separable from its proof.[100] The upshot of these reflections is that Wittgenstein acknowledges two sources of mathematical meaning: proof and application. That is, mathematical signs must be able to be employed in empirical propositions (cf. Schroeder 2021, 185-186).[101] We shall return to examine the role of application in Chapter 6, and we shall see how the distinction here is perfectly exemplified in the debate between 'descriptivism' and 'revisionism' as it relates to set theory (and thus ultimately reconcilable, when Wittgenstein's philosophy is properly understood). To return to our example of an incorrect equation of mathematics: it makes sense to think of it as having some

[100] This idea is presented in *RFM* 258ef. Wittgenstein imagines calculating machines that occur in nature, and people who use the results of these machines as we do. These people would have no access to the proofs of the results they employ, yet the results would serve a similar function in their life.
[101] The most important quotation that exemplifies this position is the following: 'I want to say: it is essential to mathematics that its signs are also employed in *mufti*. It is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics' (*RFM* 257de).
For an extensive discussion of some of the specific ways in which the idea of application is used in Wittgenstein's philosophy of mathematics, see Schroeder (2021, 183-188).

meaning, insofar as it can be applied outside mathematics and insofar as it can be checked. However, this does not make it a mathematical proposition, since it is not part of the system of mathematics. Using Rodych's terminology, it violates Wittgenstein's structuralism; an incorrect equation does not have a position in the mathematical calculus. This preserves some use for a decision procedure in Wittgenstein's explanation, albeit its role as an explanation of meaning and its role in Wittgenstein's philosophy of mathematics are greatly reduced. Indeed, instead of being the standard by which other parts of mathematics are judged, it is incorporated into Wittgenstein's explanation of mathematical propositions (and his philosophy of mathematics) generally. And, of course, this is a reflection of the decreased/restricted use of the verification principle (ultimately) even within the philosophy of mathematics.

With this alternate explanation of incorrect equations in mind (which fits more generally with respect to any decision procedure), it is reasonable to assume Wittgenstein did indeed move away from having the idea of decision procedure as an important part of his later philosophy. Instead of giving central importance to the verification principle to decide what is really a mathematical proposition, question, or conjecture, he accepts some of the ordinary distinctions we make about mathematics and seeks to explain these. This includes accepting the idea of, for example, mathematical research (in contrast to mere 'homework') and mathematical conjectures, and, together with this, giving a plausible explanation of the latter's meaningfulness both before and after being proved. Wittgenstein, in his later work, restricts the use of mathematical verificationism, as he does in the empirical case; proof does indeed give meaning to a mathematical proposition, but this is only *one* element of its meaning. In line with this, Rodych's explanation of mathematical meaning, when properly qualified, does serve as an explanation of meaning *as it relates to proof*.[102] However, Rodych, in this context,[103] neglects the idea of meaning derived from application (more broadly, from different 'uses' of mathematical signs/propositions), as well as Wittgenstein's later considerations about mathematical conjectures and incorrect equations. As we

[102] Specifically, WV_1S_2 arguably accurately captures Wittgenstein's later philosophy of mathematics best, at least in terms of the part of a proposition's meaning that derives from its proof. This is unsurprising, since Rodych sees the disadvantage of WV_1S_2 to be precisely what we have argued is the case with respect to Wittgenstein's later philosophy of mathematics: that an algorithmically decidable formula is not a proposition. This position affirms the ideas that only 'primitive propositions' (i.e. axioms) and proved propositions of a calculus are both propositions and have mathematical sense.
[103] As we shall see in Chapter 6, Rodych gives this aspect of mathematical meaning central importance when he examines set theory.

have seen, these considerations explain why mathematical verificationism is no longer central to his philosophy of mathematics in his later work.

We have examined the development of the verification principle in the intermediate period. As we have seen, this began, shortly upon Wittgenstein's return to philosophy, with the use of ideas that anticipated the explicit principle. These ideas were used to make logical distinctions within the philosophy of mathematics. That is, Wittgenstein used the idea of 'verification' to distinguish types of propositions by indicating that they mean something different. Seeing that 'verification' referred to the comparison of the proposition with the world for Wittgenstein in early 1929 (a continuity with the *Tractatus*), we proceeded to examine the connection between the *Tractatus* and verificationism. From there, we examined the relation between verificationism and the Vienna Circle. We saw that there is only evidence to support the idea that the Vienna Circle's verificationism originated from Wittgenstein. We then explored whether the doctrine of verificationism was already implicit in the *Tractatus,* and we concluded it wasn't. Although the term 'verification' is used to mark a continuity with the *Tractatus,* the doctrine of verificationism itself indicates a new development in Wittgenstein's intermediate thought. We then examined this development in much more detail: we showed that the verification principle in its explicit form was a reaction to the demise of the phenomenological language, along with Wittgenstein's commitment to the ideas that an empirical proposition is compared with the world and that this is done relative to different 'spaces'. The verification principle is meant as a way of bypassing the problem of specifying the shared logical form between language and the world. Moreover, this developed roughly concurrently with his idea of grammar (as a comprehensive discipline), and thus we see common features to both. We ended this examination by looking at how Wittgenstein ultimately limited the use of verificationism in his later work. To conclude the chapter, we investigated Wittgenstein's use of verificationism in the philosophy of mathematics (which continued later into the intermediate period). We saw how verificationism was used to delimit meaningful mathematical propositions, and how this related to his position on mathematical conjectures, problems, and questions. Using Rodych's work as a point of departure, we then considered the various contradictory elements of Wittgenstein's mathematical verificationism and, using insights from Wittgenstein's later work, assessed the limitations of Rodych's characterization of Wittgenstein's most convincing verificationist position as well as the limitation of Wittgenstein's developed intermediate period verificationist view itself. Finally, we examined how and why Wittgenstein's use of the verification principle in relation to the philosophy of mathematics was restricted.

4. Infinity

A proper understanding of the verification principle is a necessary prerequisite to an understanding of Wittgenstein's critique of the extensional conception of infinity, which in turn is necessary for an understanding of his own analysis of infinity and his application of it to other work. Having already dealt with the verification principle, we shall first examine Wittgenstein's examples of the extensional infinite (i.e., the idea of an existing infinite totality) and his arguments against them. This will be done in two sections; first, we shall explain this conception of the infinite as it relates to the empirical world, and then we shall explain it as it relates to the *a priori* discipline of mathematics. An examination of Wittgenstein's discussion of the relationship between the concepts of the infinite, and possibility and actuality, will follow. We shall then explore the confusions that underlie the extensional conception in more detail. We then turn to the *Tractatus*' influence on Wittgenstein's thought on the concept of the infinite in 1929. Given that we have already dealt with many of the general views expressed in the *Tractatus* in Chapter 1, here our focus will be on a few passages that exemplify Wittgenstein's position in 1929 and the position's relationship to the *Tractatus*. We will then focus on how this position developed in the early part of the intermediate period, first outlining the most general change in Wittgenstein's position, and then examining how this applies to specific arguments. We conclude with a critical analysis of some of the details of Wittgenstein's new position concerning infinity in 1931.

4.1 First Steps: Against the Extensional Conception of the Infinite in the World

Even disregarding the technical elements of the verification principle, all of the ways this principle relates to criticisms of infinity, and how these arguments combined relate to technical aspects of the philosophy of mathematics, it is clear that Wittgenstein was contemplating the nature of infinity in a very general way upon his return to philosophy in 1929. One of his earliest comments upon returning to philosophy includes the assertion that he had once [*einmal*] claimed that there is no 'extensional infinity' (MS 105, 23).[104] Ramsey had questioned this claim by asking Wittgenstein to imagine a man

[104] The 'extensional infinite' is the idea that there is an infinite collection of objects existing as a totality.

who lives forever.[105] According to Ramsey, this would be an example of the extensional infinite (MS 105, 23). Wittgenstein, as evidenced by *The Philosophical Remarks*, also envisions other similar examples: iron spheres that go on forever, a row of trees that never ends, and a wheel that never stops spinning (*PR* 166 and MS 105, 23). And, as part of the problem as it relates to the philosophy of mathematics, he imagines various individuals who 'select' numbers/fractions for an infinite length of time. In all cases, Wittgenstein gives a variety of related counterarguments to undermine these supposed examples of the extensional infinite. Most generally, he claims: 'You can only answer the objection "But if nevertheless there were infinitely many things?" by saying "But there aren't". And what makes us think that perhaps there are is only our confusing the things of physics with the elements of knowledge' (*PR* 168).[106] This general argument is the center of gravity for all the more specific arguments Wittgenstein gives against the extensional infinite during this time period.[107] It seeks to deny that there can be an infinite collection of objects and to link the infinite with 'elements of knowledge'.[108] It will be further examined in the context of the specific examples denying the extensional infinite that he provides.[109]

[105] Similar examples seem to all arise from discussions with Ramsey on the topic. They all deal with the possibility of an infinity in reality (i.e., an experienced infinite totality of objects or something 'never ending'). 'Reality' and 'actuality' are equivalent for Wittgenstein in this context.

[106] This is the focus of Wittgenstein's critique of the extensional conception in 1929. *At least at times* he puts his argument in terms of a denial of the existence of a fact (i.e., the existence of an infinite collection of objects). In 1931, however, as argued extensively by Kienzler (1997, 164-165), Wittgenstein, after re-evaluating his earlier comments, comes to think that it is better exclusively to frame the argument in terms of denying the sense of Ramsey's arguments/questions. This will be dealt with in much more detail later in the chapter.

[107] Giving this central importance is the only way I can see to make sense of Wittgenstein's later claims in 1931. This is explained in great detail later in this chapter. However, while some version of this argument holds an important place for Wittgenstein, it should become apparent that he also uses several other different arguments, in an attempt to delimit sense from nonsense, quite successfully. Some of these live on in his later work.

[108] Very briefly explained, 'elements of knowledge' seem to be the members of a series or expansion etc., the infinite nature of which is given by a rule/law which allows for the possibility of its unbounded construction. For reasons that are not readily apparent, Moore (2011, 116) links 'elements of knowledge' with 'things', even though Wittgenstein seems to indicate the terms stand opposed. It is also interesting to note that in the manuscript, as a further explication of the above quote, Wittgenstein says: 'The only reason why you can't say there are infinitely many things is that there aren't. If there were, you could also express the fact!' This clearly involves the *Tractatus*' thesis about the shared form between language and the world and the saying/showing distinction which is further explained in Section 4.5. Seemingly, Wittgenstein's point is that a 'proposition' about an infinite totality can't be expressed at all as evidenced by the arguments given in the first three sections of this chapter. And the reason for this is the metaphysical arguments given in the *Tractatus*.

[109] These arguments themselves are able to stand alone, and will thus be examined as such here. However, it should be noted that in the background remains Wittgenstein's views about the *Tractatus* (which lend these arguments additional support). The most general argument (quoted above) that there are not 'infinitely many things' clearly is dependent on views in the *Tractatus*, as it is backed up by the

One of the examples Wittgenstein envisions is the following:

> The situation would be something like this: We have an infinitely long row of trees, and so as to inspect them, I make a path beside them. All right, the path must be endless. But if it is endless, then that means precisely that you can't walk to the end of it. That is, it does *not* put me in a position to survey the row. (*Ex hypothesi* not).
>
> That is to say, the endless path doesn't have an end 'infinitely far away', it has no end. (*PR* 146)

One may talk about an infinite number of objects, but to think of this as an existing totality is to fall into confusion. By emphasizing the fact that an infinitely long row of trees never comes to an end, Wittgenstein rightly draws attention to the fact that there are no experiential criteria that could be possibly fulfilled that would vindicate the extensional conception. An infinitely long row of trees is precisely one that never comes to an end (not one that comes to an end at/after an infinite number of trees!). Thus, such a row of trees is not 'surveyable' for its infiniteness. At any point as one travels along the infinite row, one has not yet reached a point where one can say there are an infinite number of trees (or that the row is infinitely long), so one cannot say an infinite totality of objects has been encountered. Any imagined example used to support the extensional conception fails for similar reasons. Another one is the following:

> Imagine the following hypothesis: there is in space an infinite series of red spheres, each one metre behind its predecessor. What *conceivable* experience could correspond to this hypothesis? I think for instance of my travelling along this series and every day passing a certain number, *n*, of red spheres. In that case, my experience ought to consist in the fact that on *every possible* day in the future I see *n* more spheres. But when shall I have had this experience? Never! (*PR* 167-168)

Once again, there is no point at which one can rightfully say that one has had the experience of seeing n more spheres 'every possible day in the future' precisely because this too involves an infinite number of days; at no time has one actually encountered an infinite totality of objects. Similar arguments can be applied to all of the different examples Wittgenstein uses: Ramsey's man who lives forever, the wheel that never stops spinning, as well as the infinitely long 'straight line' of iron spheres. No matter how long one knows a man, how long a wheel spins for, or how many iron spheres one comes across, one is not justified in saying the period or number in question is infinite.

quotation in footnote 108 – which relates to the shared form of language and the world and the saying/showing distinction. This is explained further in Section 4.5.

These arguments may seem to be limited to epistemology and concern only what we can *know* about or what *we are justified in claiming* about an infinite totality of objects. And, it is true, there are some, at least *prima facie*, unclear points in Wittgenstein's thought from this time. At times, Wittgenstein seems to allow for the possibility that an existing infinite series is imaginable, such as an unending row of trees or stars, only there must be a rule/law that indicates, for instance, how the sizes of the trees are distributed to infinity. Or he imagines cases that don't involve a 'construction' at all (e.g. a wheel spinning forever). However, in these cases, the above arguments still present a devastating problem for the extensional conception. For what experiential criteria could possibly indicate that the row of trees (or stars) or wheel spinning will never end? Wittgenstein rarely brings up the verification principle in these comments (a notable exception is in 1931, when he discusses the Law of Inertia), but it is apparent, especially given what has been outlined in the previous chapter, that his arguments are meant to call to mind what a verification in the case of an existing infinite totality would look like. It is apparent that such an *existing totality* could never be experienced as such. And, thus, with the verification principle in mind, granted everything that has been said in the previous chapter, the above arguments don't concern merely epistemology, but charting the bounds of sense (in this case what can be sensibly said about experience and the infinite generally). For, as Wittgenstein accepted from the intermediate period onwards, getting clear about the possibilities of the verification of a proposition is one way of establishing its meaning (*PI*, §353). And the extensional conception of the infinite doesn't admit of a verification. This delineating of the bounds of sense is in line with the project of the *Tractatus*. Further arguments against the intelligibility of an infinite totality are given in the next section. These deal primarily with the possibility of an existing infinite totality in mathematics, making the intensional infinite all the more plausible.[110]

Wittgenstein, it is true, also suggests that there may be meaningful statements involving natural laws that importantly relate to our concept of infinity,[111] although he does not discuss this in much detail. The suggestion of natural laws making intelligible the idea of an infinitely long series of objects in the world is made in 1929 (e.g. *PR*

[110] If the intensional model can be seen to be the only viable alternative to understanding infinity in mathematics, then, especially given what has already been argued, it becomes all the more tempting to extend it to other areas as well.
[111] Moore even interprets comments from 1929 in this manner (2011, 111).

166). In 1931, in addition, he considers the Law of Inertia which can make claims that under certain conditions the movement of a body ‘will never end’ (*PR* 307). One must remember in these cases that Wittgenstein imagines natural laws as importantly similar to rules. This is shown by his use of the verification principle, which indicates that there is such a thing as falsifying the proposition, but no such thing as verifying it. This, as discussed in the last chapter, indicates at least that it is a proposition in a different sort of way (*PR* 307). As akin to a rule, then, it functions differently from a proposition that could be verified.[112]

4.2 Going Further: Against the Concept of an Infinite Totality in Mathematics

Wittgenstein also argues extensively against the idea of an infinite totality in mathematics, which becomes essential to defending the intensional conception (both within and outside mathematics).[113] Against the idea of an infinite totality, Wittgenstein states the following:

> Let’s imagine a man whose life goes back for an infinite time and who says to us: ‘I’m just writing down the last digit of π, and it’s a 2’. Every day of his life he has written down a digit, without ever having begun; he has just finished.
>
> This seems utter nonsense, and a *reductio ad absurdum* of the concept of an infinite *totality*. (*PR* 166)

It is patently absurd that one should complete listing the digits of π (made comical with the addition that the final digit is 2). In addition, in order for a person to have reached the point where he could have written the final digit of π, he would have had to start an infinite[114] time in the past. That is, he never would have begun![115] This argument can

[112] This would be a very natural development in his thought from his view in the *Tractatus* which, according to Constantine Sandis and Chon Tejedor, is that a natural law is the ‘instruction for the construction of senseful propositions within a particular natural science system’ (2017, 579). A more detailed discussion of this view or its relationship to the intermediate period view is beyond the scope of this chapter.

[113] As should be evident from Chapter 2, it is possible to dismiss outright the idea that an infinite mathematical totality involves the meanings of numerals (e.g. numbers) existing in a mathematical realm.

[114] It is, of course, noteworthy that it is necessary to imagine an infinite amount of time to make this case plausible at all. That is, to even begin to make an example of an infinite list of numbers possible, one must imagine an infinite amount of time to do so. Of course, the example is proven to be a logical impossibility, but even if it made sense, it would hardly serve as an *explanation* of an infinite totality or how one is possible. This, of course, agrees with Wittgenstein’s claim that the infinite can never be explained without making reference to itself (*PR* 158 – quoted later in this chapter).

[115] Strangely, Potter (2011, 126) sees this argument as evidence of obvious confusion on Wittgenstein’s part. He claims that Wittgenstein must have misstated his argument since π is irrational in nature and thus

similarly be applied to the previous examples, and, of course, a similar problem arises if one imagines someone, having started at a set time, attempting to write down all of the digits of π. Wittgenstein says:

> Let's imagine someone living an endless life and making successive choices of an arbitrary fraction from the fractions between 1 and 2, 2 and 3, etc. *ad. inf.* Does that yield us a selection from all those intervals? No, since he does *not* finish. But can't I say nonetheless that all those intervals must turn up, since I can't cite any which he wouldn't *eventually* arrive at? But from the fact that given *any* interval, he will *eventually* arrive at it, it doesn't follow that he will *eventually* have arrived at them all. (*PR* 167)

Once again, while it is true that eventually one will come to any particular digit of π given a long enough time to write down the digits, it does not follow that one will eventually have written down all of the digits of π. Wittgenstein's point, again and again, is to emphasize that there is no such thing in this case as writing down all of the digits of π (or even attempting to do so, since there is no possibility of success). There is, in fact, no such thing as all of the digits of π. Wittgenstein claims this repeatedly about 'all numbers': 'But you can't talk about *all* numbers, because there's no such thing as *all* numbers' (*PR* 147). It is senseless to speak of 'all numbers' as a collection when this is clearly infinite in number (e.g. the natural numbers). The same is true of π. Neither are given as an infinite collection, but their infiniteness arises with the rule that ensures ever more numbers/digits of π can always be constructed.[116] It immediately

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unending. This is to misunderstand the nature of Wittgenstein's argument. Here, Wittgenstein is clearly assuming the intelligibility of the infinite conceived as an infinite totality. Thus, he assumes π can be imagined in this way. Of course, any other infinite sequence would work just as well. Assuming it exists as an infinite totality, one must allow for the possibility of listing the last member of the series. Since the amount is infinite there would have to be an infinite number of numbers before the last member also. But this would mean that the process of listing members, which terminates with listing the final member, has no beginning. Thus, the problem is with the intelligibility of conceiving of the infinite as an existing totality. In order for one to list the last member of the series, one could never have begun!

My interpretation is clearly supported by the fact that Wittgenstein elsewhere explicitly links the idea of the concept of infinite with that which is unending. In his own argument, he realizes that π is infinite, but wishes to disprove the interpretation that would suggest one should interpret this infinity as an existing totality. Moreover, it is apparent that he is not only aware of π being an infinite decimal expansion, but that this is because π is irrational (e.g. *PR* 223). Moreover, he ultimately argues for the idea that infinity is to be regarded as that which doesn't end, so it would appear that the above argument is meant as a *reductio ad absurdum* of the conception of infinity as an existing totality.

[116] Wittgenstein at least considers the idea that the intensional model can make intelligible the imagining of an infinite number of objects in reality (although not as an existing totality – he repudiates the extensional model). For example, one can imagine an infinite row of trees precisely when one has a rule for selecting the size of such trees. It becomes impossible to imagine if one is asked to think that the size of the trees is random or when one thinks of the uniqueness of each tree (*PR* 166). Similarly, the man who lives forever or the wheel that spins forever can be more easily imagined since it does not involve the 'construction' of additional objects that at the same time must be imagined to already exist, and can be seen as taking place over time (*PR* 165-166); in the case of the never ending life, the 'forever' or

follows that writing down, for example, all of the natural numbers is a logical impossibility and not an empirical one. 'It isn't just impossible "for us men" to run through the natural numbers one by one; it's *impossible,* it means nothing' (*PR* 146). There is no sense to 'running through all of the numbers' or 'listing all of them'. These phrases have no meaning precisely because the infinite is boundless. It is impossible to list all of the natural numbers or all the digits of π because it is always, at any stage, possible to list more. And, while it may be tempting to think that such a limitation is due to human weakness, as is apparent from the above quotation, Wittgenstein rejects this suggestion.[117] For there can't be – in the logical sense – such a thing as all of the digits of π. 'All of the digits of π' means nothing. Hence, even an omnipotent being could not know all of the digits of π. As Wittgenstein says:

> 'Can God know all the places of the expansion of π?' would have been a good question for the schoolmen to ask. In all such cases the answer runs, 'The question is senseless.' (*PR* 149)

The quoted question seems similar to asking about a person's abilities, such as whether they can lift a certain weight or learn a language in a set amount of time. And something like God, one reasons, may have the abilities to do what all men can't. In reality, the questions are radically different: one asks about the physical/mental abilities of a person, the other has only the form of such a question and is actually nonsense. As indicated, we are not talking about an empirical possibility, which is indicated by the tricky expression 'the expansion of π' which, if not properly understood, leads one to think of 'infinite' as similar to a very big number, where, in fact, it is actually utterly distinct from a number. The infinite is boundless; it has no end, not one infinitely far

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'never ends' sufficiently explains its infinite nature, playing the same role as the 'ad. infin.' in mathematics. Wittgenstein never explains what, in any of these cases, would constitute experiential criteria that something indeed goes on forever. So, the one thing that still remains as a possible candidate of the manifestation of the infinite in reality has to do with very general physical laws. These can possibly involve a claim about some movement never ending precisely because of its unique use/role (which is evidenced by how they are verified). And, although these examples quite possibly have criteria to justify saying they are infinite in nature, this still does not mean these examples of infinity are actually experienced. A detailed analysis of this is beyond the scope of this chapter.

[117] Similarly, in the Lee lecture notes: 'Will three consecutive sevens ever occur in an evaluation of π? People have an idea that this is a problem because they think that if we knew the whole evaluation we should know, and the fact that we don't know is merely a human weakness. This is a subterfuge. The mistake lies in the misuse of the word infinite, which is not the name of a numeral' (*LL* 107).

away. And it is not to be compared with a number. There is a categorial distinction between numbers that are finite and the infinite.[118]

4.3 Infinity, Possibility, and Actuality

Connected with the denial of any infinite collection is Wittgenstein's linking of the concept of infinity with the concept of possibility. The concept of infinity for Wittgenstein is *essentially* something that relates to a possibility. Usually this is the possibility expressed by the rules of a symbolism, although Wittgenstein also talks about this possibility being present in the 'objects', an obvious reference to ideas in the *Tractatus* that will be discussed further below. Wittgenstein says:

> We all of course know what it means to say there is an infinite possibility and a finite reality, since we say space and time are infinite but we can always only see or live through finite bits of them. But from where, then, do I derive my knowledge of the infinite at all? In some sense or other, I must have two kinds of experience: one which is of the finite, and which cannot transcend the finite (the idea of such a transcendence is nonsense even on its own terms), and one of the infinite. And that's how it is. Experience as experience of the facts gives me the finite; the objects *contain* the infinite. Of course not as something rivalling finite experience, but in intension. Not as though I could see space as practically empty, with just a very small finite experience in it. But, I can see in space the possibility of any finite experience. That is, no experience could be too large for it or exhaust it: not of course because we are acquainted with the dimensions of every experience and know space to be larger, but because we understand this as belonging to the essence of space. – We recognize this essential infinity of space in its smallest part. (*PR* 157)

We need not deal extensively with the concepts of space and time as they relate to infinity, nor is it necessary here to deal with the *Tractatus'* influences on this paragraph (this will be done further below). Rather, we can simply focus on Wittgenstein's linking of the concept of possibility to infinity, which is anticipated by the arguments already given. As we have seen, Wittgenstein argues extensively that any *experience* is one fundamentally of the finite. Although he uses the word 'experience' in the above quotation, here, when Wittgenstein speaks of 'experience' of the infinite, he is referring to something that isn't empirical. It is not that one perceives the infinite, but rather that

[118] Wittgenstein is aware that set theory, as a branch of mathematics, extends the notion of number to include infinite aggregates. While it is true that 'number' is a family resemblance concept, and thus such an extension of the use is legitimate, it is also important to keep in mind the important differences between these 'transfinite numbers' and other common examples of numbers. This is further examined in Chapter 6.

it is ‘contained’ in the ‘objects’.[119] Empirical experience is of the facts which include finite numbers of objects, but not infinity. One can experience finite numbers of objects and yet also understand their infinite possibility. Time and space serve as examples. We continually experience finite parts of space and time, but have an understanding that these are infinite. This understanding, Wittgenstein claims, comes immediately with an understanding of space and time – no matter how expansive the nature of that which we have experienced. The infinity of space and time is simply that of an unbounded possibility: to any quantity of space or time that is imagined/experienced, a greater is still possible. This possibility arises in the symbolism, which places no limit on that which can be experienced (and so possibly represented).

Other statements support the linking of the concepts of infinity and possibility:

> You could also put it like this: it makes sense to say there can be infinitely many objects in a direction, but no sense to say there are infinitely many. And this conflicts with the way the word ‘can’ is normally used. For, if it makes sense to say a book can lie on this table, it also makes sense to say it is lying there. But here we are led astray by language. The ‘infinitely many’ is so to speak used adverbially and is to be understood accordingly. (*PR* 162)[120]

Normally, Wittgenstein wishes to emphasize, it is possible for that which is possible to be actual[121]; this is the typical way of understanding the relationship between the concepts. It makes sense to say that there can be infinitely many objects in a direction, since this merely characterizes, for example, the possibility of always listing further objects according to a rule/systematic method. It does not mean an infinite number of objects already exists. Wittgenstein’s point is that ‘infinitely many’ importantly qualifies ‘can’ where this statement incorporating ‘can’ can’t then be transformed into a statement about an actuality/reality. The ‘infinitely many’ qualifies something as a possibility (for construction or what lies in the rules of language use) and in this way

[119] As will be explained further below, this has a clear metaphysical interpretation that goes back to the *Tractatus*. It is possible to deal with these arguments largely on their own, without relying on the metaphysical baggage and philosophy of language of the *Tractatus*.
[120] Similarly, he says: ‘That is to say, the propositions “Three things can lie in this direction” and “Infinitely many things can lie in this direction” are only apparently formed in the same way, but are in fact different in structure: the “infinitely many” of the second proposition doesn’t play the same role as the “three” of the first’ (*PR* 162).
[121] Wittgenstein also uses ‘reality’ for ‘actuality’. Both terms seem to apply to simply what is the case at a certain time (when this applies to empirical matters), or, as we shall see, in the case of mathematics, what is already constructed.

functions ‘adverbially’ and not as an adjective (since it doesn’t qualify the noun).[122] This is illustrated particularly well with the idea of infinite divisibility:

> How about infinite divisibility? Let’s remember that there’s a point to saying we can conceive of *any* finite number of parts but not of an infinite number; but that this is precisely what constitutes infinite divisibility.
>
> Now, ‘any’ doesn’t mean here that we can conceive of the *sum total* of *all* divisions (which we can’t, for there’s no such thing). But that there is the *variable* ‘divisibility’ (i.e. the concept of divisibility) which *sets no limit* to actual divisibility; and that constitutes its infinity. (*PR* 158)

The infiniteness of the divisibility of a line does not consist in a sum total of divisions, since this is nonsense. Rather it consists in the possibility of always meaningfully speaking of further dividing a line, no matter how many times it has been divided. Wittgenstein goes on to talk about the possibility of dividing a line:

> And that again shows we are dealing with two different meanings of the word ‘possible’ when we say ‘The line can be divided into 3 parts’ and when we say ‘The line can be divided infinitely often’. (This is also indicated by the proposition above, which questions whether there are actual and possible in visual space.)
>
> What does it mean to say a patch in visual space can be divided into three parts? Surely it can mean only that a proposition describing a patch divided in this way makes sense. (Provided it isn’t a question of a confusion between the divisibility of physical objects and that of a visual patch).
>
> Whereas infinite – or better *unlimited* – divisibility doesn’t mean there’s a proposition describing a line divided into infinitely many parts, since there isn’t such a proposition. Therefore this possibility is not brought out by any reality of the signs, but by a possibility of a *different* kind in the signs themselves. (*PR* 159)

The mistake that underlies the confusions in this passage will be dealt with in the subsequent section. In the meantime, we shall limit ourselves to Wittgenstein’s comments about divisibility. To say that a line ‘can be divided 3 times’ and ‘can be divided an infinite number of times’ is to make categorially different claims. Here the ‘can’ refers to two different possibilities. In the case of the line, assuming one is not talking about physical space (where the actual practicality of such an outcome might be in question), the above statement indicates that the proposition that describes the divisibility of the line into three parts is meaningful. In the case of talking about the unlimited divisibility of the line, this is not to claim that there is a proposition that

[122] Seemingly, the reason for the use of ‘in this direction’ in Wittgenstein’s comment is that it suggests something unfolding. Some sense can be given to this with respect to empirical and *a priori* domains, although I am not sure Wittgenstein wholly convinces himself of this use in the empirical case.

describes a line divided an infinite number of times – for this is nonsense; there is no such proposition. Rather, it refers to the possibility that arises from the rules of the symbolism which allows for the meaningful description of a further divided line, no matter how many times it has already been divided. It states that the description of a further divided line is always meaningful, even if practically impossible. As explained in Chapter 2, it is a rule that establishes what constitutes meaningful statements of our language.[123]

We can use these arguments to further our discussion of time and space. In both cases, Wittgenstein claims, they are concepts that contain an infinite possibility. And such a possibility is not a shadow of actuality. They are not possibilities that can become actual, for, in this case, as already argued, there is no such thing as infinite time and space conceived like an aggregate or infinite set. Rather, their possibility consists in the fact that with any amount of space or time that is experienced comes the understanding that one has never reached a limit.

> That we don't think of time as an infinite reality, but as infinite in intension, is shown in the fact that on the one hand we can't imagine an infinite time interval, and yet see that no day can be the last, and so that time cannot have an end. (*PR* 163)

That is, time and space have an infinite 'form' which makes it possible to represent anything that is experienced.[124] I agree with Kienzler that Wittgenstein, by 1931, will in addition claim that no particular experience is necessary to the understanding that time is infinite. As this claim relates to the above quotation, this means that Wittgenstein will ultimately *make explicit* that reference to time's not having an end (i.e., 'no day can be the last') does not predict any event and, in particular, one that serves to define the passage of time. For numerous experiences are used to mark the passing of time (e.g. days are defined by the movement of the earth), but, Wittgenstein will argue later, this

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[123] This is nicely supported by the following: '"Possibility" is what is represented by a proposition having sense (grammar is the expression of what is possible…). An "infinite possibility" is not expressed by a proposition asserting it but by a law of construction. Infinite divisibility is not expressed by a proposition asserting that division has taken place, but by a law giving an infinite possibility of propositions asserting further division (division into a given number, not into an infinite number). To a proposition asserting a given number of divisions there is a corresponding reality: there is no infinite reality corresponding to an infinite possibility. Infinity is not a number but the property of a law' (*LL* 13-14).

[124] Elsewhere, Wittgenstein says: 'The infinity of space is the infinity of mathematical induction. It is surely clear that we express nothing *factual* by saying that space is infinite. What we know *a priori* is – here and everywhere – the *form* in terms of which we express our experiences' (*WVC* 217). It is interesting that Wittgenstein chooses to equate infinity in this case to actual mathematical induction.

does not capture what is essential to the infinity of the concept of time. This will be dealt with further later in this chapter. Now, it is simply necessary to note that Wittgenstein (rightly) rejects the notion that the infinity of time or space is extensional in nature.

As a final consideration, it is necessary to consider what Wittgenstein says about infinity, possibility, and actuality as these concepts relate to the philosophy of mathematics. Wittgenstein says:

> The rules for a number-system – say, the decimal system – contain everything that is infinite about the numbers. That, e.g. *these rules* set no limits on the left or right hand to the numerals; *this* is what contains the expression of infinity. Someone might perhaps say: True, but the numerals are still limited by their use and by writing materials and other factors. That is so, but that isn't expressed in the *rules* for their use, and it is only in these that their real essence is expressed. (*PR* 160-161)

As we have already seen, the infinite is the possibility that arises with the rules of symbolism that allow for the unlimited construction of numerals. Wittgenstein anticipates the objection that this construction is limited by the 'writing materials and other factors' and, while this is correct, it is irrelevant to his claim, for these facts in no way define the numerals. To be sure, it would be the case that were very general facts otherwise (e.g. if there was no way to record large numbers), our number-system may indeed not be very useful, but these facts do not serve as the rules that establish the meaning of the mathematical terms. In addition, from the above quotation, it may seem that the infinite possibility is dependent on the possibility of its actuality which, it would seem, is not (empirically) possible. Wittgenstein denies this. The meaning of the infinite possibility is not dependent on an unintelligible actuality. Wittgenstein says:

> Does the relation $m = 2n$ correlate the class of all numbers with one of its subclasses? No. It correlates any arbitrary number with another, and in that way we arrive at infinitely many pairs of classes, of which one is correlated with the other, but which are *never* related as class and subclass. Neither is this infinite process itself in some sense or other such a pair of classes.
>
> In the superstition that $m = 2n$ correlates a class with its subclass, we merely have yet another case of ambiguous grammar. (*PR* 161)

To think of the quoted equation as correlating a class with a subclass is to think of this correlation as already existing as an infinite totality. Of course, this is to be committed to the extensionalist conception of infinity. Against this, Wittgenstein argues that this equation simply correlates an arbitrary number with another. Whatever *m* may be, given the very simple rule expressed in this equation, it is possible to calculate the number with which it is to be correlated. Of course, each such correlation has not taken place.

> What's more, it all hangs on the syntax of reality and possibility. $m = 2n$ contains the *possibility of correlating any number* with another, *but doesn't correlate all numbers with others.* (*PR* 161)

Without the element of time, which gives 'actuality' to empirical possibilities, it may seem that there is no proper distinction between possibility and actuality in mathematics or, as Wittgenstein says, 'possibility is (already) actuality'. This, Wittgenstein argues, is mistaken. In fact,

> The word 'possibility' is of course misleading, since someone will say, let what is possible now become actual. And in thinking this, we always think of a temporal process and infer from the fact that mathematics has nothing to do with time, that in its case possibility is (already) actuality.
>
> (But in truth the opposite is the case, and what is called possibility in mathematics is precisely the same as it is in the case of time.) (*PR* 161)

The equation $m = 2n$ does not correlate a class with a subclass. Rather, it contains the possibility of correlating any number with another. Since the distinction between possibility and actuality in empirical matters seems to hinge on the element of time, it is tempting to think that the distinction must not exist in the philosophy of mathematics. But, in fact, this is incorrect; with Wittgenstein's views concerning rules, it is apparent, as is evidenced by the above quotation, that he thinks that actuality in mathematics consists in a completed construction. In contrast, possibility involves constructions that have not yet been undertaken, but which can be undertaken by following a rule (typically, as the examples we have discussed suggests, these are infinite). Thus, $m = 2n$ contains the possibility of correlating an infinite number of numbers with another, and the actuality is the numbers that have so far been correlated according to the rule.

4.4 Coming to Grips with the Infinite

Around[125] this period in Wittgenstein's thought, he began to be concerned with clearly explicating the rules of our language; these rules were considered to be part of a system – a calculus – and their expression, presented in contrast to all the other rules which govern related concepts, was to be used to perspicuously demarcate sense from nonsense and thus clarify the important concepts under investigation. This is part of what is involved in his investigations of the concept of infinity as we have examined them thus far. However, also around this time, he began to think not only that showing and arguing against confusion, but also showing exactly how such confusion arose, was important. For only showing the exact tendency of thought that led to the confusion could completely eradicate it. Engelmann has identified this method as an important part of the development of Wittgenstein's intermediate period thought and calls it the 'genetic method'. He has shown that it arose primarily in relationship to Wittgenstein's critique of Russell's causal theory of meaning, and the problems with the concept of intentionality that continued to arise (Engelmann 2013, 65-111). However, I believe the seeds of this method can already be seen in Wittgenstein's discussion of the confusions that arise around the concept of infinity. In addition to pinpointing any confusions, Wittgenstein is also interested in clearly identifying what leads to them. In what follows we shall further examine some of the trains of thought that Wittgenstein identified as leading to confusions regarding the concept of infinity.

The idea that every possibility is capable of being actual is one confusion that has been extensively dealt with in the last section. Another one Wittgenstein identifies early in 1929 concerns the interpretation of the quantifiers. When writing the *Tractatus*, Wittgenstein assumed the meaning of quantifiers to be readily apparent. Dealing with a finite number of objects, the existential and universal quantifiers were to be analyzed as logical disjunctions and conjunctions respectively. So, for example, assuming 'F' to be a decidable predicate, and the domain of discourse to consist of two objects, the expressions '(x) F(x)' and '(∃x) F(x)' could be analyzed as 'F(x) ∧ F(y)' and 'F(x) ∨ F(y)', respectively. This nicely dovetailed with the overall picture theory of language.[126]

[125] The calculus conception of language is already being developed in the *Philosophical Remarks* and reaches its most developed form in *The Big Typescript*. Engelmann pinpoints the genetic method as being employed later, most clearly in later manuscripts and in *The Big Typescript*. Although, as should be evident from what follows, seeds of this method exist already in the *Philosophical Remarks*.
[126] For example, by the lights of the *Tractatus*, general statements could quantify over an infinite number of 'objects'.

Wittgenstein, at the time of writing the *Tractatus,* thought that the interpretation of the meaning of these quantifiers could easily be extended over infinite domains. Any quantification involving a decidable predicate and an infinite number of objects was to be viewed as either an infinite conjunction or disjunction. The logistics of this were not worked out in detail, but, much like many other elements of the *Tractatus,* assumed to apply. Hence, Wittgenstein was committed, even if only implicitly, to the view of an extensional infinite that would give meaning to the quantifiers. This confusion is undermined by his more detailed reflections on the extensional infinite upon his return to philosophy, as well as by his simultaneous use of the newly employed verification principle. Given the arguments already presented in this chapter, it is simply possible to note that, assuming the unintelligibility of the extensional infinite, a universal or existential statement over an infinite domain can never conceivably be verified or falsified, respectively. It is impossible to list all of the propositions that would make up an infinite conjunction or disjunction and thus impossible to decisively establish whether all of the conjuncts apply, or whether none of the disjuncts do, in the case of universal or existential statements, respectively. Obviously, matters are otherwise when it comes to quantifying over a finite domain: the list of propositions will itself be finite and its truth or falsity readily determinable. Since the method of verification is different in the two cases of general statements over infinite and finite domains, with one truth value not being possible to determine in the case of the verification of a universal or existential proposition over an infinite domain (T or F respectively), it shows, at the very least, the different meanings of the two types of propositions.[127] With further reflection on the nature of the infinite, then, Wittgenstein seeks to give a different account of the meaning of the quantifiers. This is briefly discussed later in the chapter. In discussing the infinite, Wittgenstein regularly argues for a categorial distinction between the infinite and any number (the finite). Wittgenstein says:

> Where the nonsense starts is with our habit of thinking of a large number as closer to infinity than a small one.
>
> As I've said, the infinite doesn't rival the finite. The infinite is that whose essence is to exclude nothing finite.
>
> The word 'nothing' occurs in this proposition and, once more, this should not be interpreted as the expression for an infinite disjunction, on the contrary, 'essentially' and 'nothing' belong together. It's no wonder that time and again I can only explain infinity in terms of itself, i.e. *cannot explain* it. (*PR* 157-158)

[127] 'At the very least' because, as we have seen, at this time, Wittgenstein will even be inclined to say that such 'propositions' aren't meaningful at all, since they do not have a clear verification.

The infinite is not another number and so no number is closer to it than another. It is not a quantity at all and is not something that can be reached. *It is essential* to the concept of infinity in mathematics that, whatever is thus characterized, never ends. Insofar as it is thus defined, 'essentially' and 'nothing' 'belong together', since these two are constitutive of the meaning of 'infinite' in the philosophy of mathematics. It would be misleading to think of the 'nothing' as referring to an infinite disjunction.[128] To think this way would be to precisely fall into the confusions already discussed. 'Essentially' and 'nothing' characterize the infinite in the philosophy of mathematics precisely because such a general statement, if true, is essentially so. It is *inconceivable* that such a statement could have the opposite truth value it does. This is brought out by what constitutes a verification of such a statement. A verification does not consist in checking each individual instance (which is unintelligible), but rather consists in an inductive proof which shows the truth of the statement for an infinite number of propositions in one step. Such a statement, then, could not be the same statement it is and not retain the same inductive proof. Its meaning is inextricably connected with its proof. The details of this will be dealt with in more detail in the next chapter.

Whereas individual numbers specify determinate amounts, the infinite 'includes' everything finite (that is, what is given by a rule or concept). It therefore has a different logical role than any individual number and is *logically opposed* to *anything* finite. Hence, Wittgenstein says:

> A searchlight sends out a light into infinite space and so illuminates everything in its direction, but you can't say it illuminates infinity. (*PR* 162)
>
> It is, again, only the ambiguity of our language that makes it appear as if numerals and the word 'infinite' are both given as answers to the same question. Whereas the questions which have these words as an answer are in reality fundamentally different.
> (The usual conception really amounts to the idea that the absence of a limit is itself a limit. Even if it isn't put as badly as that.) (*PR* 162-163)

The questions that can meaningfully be answered with a numeral as opposed to the word 'infinite' are themselves distinct. Obviously 'infinite' can't be used to specify a determinate amount. Thus, questions that require determinate amounts are not so answered and *vice versa*. Or, where both answers can meaningfully be given, very different things are indicated by the different answers. This is further explained with

[128] Of course, the representation Wittgenstein has in mind for 'nothing' is '$\sim (\exists x)\ (Ax \vee Bx \vee Cx\ldots)$'.

Wittgenstein's claim that 'infinitely many' should be, so to speak, understood as functioning like an adverb. So, for example, when we say the 'number of objects lying in a certain direction is infinite' (e.g. a certain sequence governed by a rule), this must be read to mean that the 'objects' never come to an end; that is, it qualifies the 'lying', making us conclude one can forever 'follow' that direction and never reach its end. As has already been argued in the previous section, any number is capable of being actual, whereas giving the answer 'infinite' indicates that some series is unending (and related to a possibility). To liken the infinite to a numeral is to think of the absence of a limit as itself a limit.

And it is apparent that, not being a number or quantity, the infinite is not something that will eventually be reached:

> Generality in mathematics is a direction, an arrow pointing along the series generated by an operation. And you can even say that the arrow points to infinity; but does that mean that there is something – infinity – at which it points, as at a thing? Construed in that way, it must of course lead to endless nonsense. (*PR* 163)

One may use generality in mathematics to speak about an endless series. However, the fact that it is endless is not itself a point of the series. Hence, a more concrete example:

> We say we get nearer to √2 by adding further figures after the decimal point: 1.1412 - - -. This suggests that there is something we *can* get nearer to. But the analogy is a false one. What we give is a rule of accuracy: the more figures we add to 1.1412 - - - the closer will the square of the resultant figure be to 2. (*LL* 114)

The more places of √2 one gives, the closer the square of the resulting decimal is to 2. It is not that we are getting closer to the actual value which is infinite (i.e. that we are getting closer to the infinite).

As a final important confusion to avoid, Wittgenstein says:

> But what then has divisibility to do with actual division, if something can be divisible that *never is* divided?
>
> Indeed, what does divisibility mean at all in the case of that which is given as primary? How can you distinguish between reality and possibility here?
>
> It must be wrong to speak as I do of restricting infinite possibility to what is finite.
>
> For it makes it look as if an infinite reality were conceivable – even if there isn't one – and so once more as though it were a question of a possible infinite extension and an actual finite one: as though infinite possibility were the possibility of an infinite number. (*PR* 159)

If improperly phrased or understood, Wittgenstein's 'restricting' infinite possibility to what is finite can be misinterpreted to mean that he is somehow stipulating that this is the case or that it is an empirical truth that the infinite doesn't exist. With this in mind, it may seem that an infinite reality is logically possible, just that there isn't one in fact. But, as we have seen, an infinite collection of objects is *logically impossible*.

4.5 Remnants of the *Tractatus* in Wittgenstein's Treatment of Infinity in 1929

Kienzler has convincingly argued (1997, 143-174) for an important shift in Wittgenstein's thought regarding the infinite (as well as other important concepts not dealt with in this chapter) from when he first started doing philosophy again in 1929 to when he re-examined his notes from this time in 1931. Kienzler refers to this work of re-evaluation as the '*Wiederaufnahme*' ['resumption' or 'taking-up-again']. It is not the purpose of this chapter to carefully examine this shift in detail, since Kienzler has already done this. Rather, a brief synopsis of this shift will set the stage for a more careful examination of some of the ideas found in the *Tractatus* that underpin Wittgenstein's views in 1929. The subsequent section will show how the rejection of these ideas helps Wittgenstein move to the position he adopts in 1931.

Wittgenstein's new position in 1931 does not consist in a rejection of everything he wrote in 1929; many of the arguments, especially the ones focused on in this chapter, still remain effective in some way. However, with the realization of mistakes in his 1929 work, Wittgenstein's conviction regarding the proper method of philosophy is reinvigorated and reoriented. Most generally, Wittgenstein, in 1931, comes to reassert that the proper method of philosophy is not to deny facts, as he was doing *at times* in 1929, but to rather reject the meaningfulness of assertions/questions (Kienzler 1997, 164).[129] The proper province of philosophy continues not to be truth or falsity but sense and nonsense. This crucially applies to his arguments against the extensionalist conception. His new method focuses *exclusively* on denying the intelligibility of a claim. In addition, connected with this, Wittgenstein in 1929, having

[129] This is Kienzler's thesis, although I emphasize the 'at times', since Kienzler seems to think Wittgenstein's claim about denying facts (as Wittgenstein claims in 1931 – quoted later in this chapter) applies much more widely than I do. It is obvious that Wittgenstein, already in 1929, was succeeding, in most cases, to deal purely with matters of sense and nonsense (as evidenced by the first two sections of this chapter).

denied the existence of the extensional infinite, tried to reduce all infinity to intensional infinity on the mathematical model (Kienzler 1997, 152). This was a natural progression, given his focus on mathematical matters and the effectiveness of his account of the infinite in these terms.

Kienzler rightly draws attention to the fact that Wittgenstein had already argued for philosophy's proper role being limited to sense and nonsense in the *Tractatus*. However, while noting Wittgenstein's shift in views on infinity from 1929 to 1931, he fails to identify any of the views from the *Tractatus* (or their rejection) informing Wittgenstein's views on infinity in the intermediate period. This leaves a gap in the explanation of Wittgenstein's views in the intermediate period. Moreover, Kienzler seems to ignore certain aspects of Wittgenstein's thought in 1929. He makes much of Wittgenstein's statement in 1931 that claims he had repeatedly denied 'facts' rather than examined the sense of the claims being made. While I do not deny Wittgenstein's claims, as should be obvious at this point, there are numerous times that Wittgenstein denies the sense of a statement/claim in 1929 (e.g. there are no experiential criteria to make sense of the extensional conception, the concept of an infinite totality is shown to be absurd etc.), and it is with Wittgenstein's *Tractatus* views in mind that we can make better sense of Kienzler's findings.[130] It is to Wittgenstein's views in 1929 that come from the *Tractatus* that we now turn.

To further explain the role of the infinite, while displaying its categorial distinctness from any particular number, Wittgenstein says:

> Corresponding to this is the fact that numbers – which of course are used to describe the facts – are finite, whereas their possibility, which corresponds with the *possibility* of facts, is infinite. It finds expression, as I've said, in the possibilities of the symbolism. (*PR* 164)

Here we see a reference to the *Tractatus*, one that is commonly ignored in the secondary literature[131] but which still informs Wittgenstein's views in the intermediate period. Wittgenstein, at this time, still works with the saying/showing distinction, the idea of a shared form between language and the world, as well as a similar view of

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[130] Moore (2011, 115) does very briefly make reference to some of Wittgenstein's *Tractatus* views when citing one of the most important passages (*PR* 168 – already quoted) and their contradictory nature, but his goals preclude him from examining them in the context of Kienzler's claims.

[131] Shanker mentions the continuity of ideas between the *Tractatus* and *Philosophical Remarks*; he focuses primarily on how Wittgenstein's views on infinity relate to his attack on transfinite set theory and the 'actual infinite' (1987, 162-164).

infinity briefly presented in the *Tractatus*.[132] Numbers can rightfully be used in propositions to ‘describe facts’. That is, numbers can be meaningfully used in propositions that state facts (or, in the terms of the *Tractatus*, ‘picture facts’). On the other hand, infinity, according to Wittgenstein in the intermediate period, is not a number and *can’t* – as this is used to express a grammatical truth – be experienced as a totality. The numerals of mathematics, for example, insofar as the rules that govern them *always* allow for the construction of more members, ‘contain the possibility and not the reality of their repetition’ (*PR* 164). This, too, although much more elaborately argued for, agrees with Wittgenstein’s *stated* views in the *Tractatus*.[133]

The saying/showing description is made explicit here:

> Doesn’t it come to this: the facts are finite, the infinite possibility of facts lies in the objects. That is why it is shown, not described. (*PR* 164)

Seemingly, Wittgenstein’s argument for why the facts are finite depends on arguments already given in this chapter. Any ‘description’ of an experienced infinite totality of objects (i.e., a proposition about an infinite totality) can’t possibly be true. Thus, such a ‘description’ isn’t bipolar; with the bipolarity of a proposition being its essence, assuming the isomorphism between language and the world, this means that the facts themselves can’t be infinite (for there is no possibility of a corresponding fact concerning infinite collections of objects). Rather, infinity is a property of the symbolism; these can be, in the case of the concepts of space and time, reflections of formal properties of the objects (assuming the common form between language and the world). This would be an internal property itself and, therefore, could never be described, but shows itself in the symbolism.

[132] Infinity, as we have seen, was given by a ‘variable’, which lays out a ‘base term’ and the ‘operation’, the repeated application of which generates an unending series. This is, for example, what generates the series of integers. A version of this is repeated in the intermediate period: ‘The infinite number series is itself only such a possibility – as emerges clearly from the single symbol for it “(ɪ, *x*, *x* + ɪ)”. This symbol is itself an arrow with the first “ɪ” as the tail of the arrow and “*x* + ɪ” as its tip and what is characteristic is that – just as length is inessential in an arrow – the variable *x* shows here that it is immaterial how far the tip is from the tail’ (*PR* 162).

[133] Some of his views involved the concept of infinity, but were not yet worked out in detail. His interpretation of the quantifiers, as already mentioned, upon greater reflection, presented a problem. In addition, there is the matter of the axiom of infinity. As mentioned in Section 1.4, that there is an infinite number of objects could not be stated (since the existence of objects generally as well as the number of objects specifically could not be stated), but rather would show itself in the symbolism. Working out the details of this could have, once again, led Wittgenstein to his intermediate period views.

It is likely that Wittgenstein, still under the influence of the metaphysical views of the *Tractatus*, did not see this argument as a denial of any fact at the time of writing this remark. For, in the context of the *Tractatus*, this argument is not a denial of a fact. Facts are, strictly speaking, the worldly correlate (i.e., objects that stand in the appropriate relations) of a *meaningful* statement that is true and, given the metaphysics of the *Tractatus*, it is not possible to state that there are or are not an infinite number of objects. As we shall see in the next section, this makes sense of Wittgenstein's comments later in the intermediate period that relate to his mistaken interpretation of the infinite, its rejection because of his metaphysical views, and how he (wrongly) did not see this rejection as a denial of a fact.

4.6 Rejecting *Tractatus* Views: Wittgenstein's Position in the *Wiederaufnahme*

Wittgenstein's position by the end of 1931, when he re-examines many of the passages on infinity from his manuscripts from 1929, contains important changes to his earlier views, while retaining many of the same/similar arguments. For example:

> 'The merely negative description of not stopping cannot yield a positive infinity'.[134] With the phrase 'a positive infinity' I thought of course of a countable (=finite) set of things (chairs in this room) and wanted to say that the presence of a colossal number of such things can't be inferred from whatever it is that indicates to us that they don't stop. And so here in the form of my assertion I make the strange mistake of denying a fact, instead of denying that a particular proposition makes sense, or more strictly, of showing that two similar sounding remarks have different grammars. (*PR* 305-306)

Kienzler seems to take the interpretation of this quotation as self-evident. He simply states that the 'fact' Wittgenstein is denying is of the extensional infinite[135] (Kienzler 1997, 164-165). However, some explanation, I think, is warranted. Seemingly, Wittgenstein is saying that despite some of his own clarifications to the contrary, he still viewed the infinite, at least in this particular case,[136] on the model of the finite. That is,

[134] This 'description of not stopping' is the way of describing the examples initially suggested by Ramsey and explained at the beginning of this chapter.

[135] Wittgenstein does say, 'I once said there was no extensional infinity' (*PR* 304). However, he doesn't refer to extensional infinity as itself a fact (or not), and it is obvious from our reflections earlier in this chapter that he often tried to deny the *intelligibility* of the extensional infinite. Nonetheless, I think sense can be made of Wittgenstein's quotation and Kienzler's claim if one considers the important quotation stating outright that there is not an infinite number of things (*PR* 168 – quoted on p. 97).

[136] 'In this particular case' is meant to refer to cases of the infinite that are given with the phrase 'not stopping'. The paradigmatic examples would likely be the row of trees or iron spheres discussed earlier in

he viewed the 'positive infinite' as a 'colossal number of things' that couldn't be inferred from merely adding a 'negative description' of 'not stopping'. This 'colossal number' could be denied based on the arguments already given in the previous section. No negative description could foster such a 'positive infinity', because he outright denied the existence of this 'colossal number'.[137] The denial of the existence of this 'colossal number' (the denial of a 'fact'), which is mentioned early on in this chapter (*PR* 168), if properly interpreted, is dependent on views in the *Tractatus* which Kienzler does not mention.

The mistake is 'strange' because Wittgenstein, as Kienzler also notes, had a similar view of philosophy's purpose going back to the *Tractatus*: philosophy's proper domain is with sense and nonsense and not with 'facts' (Kienzler 1997, 164). Obviously, this remains important even in his reflections on infinity in 1929. Talking about 'all numbers', we have seen, Wittgenstein considered to be nonsense. And many of the arguments connected with the concept of infinity that he gives, and that we have examined in this chapter, involve denying the intelligibility of a claim or the lack of clear criteria that would give meaning to a claim about an experience of the infinite or an infinite totality. These arguments will live on in his reflections in 1931. Nonetheless, there is the outright denial of an infinite number of objects existing in reality in 1929. As I have already indicated, this argument was grounded in the views of the *Tractatus* and thus, at this point in Wittgenstein's thinking, he was denying something outright that he, as he came to further reject the metaphysical views of the *Tractatus*, would no longer view as any sort of metaphysical claim (or justified by metaphysical views). Engelmann has convincingly argued that it is in the course of examining the causal theory of meaning, the calculus conception of language, and the philosophical problems that recur when reflecting on the concept of intentionality, that Wittgenstein comes to give up the kind of connection between language and reality that he envisaged in the *Tractatus*. This takes place between 1930 and 1931 (Engelmann 2013, 94-96). With this likely came his rejection of the arguments that supported the outright denial of an

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this chapter. Given Wittgenstein's view on infinite sets in mathematics, even going back to the *Tractatus*, there is no reason he would be tempted by a 'positive infinity' in this case. Similarly, it is not clear why thinking of a wheel spinning and never stopping, or a human life never ending, would lead to a view of the infinite on the model of a 'countable (=finite) *set of things*'. Ultimately, I take the import of this quotation to refer to the problem of viewing the infinite on the model of the finite.

[137] In addition, as Wittgenstein intimates in the above quotation and given what has been present, it should be evident that he did question the (logical) possibility of an experience of something 'not stopping'. One must thus assume that an imagined mathematical 'ad. Inf.' is sufficient to make sense of Wittgenstein's example.

infinite number of ‘objects’. Instead, the focus of his arguments becomes purely to question the intelligibility of the claims for the extensional infinite as well as slightly altering some prior arguments so as to make sure they concern matters of sense rather than anything factual. This puts the goal of the *Tractatus*, albeit through different means, back at the forefront: to delimit the bounds of sense.

4.7 Wittgenstein’s Development in Greater Detail: Applications to Specific Arguments

In 1931, then, Wittgenstein more clearly questions the *intelligibility* of Ramsey’s claims and related arguments for the extensional infinite.

> ‘I once said there was no extensional infinity. Ramsey replied: “Can’t we imagine a man living forever, that is simply, never dying, and isn’t that extensional infinity?” I can surely imagine a wheel spinning and *never* coming to rest’. What a peculiar argument: ‘I can imagine…’! Let’s consider what experience we would regard as confirmation or proof of the fact that the wheel will never stop spinning. And compare this experience with that which would tell us that the wheel spins for a day, for a year, for ten years, and we shall find it easy to see the difference in the grammar of the assertions ‘…never comes to rest’ and ‘…comes to rest in 100 years’. (*PR* 304-305)

The arguments given in this paragraph aren’t totally at odds with his work in 1929. Again, Wittgenstein argues that there is no experience that can confirm a claim about the infinite. Whereas his claims about the extensional infinite, as they relate to mathematics, always focused on the nonsense of the idea of an existing totality, his arguments about the extensional infinite as they relate to experience alternated between making a claim about what is experienced itself – in the form of a metaphysical assertion about what is shown by the symbolism – and arguing for/against the intelligibility of an assertion about the infinite. By the end of 1931 his approach changes. The arguments for the extensional infinite are to be checked *exclusively* for whether they make sense, and, with the verification principle at his disposal, the different grammars of the concepts of the infinite and finite are to be investigated. Of course, these clarifications had already begun in 1929, but are now pursued in a much more rigorous fashion. In the above quotation, Wittgenstein already hints that the grammar of ‘never’ in this context must be investigated. And his continued use of the verification principle is made explicit with the idea that how the two propositions that include the infinite and finite quantities differ in verification is an indication of their different grammars. In the case of a stated time period, the proposition can readily be verified or falsified. In the case of the proposition that includes the 'never stop spinning', there is no experience that will confirm it.

> 'But we are surely familiar with an experience, when we walk along a row of trees, which we can call the row coming to an end. Well, an endless row of trees is one such that we never have this experience.' – But what does 'never' mean here? I am familiar with an experience I describe by the words 'He never coughed during the whole hour', or 'He never laughed in his whole life'. We cannot speak of an analogous experience where the 'never' doesn't refer to a time interval. And so once again analogy leaves us in the lurch here and I must try to find out *ab initio* how the word 'never' can be used so as to make sense in this case. – Admittedly such uses can be found, but their rules are to be examined in their own right. For example, the proposition that a row of trees is infinitely long (or that we shall *never* come to its end), could be a natural law of the same sort as the Law of Inertia, which certainly says that under certain conditions a body moves in a straight line with constant velocity; and here it could indeed be said that under those conditions the movement will *never* end. But if we ask about the verification of such a proposition, the main thing to be said is that it is falsified if the movement (row of trees) comes to an end. There can be no talk of a verification here, and that means we are dealing with a fundamentally different kind of proposition (or with a proposition, in a different sense of that word). (*PR* 306-307)

Insofar as speaking of a row of trees that never ends makes sense, Wittgenstein here uses the verification principle to bring out the fact that the proposition dealing with the claim about infinity is, if rightly called a proposition at all, one different in kind from, say, a claim about a finite number of trees. There is no such thing, in this case, as a verification. Similar to Wittgenstein's claims in 1929, there still aren't any criteria to indicate that the row of trees will never end. And, here, to further emphasize his point, he now also examines the grammar of 'never'. 'Never' used in a proposition with a stated time interval makes perfect sense. This is where it is at home and its meaning is determinate. However, in the case of an ordinary description, without any stated time interval, it is not at home and its meaning not readily determinable. For it is not clear what experience, in the case of a row of trees that never ends, could possible verify the 'never'. Wittgenstein, once again, contrasts this use with one in the Law of Inertia. Here, I take it, Wittgenstein means to suggest that the 'never' has a meaning given its use in the Law. As suggested in Section 4.1, this is because the Law is a proposition in a different sense of the term; it is akin to a rule.

Another example is given by Wittgenstein's reflections on time. As already indicated, Wittgenstein thinks, even in 1929, that time is infinite.[138] The reasons for this relate to his reflections on the concept as well as, in all likelihood, some of his views that go back to the *Tractatus*. In the *Tractatus*, time is a 'form' of the objects. In 1929, as we have seen, Wittgenstein argues for an intensional interpretation of time. As he explains it there, this is evident from the fact that no matter how many days having passed one imagines, there is always the possibility of more. Now Wittgenstein says:

> If we ask 'What constitutes the infinity of time?' the reply will be 'That no day is the last, that each day is followed by another'. But here we are misled again into seeing the situation in the light of a false analogy. For we are comparing the succession of days with the succession of events, such as the strokes of a clock. In such a case we sometimes experience a fifth stroke following four strokes. Now, does it also make sense to talk of the experience of a fifth day following four days? And could someone say 'See, I told you so: I said there would be another after the fourth'? (You might just as well say it's an experience that the fourth is followed by the fifth and no other.) But we aren't talking here about the prediction that the sun will continue to move after the fourth day as before, *that's* a genuine prediction. No, in our case it's not a question of a prediction, no event is prophesied; what we're saying is something like this: that it makes sense, in respect of any sunrise or sunset, to talk of the next. For what is meant by designation of a period of time is of course bound up with something happening: the movement of the hand of the clock, of the earth, etc., etc. ; but when we say 'each hour is succeeded by a next', having defined an hour by means of the revolution of a particular pointer (as a paradigm), we are still not using that assertion in order to prophesy that this pointer will go on in the same way for all eternity: – but we want to say: that it '*can* go on in the same way for ever'; and that is simply an assertion concerning the grammar of our determinations of time. (*PR* 309)

The infinity of time does not involve any particular experience or way of defining time. It is incorrect to confound any event in time, or, as is more tempting, one that serves as a defining characteristic of the passage of time, with the infinitude of time itself. This is to confuse the prediction of a fact with the possibility of a certain expression making sense. For it is not a particular experience of, for example, the sun rising and setting, or any other method of defining the passage of time, that is being predicted with talk of the infinite nature of time. Indeed, nothing is being predicted. Rather, the infinite nature of time is simply a way of expressing the meaningfulness of always being able to speak of subsequent events, no matter what paradigm is used to define the passage of time. Thus,

[138] Yet another slightly different example: 'If I say "The world will *eventually* come to an end" then that means nothing at all if the date is left indefinitely open. For it's compatible with this statement that the world should still exist on any day you care to mention. – What is *infinite* is the *possibility* of numbers in propositions of the form "In *n* days the world will come to an end"' (*PR* 153).

it is a rule of our language that constitutes the infinity of time and not a fact or prediction about the world. It is unclear whether Wittgenstein thought this to be an error he actually made in 1929; there is too little evidence in 1929 or in 1931 to confirm either that Wittgenstein did make this error or that he thought he did.[139] Nonetheless, this serves as an example of an error that could be made, and Wittgenstein's further clarifications about this concept perfectly exemplify the development of his thought at this time.

4.8 Putting it all in Context: The Details of Wittgenstein's Position on Infinity in 1931

When discussing the concept of the infinite in 1931, Wittgenstein now claims that there is not one definition of 'infinity' that serves to characterize all of its uses. Here I will outline this new view partially by way of how it is explained by Kienzler, before I challenge it below.

Like many other concepts discussed in the previous chapters (e.g. 'proposition', 'proof', 'number', etc.), Wittgenstein claims 'infinity' has varying uses in the different contexts in which it is used. Wittgenstein says:

> If you speak of the concept 'infinity', you must remember that this word has many different meanings and bear in mind which one we are going to speak of at this particular moment. Whether, e.g., of the infinity of a number series and of the cardinals in particular. If, for example, I say 'infinite' is a characteristic of a rule, I am referring to *one* particular meaning of the word. But we might perfectly well say a continuous transition of colour was a transition 'through infinitely many stages', provided we don't forget that here we are defining the meaning of the phrase 'infinitely many stages' *anew* by means of the experience of a colour transition. (Even if by analogy with other ways of using the word 'infinite'.) (*PR* 304)

In 1929, Wittgenstein was tempted by the idea that the only legitimate form of the infinite was intensional. Thus, for example, he imagines even empirical cases of infinity on the intensional model: an infinite row of trees can be imagined when one similarly imagines a way that the sizes of the trees are determined in all cases. Wittgenstein thus simultaneously imagines a rule that serves to establish the height of every tree in an infinite number of cases. In 1931, as evidenced by the above quotation, Wittgenstein

[139] It is true that in 1929 he does say (full quotation on p. 106): 'yet [we] see that no day can be the last'. However, whether he was *committed* to the false interpretation of this is not clear.

now considers how the word 'infinity' is actually used in the varying contexts in which it appears. Kienzler has extensively examined these uses (1997, 160-174). For our purposes, a brief overview should suffice to adequately indicate Wittgenstein's new approach.

Wittgenstein's description of the intensional infinite within mathematics does not change. Infinity remains the property of a law, and is made evident by the possibility of *always constructing* further numbers/members/decimal places according to the rule. In this sense, it is unbounded. But now Wittgenstein does not try to explain all possible uses of the word on this model, but rather purportedly examines them according to their own context. The meaning of 'infinity', Wittgenstein once again emphasizes at this time, can be misunderstood if it is thought of as being an immense quantity (i.e., having the same logical role as a very large finite quantity). It would be similar to making the mistake of thinking the answer 'at different times' is the same sort of answer as 'at twelve o'clock' in answer to the question 'At what time are they dining?'. The answer in the first case indicates there isn't just one answer and puts off the specifics, whereas the second answer is definitive in nature and does not require further questions or answers (Kienzler 1997, 170). 'Infinite', Kienzler claims, can have a similar role in ordinary use. Wittgenstein, for example, imagines a fairy in a fairy tale who promises a person as much gold as one can wish for (see MS 113, 97v). In this sense, Kienzler claims, the person is given an 'infinite freedom' to choose exactly how much gold the person wants. In this case, the amount one can choose, and even possibly of one's ability subsequently to wish for more, is unlimited and unrestrained and is, in this sense, infinite in nature (Kienzler 1997, 171-172). Of course, a person could not wish for an infinite amount of gold. An 'infinite amount' is not an amount at all, and therefore not something that could be wished for. The 'infinite', in this case, qualifies the way the choice is conducted and does not qualify the amount (i.e., it functions like an adverb rather than an adjective). This agrees with Wittgenstein's observations regarding confusions around the infinite that have been already discussed.

Wittgenstein uses another example: 'The possibility of forming decimal places in the division 1 ÷ 3 is infinite' (*PR* 313). Once again, it is the freedom to calculate this decimal to as many places as one wants that gives it its 'infinite' possibility. This possibility, of course, involves the freedom to choose *to how many places* one constructs the decimal expansion. This, Kienzler claims, is not the same as the infinite possibility given by the rule which allows for the possibility of always constructing further places of the decimal (Kienzler 1997, 172). These two examples, among others,[140] nicely illustrate an important change in position Wittgenstein makes in this later period of his thought. His decisive rejection of the extensional interpretation in the philosophy of mathematics, and his subsequent reduction of all uses of the concept of infinity to the intensional model, leads him in 1929 to declare that all infinity relates to possibility and not actuality. With the above two examples, Wittgenstein now rejects this position. For these two, and similar, examples, he now claims it is possible to say they do indeed relate to an infinite reality.[141] He believes that the infinite, in this case, relates to the unlimited choices possessed by the people in question and thus can be considered very real.[142] It was only, Kienzler thinks, Wittgenstein's focus on the mathematical examples and how they frame the debate (most importantly extensional *or* intensional interpretations) that led him to state that infinity always involves simply a possibility. By thinking all debates regarding infinity concerned only the extensional or intensional interpretations, he artificially extended elements of *that* mathematical debate to other areas that involved infinity. But only the most superficial interpretation would consider the possibility of the infinite freedom to choose a certain amount of gold as wishing for an infinite amount of something. In contrast, in these cases, Wittgenstein thinks, 'infinity' does not relate to infinite collections of objects in *any way*, so can't be explained using this, and related, terminology. Of course, 'extensional' and 'intensional' can still be useful in the philosophy of mathematics, and it is still apparent what side Wittgenstein would take in that debate, but the problems/explanations that relate to this debate, Wittgenstein now thinks, can't be thought to apply to all the different ways in which 'infinity' is used.

[140] As suggested by quotations in this chapter, Wittgenstein also uses the example of colour transitions and rulers with 'infinite curvature'.

[141] Similar examples can be found in other quotations: 'We say "the world will come to an end some day", and in using "some day" we think or feel that we have grasped the infinite. We know how we normally use "some day", "the day after tomorrow", "next week" and so on. But when we say "the world will come to an end some day" what we give is not a disjunction but an indefinite "and so on". There is infinite possibility, but no infinite reality – so we often feel. The infinite does not stand for a number or quantity. Compare a ruler with infinite radius of curvature, i.e. straight. If you give a promise to provide any amount of cash asked, your promise is infinite. "Proof" and "proposition" in mathematics are used in a number of different senses' (*LL* 107-108).

[142] I have carefully chosen the word 'real' based on what Wittgenstein (most importantly, MS 113, 97v) and Kienzler say. Obviously, even here, Wittgenstein is *not* arguing for the actual infinite. Instead, he is arguing for uses of the word 'infinite' that apply to reality. That is, he now examines possible *everyday uses* of 'infinite' instead of what only belongs to mathematics.

There are limitations of Wittgenstein's developed position in 1931. While 'infinite' can be used in the different ways he now suggests (at the very least by stipulation, if nothing else justifies them), it gives no credence to the actual infinite and his point seems to be partly obscured by his choice of expression. By saying that it was 'misleading' to claim that 'infinity is only a possibility, not a reality' (MS 113, 97v), it could be easily thought that he is now arguing for some version of an actual infinite, but really he is merely arguing that there are legitimate everyday uses, or uses that can be applied to the world, of 'infinite'.[143] In every one of the examples he gives, the meaning of 'infinite' can be rephrased in terms that relate to a possibility. Choices of amounts of gold and lengths of decimal expansions are obvious (and his best examples!).[144] One has the unlimited choice of any amount or length of decimal expansion one wants, not an actual infinity of choices of amount or decimal expansions. The complications of the colour transition example are dealt with below, but it is apparent Wittgenstein does not think one sees, or is even capable of seeing, an 'infinite' number of different shades of a colour. Finally, a ruler possesses an infinite radius of curvature by way of definition. This is a clear mathematical use and 'infinite', in this case, seems to be a placeholder to give a value to any possible use of 'radius of curvature', rather like 'undefined' (and indeed why 'undefined' is sometimes used instead). Seemingly, the idea is that a circle with a big enough radius would have a straight line as part of its curve. A bigger and bigger circle would approach this, but could obviously never reach it. 'Infinite' in this case serves as a limit within mathematics.

The bigger concern is that all of Wittgenstein's examples, and the colour transition[145] and 'ruler with infinite curvature' examples in particular, seem to get their sense from mathematics. In the case of colour transition, 'infinite' could easily be *defined* as the possibility of the further divisibility of a space that contains a colour transition (on the model of the divisibility of a line). However, Wittgenstein seems to deny this in 1929 (*PR* 165) and in 1931 (*PR* 304). In 1929, Wittgenstein suggests that there is a conflict between continuity and visual discrimination of continually divided

[143] Hence, there is now also a distinction between his use of 'reality' and 'actuality'. Uses of 'infinite' now apply 'in reality', but are still not examples of the 'actual infinite'.

[144] Here Wittgenstein would emphatically disagree. The 'infinite' choice is a reality (*PR* 313). Of course, *that a choice exists* is a reality, although the choice itself relates to a possibility.

[145] Wittgenstein is more hesitant to use the term 'infinite' at all when discussing colour transitions in 1929, perhaps because he can only conceive of it on the extensional model: 'We see a continuous colour transition and a continuous movement, but in that case we just see *no* parts, *no* leaps (not infinitely many)' (*PR* 165).

segments of colour transitions (*PR* 155-157). He is also more hesitant to use the term 'infinite' in relation to colour transition at all (*PR* 165). Nonetheless, he does suggest that the description can be so employed (e.g. *PR* 157), although, even then, it is not clear exactly why. He could think that the visual experience of what we would call 'continuity' (also referred to as 'not seeing discontinuity' (*PR* 157)), based on analogous uses of the word 'infinite', by itself justifies the ascription (as he seems to suggest in 1931 – *PR* 304) or he could be suggesting that the possibility of further division, regardless of our ability to discern it, justifies it (as he might be suggesting in 1929 – *PR* 156). Taking away the possibility of some mathematically related interpretation of the colour transition example, it becomes much more difficult to imagine precisely what justifies the use of 'infinite' (thus it truly seems like a stipulation, as Wittgenstein suggests it is).[146] The other examples are similarly artificial.[147] It is unclear how giving someone the freedom to choose any amount of gold or expand a decimal expansion to any number of places would be 'infinite', *if not for* the possibility that for any amount or decimal place chosen, another larger amount/further decimal expansion could be chosen (a possibility allowed for by mathematical symbolism). While Wittgenstein is doubtless correct that these areas of mathematics can be used in the *description* of reality, it is impossible to deny that these specific examples of uses of 'infinite' seem to get their sense from mathematics (and may, as in the case of dividing a line, be affected by practical requirements or physical limitation when applied to the world). With respect to the ruler example: unlike 'straight' or 'circular', 'radius of curvature' is not normally used in everyday descriptions at all, and insofar as it is used in making measurements, rarely would one even need to speak of the 'infinite radius of curvature' (i.e., it would rarely, if ever, actually be used in a measurement). Thus, this seems like a clear case where one can import the terminology from mathematics, even though its use, and therefore purpose, when making empirical statements is lacking. Therefore, it would appear that Wittgenstein is unable to think of examples of the use of 'infinite' *apart from the mathematical use*.

Indeed, better examples than those Wittgenstein gives seem possible to imagine: for example, giving someone a choice to do whatever they wish, or, similar to a decimal

[146] Moreover, 'infinite' does not seem to add anything that 'continuous' doesn't already express.
[147] Hence the reason why talk of a ruler with 'infinite radius of curvature' can be used to jest, as Wittgenstein suggests in later work (*LFM* 142).

expansion, allowing someone to continue a non-mathematical related activity for as long as they wish (e.g. drawing rabbits). While Wittgenstein's choices of example are somewhat artificial and his choice of wording as to what they prove somewhat misleading, it is no doubt an important development, and one perfectly exemplifying this stage in his philosophy generally, that he began to examine – what he took to be – the ordinary, everyday, uses of 'infinite'.

We have carefully examined Wittgenstein's positions on the concept of infinity and how they evolved from when he returned to doing philosophy in 1929 through to the end of 1931. This involved examining Wittgenstein's rejection of the extensional infinite as it applies to the empirical world and mathematics, and his conception of how the infinite importantly relates to possibility and actuality. I argued that confusions that underlie the extensional interpretation were carefully identified by Wittgenstein and that his approach serves as an early example of what would develop into the 'genetic method'. In addition, I argued that Kienzler's account of the *Wiederaufnahme* is in general correct, although it is usefully further explicated using ideas from the *Tractatus*, and by observing some more details of Wittgenstein's thought in 1929. The rejection of certain views from the *Tractatus* that lived on into 1929 led to the demise of important parts of his account of the infinite from that time and to his position in 1931. In addition, we examined the details of Wittgenstein's position in 1931, much of which is not convincing. It is to the more technical applications of the verification principle and Wittgenstein's analysis of the concept of infinity to topics in the philosophy of mathematics that we now turn.

5. Inductive Proof

In the preceding chapters both the verification principle and Wittgenstein's analysis of the concept of infinity have been extensively discussed. These developments of Wittgenstein's intermediate philosophy naturally arose out of, or in response to, his views in the *Tractatus*. The foundational role these elements played in the development of his intermediate views generally was also extensively examined. In addition to the generic influence of these elements on Wittgenstein's intermediate philosophy, and how they specifically brought about important changes and developments to Wittgenstein's philosophy of language, epistemology, metaphysics, and the philosophy of mathematics, the new insights were also used to deal with important *specific problems of the time* within the philosophy of mathematics that were not previously a concern for Wittgenstein in his earlier work. There is one principal reason for this: Wittgenstein's return to philosophy coincided with a greater awareness, fostered by both lectures and conversations, of contemporary problems and debates within the philosophy of mathematics. That return happened, as we have seen, around the same time that he attended meetings of the Vienna Circle, went to a lecture of Brouwer's, and met with Ramsey regularly.[148] It was through these influences that Wittgenstein would have become aware of the most important debates and controversies at the time within the philosophy of mathematics.

In what follows, we shall investigate Wittgenstein's intermediate views on inductive proofs. To set the stage, we begin with a very brief review of Wittgenstein's views on the meaning of statements containing unbounded quantifiers as this relates to the verification principle. An introduction of proof-schemas and inductive proofs will follow, with Skolem's recursive proof of the associative law serving as an example. We will then focus on the ways in which misinterpretations of the proof can lead to confusion and then proceed to give a positive characterization of the proof. In the next section, we shall compare decision (check) procedures to inductive proofs, and give an account of generality as it relates to inductive proofs. With reference to the preceding

[148] As is well known, the Vienna Circle had several practicing mathematicians whose influence would have been felt throughout the group. Kurt Gödel, one of the leading mathematicians at the time, not to mention a defender of Platonism, was one such member. Hans Hahn and Karl Menger were two more. The problems of the day would have doubtless come up through Wittgenstein's meetings with the Circle, in particular with Waismann and Schlick, the former who was clearly engaged in his own reflections on the philosophy of mathematics. Moreover, it was through Waismann and Schlick that Wittgenstein was encouraged to attend Brouwer's lecture and where his interest in some of the philosophical problems in the philosophy of mathematics was clearly fostered.

clarifications and relevant secondary literature, we then explore in exactly what way, and to what extent, Wittgenstein's work constituted a refutation of Skolem's work. We conclude the chapter by highlighting the importance of Wittgenstein examination of inductive proof to his philosophy of mathematics and, more generally, to his philosophy as a whole.

5.1 Dealing with Infinity: Quantifiers, Verification, and Proof-Schemas

As we have seen, Wittgenstein rethinks the explanation of the meaning of the quantifiers as applied to infinite domains. Originally, the interpretation of the meaning of the universal and existential statements was thought to be easily extendable from the finite to the infinite case. Thus, the meaning of universal and existential statements quantifying over an infinite domain was thought to be an infinite conjunction and disjunction respectively. From what has been established in the previous chapters, since an infinite conjunction or disjunction, on the finite model, are now thought not to allow for verification, the idea of such a statement is considered to be meaningless. On this basis, Wittgenstein counsels against speaking of 'all numbers', if this universal generalization is conceived as an extension;[149] instead, a new way of establishing the 'truth' of such propositions containing this expression is required. Two quotations, which clearly refer back to themes examined in the previous chapter, serve to introduce the aforementioned topic. Wittgenstein says:

> But if I only advance along the infinite stretch step by step, then I can't grasp the infinite stretch at all.
>
> So I grasp it in a different way; and if I have grasped it, then a proposition about it can only be verified in the way in which the proposition has taken it.
>
> So now it can't be verified by putative endless striding, since even such striding wouldn't reach a goal, since of course the proposition can outstrip our stride just as

[149] As we will examine in more detail below, Wittgenstein actually recommends, in the case of inductive proofs, not using the phrase at all. As with all natural language descriptions (what Wittgenstein also calls the 'prose' and contrasts with just the mathematics – the 'calculus') that accompany a proof, it is of no assistance to understanding the proof (and can lead to confusion, especially if misunderstood) (*PG* 410, 422). Thus, any statement involving 'all numbers', as was suggested in the last chapter, must be understood in *a very particular* way or is nonsense. For our purposes here, the understanding of 'all numbers', insofar as it has any meaning at all, clearly relates to the possibility of a construction of a proof-schema, in the form of an induction. In addition, related to this, the induction does not state anything about its infinite possibility of application, but rather *shows* this possibility. The induction shows the possibility of an infinite number of proof constructions and does not itself prove an infinite number of proofs. For, as examined in the last chapter, such a suggestion is *logically* impossible.

> endlessly as before. No: it can only be verified by *one* stride, just as we can only grasp the totality of numbers at *one* stroke. (*PR* 146)

With this, as Frascolla notes, is connected the fact that any such connection between a decidable predicate and its applying to 'all numbers' must be a necessary one and not one having occurred by 'mere chance'. For the idea that a decidable predicate could be true of an infinite sequence, but not shown by a formal (internal) connection, would require some such idea as a 'mathematically irreducible given' that could only be shown to apply by the successive application of the decision procedure to an infinite number of cases (cf. Frascolla 1994, 75). As Wittgenstein says: 'The expression "by chance" indicates a verification by successive tests, and that is contradicted by the fact that we are not speaking of a finite series of numbers' (*PG* 457). It followed from our reflections on infinity and the verification principle that a check procedure for an infinite series of numbers would be impossible and thus such a statement, if dependent on one, is meaningless. Thus, Frascolla says:

> The sharp opposition of universality and contingency in arithmetic is founded on the distinction between the existence of a general mathematical result showing the rule according to which, for any given *n*, a proof of '**P**(**n**)' can be constructed, and the mere verification, case by case, of the truth of single propositions '**P**(**n**)'. (Frascolla 1994, 75)

In mathematics 'necessary' and 'all' go together. So what began as a critique of extensionalism (largely arising because of the need to clarify the concept of infinity) develops into a conception of universality in mathematics essentially relating to a general rule of sign construction. Where it is possible to state something about 'all the natural numbers', this applies because it is possible to 'survey' all of them in *one* step. Such a process of 'surveying' requires a *form*, that is, as we have seen in the previous chapter, the unlimited possibility of sign construction in accordance with a rule (cf. Frascolla 1994, 75). For example, for the universal generalization (e.g. '(x) P(x)') a proof-schema is provided by which, for any *n*, the proof of any individual 'Pn' can be obtained. For Wittgenstein, there are two types of proof-schemas: one is an algebraic proof-schema, the other is that of complete induction. The algebraic proof-schema provides a uniform method for proving an algebraic equation (e.g. in 2x = x + x, one must only replace the variable 'x' with a numeral *n*) (cf. Frascolla 1994, 75). In an algebraic proof, it is shown that it is possible to substitute any numeral *n* for the variable x, yet the result will be preserved. The proof-schema essentially proves that the result will hold for any number. As Wittgenstein says, 'an algebraic proof is the general form of a proof which can be *applied* to any number' (*PR* 144: n. 1).

The second type of proof-schema is complete induction.[150] This includes the proof of the base step 'P(1)' and the proof that shows the form of the transition from the proof of 'P(n)' (*assuming* that 'P(n)' can be proved for any arbitrary *n* to the proof of 'P(n+1)' (i.e., the rule by which the proof of any proposition P(n+1) can be generated from the proposition P(n)). Instead of a step-by-step verification, as aptly applies in the finite case, the idea of a property applying to 'all numbers'[151] only has meaning in relation to a 'sign-process' which allows us to 'survey' them in *one step*. So, in the case of any universal generalization, e.g. '(x) P(x)', the proof consists of a proof-schema which allows us, for every decidable predicate 'P(x)', to obtain, for every *n*, a proof of the singular proposition 'P(n)'. In this way, an inductive proof serves as a general term for the infinite series of proofs of the singular propositions 'P(1)', 'P(2)' etc. That is, in the same way that '[1, ξ, ξ +1]' is the general term for the series of natural numbers (as discussed in Chapter 1) (Frascolla 1994, 75-76).

5.2 Wittgenstein's Focus on Skolem and Inductive Proofs: Preliminary Comments

It may not immediately be clear why Wittgenstein spent as much time as he did on inductive proofs, or Skolem's proof *in particular*. While there would have been, I conjecture, some element of chance that facilitated this focus,[152] there were obviously clear connections/relationships between Wittgenstein's work on infinity, as well as his philosophy of mathematics generally, and Skolem's own work. Arguably most important, a central goal of Skolem's work overlapped with Wittgenstein's: to eliminate the use of the unbounded universal and existential quantifiers, that is, the use of quantifiers that range over infinite domains.[153] Thus, the technique of proof Skolem

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[150] Wittgenstein also calls this a 'recursive proof'.
[151] This is sufficient as a preliminary explanation of how an unbounded universal generalization can have meaning.
[152] While it is clear that Wittgenstein owned Skolem's work (Marion 1998, 98 – Marion in turn references *PR* 195-196: n.1), it is not exactly clear what brought about his acquaintance with the work. According to Goldfarb (2018, 245), Skolem's paper was obscure, so he suggests that Wittgenstein was perhaps told about it by Schlick or Waismann.
[153] This is, indeed, manifested in the very title of Skolem's paper: 'The foundations of elementary arithmetic established by means of the recursive mode of thought, without the use of apparent variables ranging over infinite domains'. It is beyond the scope of this chapter to go into the details of this formal system. The following should also be noted: neither thinker was the first to embark on the clarification of

uses is designed, as Marion notes, to be an alternative development of arithmetic that avoids the paradoxes of the theory of types (Marion 1998, 98). It is thus meant as a 'finitist solution' to paradoxes naturally arising from the mathematical notation of the time.[154] This shared goal of avoiding the unbounded quantifier could have very easily piqued Wittgenstein's interest and encouraged him to read Skolem's work, at which time inadequacies of Skolem's own ordinary language descriptions of what he had done, or possible misinterpretations of Skolem's results that could arise because of a misunderstanding connected with the concept of infinity, likely became the subsequent focus of Wittgenstein's work. It was likely in the context of this analysis (perhaps along with his discussion of works in set theory) that his intermediate period distinction between 'calculus' and 'prose' was developed.[155] As I hope to make clear, it is only the 'prose' of Skolem's work, or confusions that could arise because of it (or the symbolism used in the proof), that Wittgenstein seeks to clarify. Thus, as will become apparent in this chapter, although there were obvious shared aims brought on by concerns surrounding the concept of infinity, Wittgenstein's insights into the philosophical problems generally, as well his specific use of his idiosyncratic saying/showing distinction as a means of analysis in this context, led to some more specific criticisms of Skolem's work itself (of the 'prose') or, at least, anticipatory clarifications to serve as a prophylactic against possible confusions that could arise from Skolem's work. While Skolem's work was doubtless the focus and starting point for Wittgenstein's reflections, a concern with inductive proofs more generally would have been a natural progression.

Finally, it is important to note why Wittgenstein spends so much time on the analysis of inductive proofs. At first glance, it may appear that he is much more concerned with inductive proofs than he is with algebraic ones, even though the latter

propositions containing quantifiers ranging over infinite domains. Similar ideas can be found in the work of Hermann Weyl and David Hilbert. See Marion (1998, 85-90) for more details.

[154] It is beyond the scope of this chapter to discuss the details of Skolem's project. It should suffice to indicate that free variable formulas replace the use of the unbounded quantifiers. The elimination of the quantifiers limits the formal expressibility of this system of arithmetic, a limitation that is made up for, as best as possible, through the use of recursive functions and inductive techniques.

[155] In fact, the calculus/prose distinction was rarely articulated using precisely those terms. It would appear that Wittgenstein uses 'prose' [*Prosa*] more in relation to inductive proof and 'calculus' [*Kalkül*] more in relation to set theory ('calculus' is often employed in the inductive proof sections, but this is to emphasize that there is a new calculus created by the proof and not a contrast with 'prose'; and 'word language' or 'theory' can replace 'prose' in discussions of set theory). Nonetheless, the distinction articulated with the use of these concepts is often (implicitly) appealed to. Most importantly, the use of 'prose' first seems to be used in the context of discussing inductive proof (and other closely related topics – e.g. the Sheffer stroke) (MS 108, 14).

too can be seen to involve infinite series. In addressing this, first, it is important to note that Wittgenstein does indeed talk about algebraic proof and that there is important overlap (and thus consistency) with respect to his views on inductive and algebraic proofs. Indeed, Wittgenstein's views on inductive proofs can be seen as a very natural extension from what he says about algebraic proofs (in the context of his overall philosophy at the time).[156] The relative complexities between the different techniques corresponds to a need for more extensive analysis relative to Wittgenstein's overall philosophy. In comments made early upon his return to philosophy in 1929, Wittgenstein addresses algebraic proofs (MS 106, 190 – quoted above as *PR* 144: n. 1). This is a natural topic given his consideration of infinity. In a closely positioned remark, Wittgenstein contrasts 'every' [*jede*] with 'all' [*alle*] proceeding 'number(s)' (MS 106, 186), which, once again, indicates his problem with an extensional view of the infinite. It loosely contrasts the idea of a successive operation with a totality ('all' stands for a totality). This view of the algebraic proof thus indicates that the proof will continue to hold for 'every' number, which is shown by the 'general' form of the proof. The analysis of an inductive proof is dependent on this. For one, the inductive proof contains elements of an algebraic proof. Parts of an inductive proof clearly use the algebraic proof procedure(s). But, in addition, the inductive proof goes beyond the algebraic by being not just the general form of a proof that can be applied to any number, but by being the general form of a series of proofs. This allows much greater application of the proof, beyond the simple deriving of equations within algebra itself, by providing a pragmatic justification for the application of rules of algebra to arithmetic. And, as we shall see, when accepted, it provides a new and independent criterion for what counts as a correct arithmetical calculation. Unlike an algebraic proof which involves the transformation of equations by strict rules of substitution, the inductive proof importantly deviates from such (comparatively) straightforward proof techniques (to be discussed extensively below). Insofar as this template for a series of proofs can be properly represented with the systematic representation of arithmetical numerals (typically '1'), the inductive proof serves as a 'link' between algebra and arithmetic. Inductive proofs serve as a justification for the application of algebraic laws

[156] This is evident, among other things, from the use of the saying/showing distinction in both. How one interprets an algebraic proof, that one takes it as applying to 'every number', eventually requires simply seeing what is meant (as this corresponds to 'showing'). This is similarly the case with the inductive proof at the stage of seeing the infinite series of proofs that could be constructed as this is shown by the inductive proof.

to arithmetic. The algebraic proof techniques are, therefore, not only used to bring out similarities with inductive proof techniques, but are used as an essential point of comparison to bring out what is unique to inductive proof.

The analysis of inductive proof makes up some of the earliest sustained discussion of a mathematical topic in Wittgenstein's intermediate period (i.e., MS 105, 73-105), and his reflections on this topic continue into the latter parts of the period. Moreover, as we shall see, these reflections lead to important developments in Wittgenstein's philosophy generally. It is to the details of these matters that we now turn.

5.3 Inductive Proof: An Example

In order to achieve a better understanding of inductive proofs generally, as well as to be readily able to understand references to specific parts of the proof, it is best now to look at a specific inductive proof. Wittgenstein extensively makes reference to Skolem's proof (a proof of the associative law of addition) when making comments about inductive proofs[157], and this proof is even extensively quoted in *Philosophical Remarks* by the editors in order to aid the reader, so I will quote that material in full:[158]

> I will introduce a descriptive function of two variables a and b, which I will designate by means of a + b and call the sum of a and b, in that, for b = 1, it is to mean simply the successor of a, a + 1. And so this function is to be regarded as already defined for b = 1 and arbitrary a. In order to define it in general, I in that case only need to define it for b + 1 and arbitrary a, on the assumption that it is already defined for b and arbitrary a. This is done by means of the following definition:
>
> Def. I. a + (b + 1) = (a + b) + 1

[157] Of course, it should be noted that the method of inductive proof is used in other areas of mathematics and thus doesn't always have numbers making up its 'base case'. Being a common proof procedure in logic too, it needn't concern only algebra or algebra's connection to arithmetic. In reference to logic, it once again allows one to discern the common form held by an infinite number of proofs, proving that systems (e.g. the propositional calculus) are, for example, sound and complete. It involves the construction of a rule by which the possibility of an infinite number of proofs *within the system* can be discerned. Clearly, in this case too, the proof technique can be seen as a way of connecting areas of mathematics/logic. The examination of more complex inductive proofs within different areas of mathematics can help one, I suggest, as Wittgenstein recommends doing, see the inductive proof 'naively' (*PG* 415). Being able to see the proof naively allows one to see *how* the proof importantly determines a new application for a certain sign construction and serves as a prophylactic against various confusions that can arise with the translation of certain parts of a particular proof into ordinary 'prose' language.

[158] This material is quoted from *Philosophical Remarks*, which, in turn, takes the material from van Heijenoort (1967, 302-306). The van Heijenoort was referenced in order to check the details of the 'prose' (i.e., the way what is occurring in the proof is described in ordinary language) throughout.

In this manner, the sum of a and b + 1 is equated with the successor of a + b. And so if addition is already defined for arbitrary values of a for a certain number b, then by Def. I addition is explained for b + 1 for arbitrary a, and thereby is defined in general. This is a typical example of recursive definition.

Theorem I. The associative law: $a + (b + c) = (a + b) + c$

Proof: The theorem holds for c = 1 in virtue of Def. I. Assume that it is valid for a certain c for arbitrary a and b.

Then we must have, for arbitrary values of a and b

(α) $$a + (b + (c + 1)) = a + ((b + c) + 1)$$

since, that is to say, by Def. I $b + (c + 1) = (b + c) + 1$. But also by Def. I

(β) $$a + ((b + c) + 1 = (a + (b + c)) + 1$$

Now, by hypothesis, $a + (b + c) = (a + b) + c$, whence

(γ) $$(a + (b + c)) + 1 = ((a + b) + c) + 1$$

Finally, by Def. I we also have

(δ) $$((a + b) + c) + 1 = (a + b) + (c + 1).$$

From (α), (β), (γ) and (δ) there follows

$$a + (b + (c + 1)) = (a + b) + (c +1)$$

whence the theorem is proved for c + 1 with a and b left undetermined. Thus the theorem holds generally. This is a typical example of a recursive proof (proof by complete induction). (*PR* 194-195: n. 1)

First, we can see that the inductive proof of the associative law of addition begins with the recursive definition of addition. Once there is a definition of the successor function, the addition function can be defined for the case of an arbitrary a and b=1. It is then necessary to only worry about its meaning for an arbitrary a and b + 1. This meaning is established through Def. I., which defines the meaning of a + (b + 1) as essentially being the successor of (a + b). This serves to recursively define the addition function. It is then necessary to prove the theorem. Assuming c=1, the theorem holds in virtue of Def. I. *Assuming* it holds for an arbitrary c, it is then necessary to show it holds for c + 1. Repeated applications of Def. I, in combination with the assumption, allows one to prove the proposition that a + (b + (c +1)) = (a + b) + (c + 1).[159]

[159] Wittgenstein summarizes it as follows: 'The proof shows that the form 'a + (b + (c + 1)) = (a + b) + (c + 1)'… 'A (c + 1)' follows from the form 1) 'A(c)' in accordance with the rule 2) 'a + (b + 1) = (a + b) + 1' … 'A(1)'. Or, what comes to the same thing, by means of the rules 1) and 2) the form 'a + (b + (c + 1)' can be transformed into '(a + b) + (c + 1)'. This is the sum total of what is actually in the proof. Everything else, and the whole of the usual interpretation, lies in the possibility of its application. And the

For ease of reference, the following should be noted: in *Philosophical Grammar* (p. 397) Wittgenstein uses 'A' to refer to the associative law itself (i.e., '$a + (b + c) = (a + b) + c\ldots$'), while the equations that constitute what is called 'B' (the 'recursive proof of A') are, rather confusingly, labelled in a different way: Def. 1 is indicated with 'α', a concatenation of α, β, and γ (i.e., '$a + (b + (c +1)) = a + ((b + c) + 1) = (a + (b + c)) + 1$') is indicated with 'β' and δ is indicated with 'γ'.[160] Thus, we have:

$$A = \quad a + (b + c) = (a + b) + c\ldots$$

$$B = \left\{ \begin{array}{l} \alpha\ a + (b + 1) = (a + b) + 1 \\ \beta\ a + (b + (c + 1)) = a + ((b + c) + 1) = (a + (b + c)) + 1 \\ \gamma\ (a + b) + (c + 1) = ((a + b) + c) + 1 \end{array} \right.$$

5.4 Inductive Proof: Preliminary Clarifications

Wittgenstein makes many crucial conceptual observations regarding this proof. First, he notes that the associative law has the form of a definition. Granted this is the case, he inquires into what sense it makes to deny it (or prove it). He even suggests that such a 'law' does not have a sense at all and therefore isn't a proposition.[161] As Wittgenstein says:

> Of course, a definition is not something that I can deny. So it does not have a sense, either. It is a rule by which I can proceed (or have to proceed). (*PR* 194)

As an algebraic law, once it is laid down, it is a rule of the algebraic system. That is, it contributes to the determination of what is a meaningful sign transformation and is not

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usual mistake, in confusing the extension of its application with what it genuinely contains' (*PR* 193-194).

[160] Alternatively, Wittgenstein also uses the following equations to represent the general form of B (with the general form of A being '$\varphi n = \psi n$'):
$\alpha\ \varphi(1) = \psi\ (1)$
$\beta\ \varphi\ (c + 1) = F\ (\varphi\ (c))$
$\gamma\ \psi\ (c + 1) = F(\psi\ (c))$
Wittgenstein refers to this general form using 'R'.

[161] Wittgenstein's reasons for claiming this seem clearly to call back to his views in the *Tractatus*. It makes no sense to deny a basic law of algebra, since a basic law, giving meaning to the signs constituting it, does not itself assert anything and, with this, isn't bipolar. It thus isn't a proposition. To deny a law would be to exclude a sign construction (i.e., eliminate a rule) and thus one would be operating with a different formal system.

itself proven by some other sign transformation. In certain cases, rules can be reduced to other rules of the system, but it is unclear what could be meant by claiming that some *basic law* is proved by some sign transformation.[162] Its role as a basic law is simply stipulated. Thus, Wittgenstein says: 'If we ask "Does a + (b + c) = (a + b) + c?", what could we be after? Taken purely algebraically, the question means nothing, since the answer would be: "Just as you like, as you decide."' (*PR* 203). Moreover, Wittgenstein explains the role of the definition/rule in the context of the system of mathematics:

> We cannot ask about that which alone makes questions possible at all.
> Not about what first gives the system a foundation.
> That some such thing must be present is clear.
> And it is also clear that in algebra this first thing must present itself as a rule of calculation, which we can then use to test the other propositions. (*PR* 203)[163]

A rule of algebra is itself something that gives meaning to sign transformations within algebra. In this way, whatever an inductive proof is, it can't be seen as something that would somehow provide a foundation for the basic rule. For the basic rule itself contributes to the determination of the system in which other propositions can then be 'tested'. The basic rules of the system are what make it possible for propositions to be 'tested' at all and are not therefore capable of similar 'testing'.

Of course, this fact, as Wittgenstein is careful to emphasize, must mean that an 'inductive proof' involves a distinct use of the word 'proof'. At the very least, an inductive 'proof' is a proof *in a different way* from a typical algebraic or arithmetical proof. In order to further explain this, Wittgenstein does indeed draw attention to some of the idiosyncrasies of an inductive 'proof'. He says:

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[162] Wittgenstein gives the example of a system where a derived mathematical proposition that is proved is appealed to and used to prove other mathematical propositions, but in which the chain of deductions can ultimately be followed back to the definitions, and ultimately, the 'primary signs' used in the definitions (*PG* 423-424). It is important to note that Wittgenstein sees the inductive proof as having a very different role from that of a reduction.

[163] It is interesting to note that the relationship between mathematical rules and propositions, as it is expressed here, clearly is very similar in structure to what Wittgenstein later argues about the relationship between world-picture 'hinge propositions' and empirical propositions in *On Certainty* (e.g.§162-163). There he argues that there must always be world-picture 'hinges' that 'stand fast'. These 'hinges' serve as the necessary backdrop without which there could be no empirical propositions. More generally, the doubting, knowing, and testing of empirical propositions can only exist on the basis of 'hinges' that 'stand fast'. 'Hinges' are like rules and foundational in nature. Interestingly, this position seems anticipated by, and perhaps inspired by, Wittgenstein's reflection on the relationships between rules of calculation and mathematical propositions in the intermediate period. Epistemological concerns were still rather sporadic for Wittgenstein during the intermediate period, as they were for him in the *Tractatus* (e.g. *TLP* 6.51).

> In that case [that 'the calculation gives the result that a + (b + c) = (a + b) + c (and no other result)'] the general method of calculating it must already be known, and we must be able to work out a + (b + c) straight off in the way we can work out 25 × 16. So first there is a general rule taught for working out all such problems, and later the particular cases are worked out. – But what is the general method of working out here? It must be based on general rules for signs (– say, the associative law –). (*PG* 395)

An inductive 'proof' of 'general propositions' such as the algebraic law is logically distinct from other proof methods. When discussing a proof of other equations, such as those found in arithmetic, a proof consists of transforming the left side of the equation into the right side by means of established arithmetical rules. A proof, in this case, is like the check procedure where the established rules of the system give meaning to the signs and determine the rules that must be followed in order to obtain a proof (and what will count as a miscalculation).[164] It makes no sense to 'work out' the definition since, in the sense of 'proof' that involves transforming one equation into another according to strict laws, any transformation would obviously presuppose the use of the algebraic law in the 'proof' itself. Thus, as Wittgenstein says:

> To check 25 × 25 = 625 I work out 25 × 25 until I get the right hand side; – can I work out a + (b + c) = (a + b) + c, and get the result (a + b) + c? Whether it is provable or not depends on whether we treat it as calculable or not. For if the proposition is a rule, a paradigm, which every calculation has to follow, then it makes no more sense to talk of working out the equation, than to talk of working out a definition. (*PG* 395)

The basic rules of the system are what constitute what will be considered a proof within that system. Since the algebraic law is a basic rule of the system it is unclear what would be appealed to in order to 'prove' it. Once again, this is to emphasize the difference meanings of 'proof' in these cases.

5.5 The Variables in an Inductive Proof: Further Clarifications

In addition to the distinct ways in which A [a + (b + c) = (a + b) + c] and the equations in B can be seen to be proved or correct and, in turn, in what way B thus proves A, Wittgenstein also makes reference to an important related potential for confusion. The temptation to think that A is proved by the equations in B is fostered by the use of the

[164] 'What makes the calculation possible is the system to which the proposition belongs; and that also determines what miscalculations can be made in the working out. E.g. $(a + b)^2$ is $a^2 + 2ab + b^2$ and not $a^2 + ab + b^2$; but $(a + b)^2 = -4$ is not a possible miscalculation in this system' (*PG* 395).

same variables (i.e. identical signs) being used in the two cases. Wittgenstein makes several important points on this topic. One comment appears in *Philosophical Remarks* and several others appear in *Philosophical Grammar*, although the earlier comments quoted do not make it into a typescript and the last comments that we will consider were 'omitted' by Wittgenstein from the relevant manuscript (seemingly the editors of *Philosophical Grammar* mean that these comments did not make it into later work). Nonetheless, they are useful in trying to piece together his position on this topic, although their limitation as source material must always be remembered. It is especially important to understand the thrust of these comments because otherwise they could, without a proper understanding, appear as potential criticism of Skolem's proof (or its 'definition') itself, something which has been claimed about Wittgenstein's comments more generally (see the final section for more details).[165] Moreover, the *details* of these comments have largely, if not entirely, been ignored in the secondary literature.[166]

In *Philosophical Remarks*, Wittgenstein says:

> In Skolem's proof the '*c*' doesn't have any meaning *during* the proof, it stands for 1 or for what may come out of the proof, and *after* the proof we are justified in regarding it as some number or other. But it must surely have already meant something in the proof. If 1, why then don't we write '1' instead of '*c*'? And if something else, what? (*PR* 194)

While I am still somewhat unsure of the exact meaning of part of this quotation[167], at least part of its purpose is to draw attention to the fact that it is only after the proof that the 'c' can be taken to mean 'some number or other' – whereas, within algebra, it can

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[165] In reference to the technical elements of the proof itself, the issue of the variables used is the only thing Wittgenstein mentions as possibly leading to confusion (and there is no evidence that he is arguing for a change of notation). With respect to the ordinary language ('prose') description(s) of the proof, he clearly also challenges the idea that arithmetic could be given a foundation. Although it is 'arithmetic' that Skolem speaks of giving a foundation to, it is evident, from his proof, that this would be done by appealing to algebra. Algebra and arithmetic are separate systems, and thus it is not clear what would be meant by giving one a foundation in the other, although, as is argued, inductive proofs do connect the two systems. This is similarly emphasized by Wittgenstein through his claims that the basic laws of algebra similarly can't be proven (or given a foundation). Wittgenstein's criticisms of foundationalism, especially logicism, begin in the intermediate period, and develop further in his later work. See Schroeder (2021, Ch. 3) for further details.

[166] Shanker (1987, 201) simply directs the reader to the relevant passages and Frascolla (1994) and Marion (1998) do not make reference to these passages at all. While Rodych (2000a, 259-260) does note a difference in meaning of the variable in the inductive step and the 'proxy statement' of an inductive proof (between 'any arbitrary number' and 'any particular number', respectively), his analysis does not go any further than this. Yet these passages have central importance with respect to providing further support to Wittgenstein's distinction between the different senses of 'proof'.

[167] Specifically, it is unclear to me exactly what Wittgenstein means by 'or what may come out of the proof'. I assume he does not mean what is established by the proof, since this is clearly what is meant by the phrase 'after the proof'.

have other variables or symbols as its meaning (in line with rules of algebraic substitution), it can only mean ‘some number or other’ by having recourse to the inductive proof itself, (i.e., ‘after the proof’). This clearly follows from the claim that the proof’s infinite application is shown by the inductive proof and that the associative law itself does not appear as the final line in the proof (both claims which will be more extensively discussed below).

This interpretation is supported by what Wittgenstein further writes in *Philosophical Grammar*, and thus these arguments are a useful tool for understanding Wittgenstein’s position even though they didn’t make it into the amended typescript. Wittgenstein notes that the variables in α are not used in the same way as the variables that appear in the equations β, γ (*PG* 408). As we have seen, typically he emphasises the different roles of A and B, but this argument clearly is meant to show that, even within B itself, the variables are not used in the same way. This is evident from the fact that Def. 1 (i.e., α) allows the substitution of equations with different variables (but the same form), whereas the aforementioned equations in the proof do not allow a replacement of different variables, but themselves are the transformation of variables with reference to α (and already established rules of algebra). So even within B itself, the equations can’t be thought to be a simple matter of transformation of equations according to strict rules of algebra. I believe this is an example of what Wittgenstein refers to as the ‘cross-section’ through the equations of B (*PG* 412), which is unique to the inductive proof. The latter two equations in B make use of A in a particular way, and are not a matter of a simple deduction. The equations in B have different roles, which must simply be ‘seen’, and their correctness and the correctness of their interrelationship is *shown* by the entire proof.

This specific topic is again discussed in an Appendix to *Philosophical Grammar* (p. 446), the comments of which were ‘omitted’ from the manuscript (seemingly the editors mean that these comments did not live on in subsequent work). First, Wittgenstein reiterates what has been already discussed: in the case of algebra, any instantiation of the algebraic rule given by A can be regarded as justified by A (and any transformation thereby). This clearly follows from the basic rules of algebra. But, as already noted, B can’t be seen as a justification of any algebraic statement (here Wittgenstein makes reference to an algebraic statement with different variables, although we have already seen that the same applies even with identical variables).[168] This position then is further supported with the claim that not all of the variables represented by the same signs themselves are identical[169] in A and B. To accomplish this, Wittgenstein imagines placing numbers in the equations of B to bring out that the variables (specifically c) aren't the same in A and B. He says:

$$\left.\begin{array}{l} \alpha\ 4 + (5 + 1) = (4 + 5) + 1 \\ \beta\ 4 + (5 + (6 + 1)) = (4 + (5 + 6)) + 1 \\ \gamma\ (4 + 5) + (6 + 1) = ((4 + 5) + 6) + 1 \end{array}\right\} \ldots W$$

> but that doesn't have corresponding to it an equation like A_w:
> 4 + (5 + 6) = (4 + 5) + 6! (*PG* 446)

Seemingly, Wittgenstein's argument is that if the variables were being used in the same way, the replacement of the variables by numbers would preserve the relationship between the equations (i.e., A and B). That is, the equations in B with variables replaced with numbers would serve as a 'proof' of the replacement of variables by numbers in the equation A.[170] Since this is not the case, the variables must have different meanings. I take this to be a variant of the argument already provided, although with different substitutions being considered.[171]

Wittgenstein continues by emphasizing the different uses of the variable c:

> All that is meant by what I've written above is that the reason it looks like an algebraic proof of A is that we think we meet the same variables a, b, c in the equations A as in α, β, γ and so we regard A as the result of a transformation of those equations. (Whereas of course in reality I regard the signs α, β, γ in quite a different way, which means that the c in β and γ isn't used as a variable in the same way as a and b. Hence one can express this new view of B, by saying that the c does not occur in A.) (*PG* 446- 447)

[168] The tension in Wittgenstein's own thought regarding when 'justification' can be properly applied is outlined and explained in the final section (5.10).

[169] It is interesting to note that, while Wittgenstein in the last two passages talked about a variable having 'meaning' (what seems to indicate that it stands for or denotes something else) or variables being 'meant' in different ways, here he exclusively discusses the variables in terms of 'being the same' or 'identical', or being 'used in the same way' or being given a different 'function'.

[170] This is what I assume must be meant by 'corresponding equation'.

[171] Whereas Wittgenstein typically emphasizes the different roles of the equations by considering in what way the equations license transformations, here he examines how, even considered purely as arithmetical equations, they also wouldn't have the same relationship to each other that one would assume (and thus the variables don't all mean the same). Thus, the argument seems important from the standpoint that it makes clear that the inductive proof can't be viewed purely as an exclusively algebraic *or arithmetical* proof, but rather one that importantly involves both systems (and thus connects them). Nonetheless, it should not be forgotten that this argument was omitted.

This last quotation closely agrees with what has been already argued. Wittgenstein argues that the use of ‘c’ is not the same in A and the equations of B. Whereas in A all three variables are merely placeholders to indicate a legitimate transformation of another equation with a similar form, in β and γ the use of the variable ‘c’ must be considered a placeholder for some specific number and also be the *same*. Yet, as indicated in the *Philosophical Remarks*, what this meaning is is unclear. It could be ‘1’ or some other number, but it is clear that it must be some *specific* number, for it is only at the *end* of the proof that one is justified in thinking that it can meaningfully be ‘some number or other’. As Wittgenstein also expresses it, c is the ‘hole through which the stream of numbers has to flow’ (*PG* 447). This clearly brings out how its use is different from the c in A. Using the same variables, without the proper attention to their logical dissimilarity, can lead to the confusion that one set of equations proves the other equation – in the same sense.

All of this is clearly meant to once again bring out the logical dissimilarity of the c from the other variables and, in this way, emphasize the logical dissimilarity with respect to what way B ‘proves’ A in comparison to how this is undertaken in regular arithmetical or algebraic equations (e.g. at least the last two equations of B itself). Thus, as Wittgenstein says, ‘For if the proposition is a rule, a paradigm, which every calculation has to follow, then it makes no more sense to talk of working out the equation, than to talk of working out a definition’ (*PG* 395). Moreover, unlike a typical proof (e.g. an algebraic or arithmetical transformation according to strict rules), the associative law itself never appears as the final line in the proof. So, granted that Wittgenstein is right about this, what is an inductive proof and what does it prove?

5.6 Inductive Proofs: The Positive Characterization

There are a couple of important quotations that, given what has been said, serve to introduce this topic:

> A recursive proof is only a general guide to an arbitrary special proof. A signpost that shows every proposition of a particular form a particular way home. It says to the proposition 2 + (3 +4) = (2 + 3) + 4: ‘Go in *this* direction (run through this spiral), and you will arrive home.’ (*PR* 196)

> An induction doesn't prove the algebraic proposition, since only an equation can prove an equation. But it justifies the setting up of algebraic equations from the standpoint of their application to arithmetic. (*PR* 201)

From the steps of the inductive proof, it can not only be seen that the associative law can be proven for an arbitrary a and b, but also, by the end of the proof, an arbitrary c. For the inductive proof shows the common form that any particular proof will take. In this way, it points beyond itself to an infinite series of proofs, where in every new row of the proofs 'c' takes the place of 'c +1' and is transformed into its successor (i.e., '1' is added).[172]

It is in this way that the inductive proof 'justifies the setting up of algebraic equations from the standpoint of their application to arithmetic'. For the inductive proof is the general form – what is common, for example, to the proof involving any specific number and its successor – of the proof for any particular relevant arithmetical equation. The associative law does not appear as a final line in the proof because it is not being derived from something else and because it is the inductive proof that *shows* its applicability (which the law must have continual 'recourse to' – *PR* 203) ; it is not a meaningful proposition (*PR* 198) and indeed can be seen as akin to a name, or going proxy, for what the proof actually shows (*WVC* 135; cf. Rodych 2000a, 259). Rather, the inductive proof can be seen as establishing *another way in which the law can be applied*. That is, the inductive proof shows the law's infinite application to arithmetic – the necessary preparation to be able to apply the law to numbers.[173] Moreover, with this in mind, it is clear that the inductive proof does not serve, as other examples Wittgenstein gives, to 'continue a system of proofs backwards' – what I take Wittgenstein to mean as providing a foundation for the system and/or proving 'fundamental' propositions of the system by a smaller set of propositions *within* the system – but rather forges a connection with another system of mathematics. Thus, Wittgenstein says:

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[172] Seemingly this can be understood in two slightly different, but equivalent, ways. First, one can continue to add variables (e.g. 'd') which stand for '1' to each new line of proofs (using 'ξ' as a 'stop-gap for what only emerges in the course of the development' – *PR* 197). It should be apparent that, on the model of the earlier list of equations, the equations can be readily transformed to preserve the truth of the equation. This is clearly what Wittgenstein is trying to show with the list of equations with arrows in the quotation just below. Alternatively, this means that, while the proof clearly is only proven for 'c +1', the structure of the proof clearly *shows* it is also proven for 'c + 2', 'c + 3' and so on, since all of these quantities will find themselves as terms in the series of proofs.

[173] In addition to what has already been quoted, this is most clearly expressed in Moore's recently published notes: 'We haven't proved it as algebraic formula; it is only a postulate as such, i.e. a rule of a game. What we've proved is that the rule applies, if a, b & c are numbers' (*NM* 56).

It is something like this: all that the proof of a *ci-devant* fundamental proposition [the associative or commutative law derived from a set of propositions within the system] does is to continue the system of proofs backwards. But the recursive proofs don't continue backwards the system of algebraic proofs (with the old fundamental laws); they are a new system, that seems only to run parallel with the first one. (*PG* 425)

They do not serve as a foundation for the stipulated laws of algebra, but rather serve as a way of preparing the laws of algebra to be applied to arithmetic.[174] It is the inductive proof that establishes that the law can be applied to numbers; and this linking of systems thereby creates a new system. As will be discussed in more detail below (Section 5.10), this application is essential to understanding algebra as more than a mere calculus, for it thereby gives algebra an extra-mathematical application (algebra thereby has a connection to arithmetic which is used in empirical descriptions) and is one component of what leads Wittgenstein also to speak of the associative law as having meaning.

With the construction of an inductive proof, it is unnecessary to actually undertake all of the individual proofs to be able to see the law's correctness in relation to arithmetic and in this way we avoid the problems that arise with the verification principle in relation to propositions making reference to the infinite. Wittgenstein says:

I know a proof with endless possibility, which, e.g., begins with 'A(1)' and continues through 'A(2)' etc. etc. The recursive proof is the general form of continuing along the series. But it itself must still prove something since it in fact spares me the trouble of proving each proposition of the from 'A(7)'. But how can it prove this proposition? It obviously points along the series of proofs

$$a+(b+(\xi+d)) = (a+(b+\xi))+d = ((a+b)+\xi)+d = (a+b)+(\xi+d)$$

$$a+(b+((\xi+d))) = (a+(b+(\xi+d)))+d = ((a+b)+(\xi+d))+d = (a+b)+((\xi+d)+d)$$

[174] For more extensive arguments for why 'recursive proof does not reduce the number of fundamental laws', see the section with that name in *Philosophical Grammar* (i.e., *PG* 425-426). It is important to note that Wittgenstein always insists on the algebraic law's 'independence' even *after the proof*. The rule continues to apply to algebra, and is not a derived rule, but now also has an application to arithmetic. The fact Skolem talked about the 'foundations of elementary arithmetic' is, according to Wittgenstein, an ordinary description of his proof that betrayed confusion.

> This is a stretch of the spiral taken out of the middle. (*PR* 196-197)[175]

In the case of this algebraic law, the inductive proof has an important relationship with arithmetic. It is the general form a proof of any particular proof of the associative law will take. In this way, it is representative of numerous proofs and rightly called by Frascolla a 'proof-schema'.[176] Once the general form of the proof is captured by the inductive proof and its infinite application is observed, it is unnecessary to actually undertake each particular proof to be sufficiently convinced of an arithmetical instance of the law's correctness.[177] In this way, it is the inductive proof that establishes an arithmetical law to the effect that, if one undertakes such-and-such a calculation, one *must* achieve such-and-such a result (cf. Frascolla 1994, 83). It itself does not serve as the proof of an arithmetical proposition, but it does show the form any such proof must take and thus serves as a 'general guide to an arbitrary special proof' (*PR* 196). Once the induction is accepted, the associative law can indeed be treated as a law (i.e. a new criterion for correctness) that applies to arithmetical equations and any calculation that does not agree with the associative law can be rejected as incorrect (cf. Frascolla 1994, 83). Unlike a typical proof, the inductive proof does not operate from, or on the basis of, anything more fundamental than the associative law itself is for algebra. And, after the three steps of the inductive proof are undertaken, it is unnecessary to make longer and longer chains of equations. Anything after the initial spiral (as Wittgenstein also calls it) is superfluous (see *PR* 199), for the shared general form that all such proofs will share can be readily seen.

Wittgenstein is careful to emphasize that this infinite possibility of application is itself not something that can be proved, and thus also not something stated by the proposition[178] itself, but rather *shows* itself in the inductive proof. Wittgenstein emphasizes these points in at least a couple of important passages:

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[175] The other representation of the proof as a spiral appears in *PR* 199.

[176] Wittgenstein himself clearly makes reference to this with his inquiry into whether the inductive proof is a single proof or, as he clearly intimates is actually the case, a 'certain arrangement of proofs' (*PG* 399), a proof of 'every proposition of a certain form' (*PG* 400) or a 'series of proofs' (*PG* 430) – the latter two phrases which seem particularly apt.

[177] But this is not to say that the proof-schema actually proves all the individual arithmetical propositions of that form. This is a logical impossibility, as discussed in the last chapter. It is the general form of the proof, given by the induction, *together with the individual proposition*, which supplies one with a proof (*PG* 430).

[178] 'Proposition' here, of course, is being used loosely. Wittgenstein, as has been touched upon above, rejects the idea that the associative law is a proposition at all – for a variety of reasons. This is the case both when it is considered simply in relation to algebra and when it is considered in relation to the induction.

> If we now suppose that I wish to apply the theorem to 5, 6, 7, then the proof tells me I am certainly entitled to do so. That is to say, if I write these numbers in the form ((1 + 1) + 1) etc., then I can recognise that the proposition is a member of the series of propositions that the final proposition of Skolem's chain presents me with. Once more, this recognition is not provable, but intuitive. (*PR* 195)[179]

> An algebraic proposition must always gain only arithmetical significance if you replace the letters in it by numerals, and then always only *particular* arithmetical significance.
> Its generality doesn't lie in itself, but in the possibility of its correct application. And for that it has to keep on having recourse to the induction.
> That is, it does not assert its generality, it does not express it; the generality is, rather, shown in the formal relation to the substitution, which proves to be a term of the inductive series. (*PR* 203)

The generality of what is 'proved' for Wittgenstein is shown by the inductive proof; at the same time, in agreement with what was explained in Section 3.2.4, it is this proof that will ultimately be understood, by Wittgenstein, as conferring meaning on the statement itself. From the proof, one can see that the result holds with the substitution of any number. That is, any choice of number can be seen to find itself in the series of proofs and, with this, to be amenable to the proof procedure. As is similar with many statements in mathematics that derive their sense from their proof, and in contrast to a statement that belongs to a system of mathematics and is amenable to a check procedure, the inductive proof confers (a new) meaning on the law, for, prior to the inductive proof, the law did not have an application to arithmetic (and thus this additional use). This tension in Wittgenstein's thought between his view in the intermediate period that an inductive proof shows the infinite applicability of the 'proposition' it 'proves' and his view, even beginning in the intermediate period, that the proof *gives meaning* to the proposition that is proved, is further discussed in section 5.10.

[179] Of course, most notable in this quotation is the fact that Wittgenstein speaks of the recognition of the proposition being a member of the series of propositions, but this not being provable. This is another way of drawing attention to the infinite applicability which the inductive proof *shows*. Moreover, this is one of the few passages that clearly makes reference to the recursive nature of the proof. The addition function itself can be given as a recursive function, which means all expressions of numerals can be given as sequences of 1's. Envisioned in this way, the recursive nature of the proof should be *even more perspicuous* (as the continual addition of 1's to make up the series of proofs).

5.7 'Showing' Revisited: Additional Examples

Of course, it is noteworthy that Wittgenstein once again speaks of the technical sense of 'showing'. Here, we see, it still occupies a *central role* in his philosophy. In the inductive proof sections, Wittgenstein uses three examples to explicate his conception of showing: (1) multiplication generally, (2) periodicity, and (3) the particular application of a general rule.

(1) Wittgenstein's first example concerns fundamental rules of our number system generally. An induction in this case is fairly obvious, and something that is clearly realized from a fairly early age and doesn't involve a specific proof. Multiplication consists of the standard rules learned by children at a young age; these simple rules apply to numbers, no matter how large, which themselves give numbers as results. For Wittgenstein, this can't be stated in a mathematical proposition, but is rather shown by the rules as they are actually employed. Of course, this feature of our number system is typically realized early on, as it is understood that an infinite number of numbers can be multiplied together (the meaning of which, from the last chapter, is to refer to an *infinite possibility of choice of number*).

However, one may think that an inductive proof could be constructed to show this property.[180] One could, for example, show that multiplying the single digit numbers together clearly produces a number and then undertake an induction on the size of the numbers being multiplied. However, given the foundational nature of multiplication rules, and everything else Wittgenstein suggests, there would be no purpose to such a proof, given the logical priority these different mathematical techniques have within our system of mathematics (i.e. the rules for multiplication in comparison with inductive proofs).[181] If one did have any question about the inductive property of the rules for multiplication, then it would be of no value to provide an inductive proof to convince one of this.[182] If one did not understand this basic use of mathematical signs, clearly a more complex mathematical method, which presupposes understanding of the laws of multiplication, could not serve to make explicit the rules of the multiplication system.

[180] What follows is my attempt to make sense of Wittgenstein's comments on this subject (*PR* 204-205).
[181] This harkens back to a quotation already discussed (i.e., *PR* 203). Clearly the rules of multiplication allow one to undertake a whole host of calculations, and 'test' various propositions for their correctness. And the relative primacy of this system to all of our mathematics should be obvious.
[182] It would seem that someone who didn't understand the inductive property really doesn't understand the rules of multiplication. Wittgenstein talks about a 'clarification of the symbolism and an exhibition of an induction', by which, Wittgenstein likely means it would involve the practice of using the rules in diverse and sufficiently complicated examples.

In this case, the infinity of the multiplication system readily shows itself to anyone who has learned the rules of multiplication.

(2) Periodicity is similar. In addition to the above quotation, Wittgenstein comments:

> Suppose that people argued whether the quotient of the division 1/3 must contain only threes, but had no method of deciding it. Suppose one of them noticed the inductive property of 1.0/3= 0.3 and said: now I know that there must be only threes in the
> 1[183]
> quotient. The others had not thought of *that* kind of decision. I suppose that they vaguely imagined some kind of decision by checking each step, though of course they could never have reached a decision in this way. If they hold on to their extensional viewpoint, the induction does not produce a decision because in the case of each extension of the quotient it shows that it consists of nothing but threes. But if they drop their extensional viewpoint the induction decides nothing, or nothing that is not decided by working out 1.0/3 = 0.3, namely that the remainder is the same as the dividend. But nothing else. 1
> Certainly, there is a valid question that may arise, namely, is the remainder left after this division the same as the dividend? This question now takes the place of the old extensional question, and of course I can keep the old wording, but it is now extremely misleading since it always make it look as if having the induction were only a vehicle – a vehicle that can take us into infinity…
>
> Of course the question 'is there a rational number that is a root of $x^2 \times 3x + 1= 0$?' is decided by an induction; but in this case I have actually constructed a method of forming inductions; and the question is only so phrased because it is a matter of constructing inductions. That is, a question is settled by an induction, if I can look for the induction in advance; if everything in its sign is settled in advance bar my acceptance or rejection of it in such a way that I can decide yes or no by calculating; as I can decide, for instance, whether in 5/7 the remainder is equal to the dividend or not. (*PG* 402-403)

At some point, the inductive property of 1/3 was realized.[184] Through the introduction of a new sign (i.e. '·'), which was defined as the dividend being equal to remainder of a division, a new mathematical system was created. As a result, the inductive property could show itself in the system and serve as a way of deciding the question as to whether, for example, a 4 lies in the decimal expansion of the result of the division. Previously, prior to the creation of this new system, it would not be possible to even ask the question of whether a 4 lies in the expansion. For, independently of actually finding

[183] The notation '1', taking its place on an individual line, represents the remainder of the division.
[184] It is not sufficient, Wittgenstein emphasizes, for one simply to 'realize' the remainder is equal to the dividend; rather, the inductive property must have also been 'seen'. That is, one would need to 'see' that the continued division of the remainder results in a repeated infinite decimal expansion. So, the discovery of the periodicity really comes with the introduction of a new sign and a new system of division. For then it becomes possible to ask about an inductive property (insofar as this can be ascertained from a division), prior to any inductive property showing itself (*PG* 404).

any 4's, there was no method to answer the 'question' and therefore it wasn't a meaningful question at all. This, once again, follows from the results of previous chapters.

Therefore, in contrast to what will later be discussed, periodicity offers a simple way of searching for an induction (according to whether the decimal is periodic or not). In examples like this one, it is the case that there is a check procedure for inductions and thus a perfectly meaningful question prior to the actual determination of whether the decimal is periodic or not. For, when wishing to establish whether a decimal is periodic, and contains an important inductive property, it simply suffices to note whether the remainder of the fraction is equal to the dividend. Given the possibility of establishing this, the question (because of the obvious possibility of establishing an answer) gains a sense independent of the actual proof of its inductive property, in contrast to equations that require inductive proofs. So it is clear, as it was to Wittgenstein, that there are individual cases where an induction, or the lack thereof, can be determined by a check procedure. At the same time, these cases are clearly distinct from those cases that are referred to as 'inductive proofs', as shall be discussed further below.

(3) Finally, Wittgenstein likens the recognition of the infinite application shown by an inductive proof to the realization of what constitutes the particular application of a general rule:

> Neither can I prove that a + (b + 1) = (a + b) + 1
> is a special case of a + (b + c) = (a + b) + c
> I must see it.
> (No rule can help me here either, since I would still have to know what would be a special case of this general rule.) (*PR* 198)

The relationship between the general and particular, in this case, is one that can't be stated, but is shown. There is no further rule (or clearly, in the terms we have been using, 'assertion') that allows one to *infer* that one has made the transition from general to particular correctly (or that shows one how to do so if one has not). As is often the case with the concept of 'showing', it must simply be 'seen'. Of course, part of the reason for bringing attention to this example in particular is because it serves to illustrate what is special about the inductive proof. One does not infer from B to A, rather one simply must see what is shown by B. It is very interesting to note that it would appear that here lies one of the earliest references to what would later become the rule-following considerations (cf. Frascolla 1994, 114). Given the complexity of the rule-following problem (at least in terms of what it becomes), it is unsurprising to find Wittgenstein making reference to the saying/showing distinction at this point.

5.8 Inductive Proofs, Decision Procedures, and Generality

Wittgenstein is very careful to distinguish between general check procedures and inductive proofs. One of the most important passages on the topic is the following:

> So he has seen an *induction*! But was he *looking for* an induction? He didn't have any method for looking for one. And if he hadn't discovered one, would he *ipso facto* have found a number which does not satisfy the condition? – The rule for checking can't be: let's see whether there is an induction or a case for which the law does not hold…
> …
> Prior to the proof asking about the general proposition made no sense at all, and so wasn't even a question, because the question would have made sense if a general method of decision had been known *before* the particular proof was discovered.
> (*PG* 400-402)

In accordance with what was discussed about the verification principle and infinity in Chapters 3 and 4, there is no such thing, according to Wittgenstein, as 'looking for' an induction, since there are no strict rules that one can follow to determine whether there is *or is not* an induction. Trying to construct an inductive proof is unlike trying to determine the correctness of an equation. In the latter, there are strict rules that will give one a determinate answer, whereas, in the former, the general form of an inductive proof, provided by R, can provide *no decisive procedure* as to how one is to be constructed. While there is a clear criterion for the 'truth' of the algebraic law (i.e. the inductive proof), there is no criterion for its 'falsehood'. A proof may not be constructed, but this does not mean that one *can't* – in the grammatical sense – be constructed. Without a clear criterion for 'truth' *and* '*falsity'*, it is clear that, for Wittgenstein in the intermediate period, the general form of an inductive proof can't confer meaning on the mathematical law independent of the particular inductive proof which shows the law's application (in contrast to a check procedure).[185]

[185] As suggested already in Section 3.2.2, if the laws of logic don't hold (e.g. the law of excluded middle), the expression is not a proposition (*PG* 400).

One can start with the idea of constructing an inductive proof, but, because of its role as a meaning-conferring extension of mathematics, there is no way to specify exactly how this proof is to be constructed based on the general form of the inductive proof. As Wittgenstein says:

> So when we said above we could begin with R, this beginning with R is in a way a piece of humbug. It isn't like beginning a calculation by working out 526 × 718. For in the latter case setting out the problem is the first step on the journey to the solution. But in the former case I immediately drop the R and have to begin again somewhere else. And when it turns out that I construct a complex of the form R, it is again immaterial whether I explicitly set it out earlier, since setting it out hasn't helped me at all mathematically, i.e. in the calculus. So what is left is just the fact that I now have a complex of the form R in front of me. (*PG* 416)

Just being given the algebraic law (a + b) + c = a + (b + c) and the general form of an inductive proof R is not sufficient to guide one in constructing an inductive proof. The specific functions that are to be used, and the setup and relationship between the equations, is only understood upon the actual construction of the proof (which shows itself). Prior to the proof, similar to what was discussed in Chapter 3, the algebraic equation has no meaning precisely because it did not have such an application to arithmetic.

Thus, Wittgenstein says:

> If I said that the proof of the two lines of the proof justifies me in inferring the rule a + (b + c) = (a + b) + c that wouldn't mean anything, unless I had deduced that in accordance with a previously established rule. But this rule could only be
>
> $$\left.\begin{array}{l} F_1(1) = F_2(1),\ F_1(x + 1) = f\{F_1(x)\} \\ \qquad\qquad\qquad\ F_2(x + 1) = f\{F_2(x)\} \end{array}\right\} \quad F_1(x) = F_2(x) \ldots (\rho).$$
>
> But this rule is vague in respect of F_1, F_2 and f. (*PG* 409)

As Wittgenstein ultimately argues, it is only the lining up of B next to A which allows one to see what the proof shows about how A is now to be understood (which is comparable to the final 'showing' example given above). This is because B, as we have seen, does not simply contain a deduction of A, but rather, through a 'cross-section' (*PG* 412), where the equations in B are used in different ways, *shows* a different use of A. Exactly how this is all constructed thus can't be known in advance of the construction. Exactly how A is to be understood is given by the proof itself, which can only be clearly seen (as this corresponds to what is *shown*) with the actual construction of the proof, and not by any abstract form of the proof. For the abstract form of the proof, Wittgenstein argues, is too ‘vague’ to determine how the proof is to be actually constructed (or whether one is even possible).[186]

From this, it can be clearly stated exactly what is meant by generality for Wittgenstein (as it relates to inductive proofs).[187] While it is common to talk of ‘all numbers’ in this case, this is, in accordance with what has already been extensively discussed in the last chapter, really a way, in so far as it can be understood to have meaning at all, of speaking of the infinite possibility of sign construction in accordance with a rule. One can speak of ‘all numbers’, if one wants an expression of ordinary language to ‘describe’ what one has done (and Wittgenstein argues there is really *no phrase* that could accurately do this job – *PG* 410, 422). But it is only on the basis of what is done in the finite case (a step-by-step check of number) that, in the infinite case, such a translation of what is established is justified. It is based on an *analogy* with the finite case, where a step-by-step check can be done, that one is naturally inclined to say one has proved something about ‘all numbers’[188], but then, the expressions of ordinary language *must not confuse* one *subsequently* about what has been done.[189] This can only be avoided by ‘seeing’ what the proof actually *shows*, including its infinite application, which is entirely given by the individual inductive proof. In the last chapter, we saw that speaking of ‘all numbers’ – in so far as this could be given meaning at all – was really a way of speaking about the infinite possibility of sign construction. But whereas, in the previous chapter, the problem concerned rules for generating sequences of numbers themselves, here, clearly, the concern is the possibility of the *construction of proofs as these relate* to an infinite possibility of numbers that could appear as terms

[186] The same goes for ‘checking’ an already constructed inductive proof. The point is that one must always look at the *individual* proof and how its constituent equations relate to understand exactly what a given proof shows (see *PG* 414-415, and 417, for additional arguments on this topic). The general form of an inductive proof is insufficient to decide whether an individual inductive ‘proof’ is indeed a proof.
[187] It is important to note that, at least at times, Wittgenstein even rejected using the term ‘generality’ at all in relation to inductive proofs. Thus, he also speaks of an inductive proof being a ‘check’ of an algebraic proposition’s structure, rather than its generality (*PG* 396). His point is clearly to block, once again, an extensionalist interpretation of the proof. Insofar as no ordinary language descriptions could be adequate to the task, it is not so much the choice of words that is the problem, but how this choice can lead to confused interpretations of this proof. So long as there is no philosophical confusion, it would seem that Wittgenstein would think ‘generality’ can be used (as he himself does – e.g. *PR* 202-203).
[188] ‘Here the connection with generality in finite domains is quite clear, for in a finite domain that would certainly be a proof that f(x) holds for all values of x, and *that* is the reason why we say in the arithmetical case that f(x) holds for all numbers’ (*PG* 406).
[189] Because of this prevalence to fall into confusion, Wittgenstein argues it is better simply to get rid of the ordinary language descriptions altogether, since it will make the actual mathematical relationships clearer (when one simply looks at the proof) (*PG* 422).

that could be proved. The iterative nature of the inductive proof is the general form of the infinite list of possible proofs. It is for this reason that Wittgenstein rejects the idea that an inductive proof proves anything about 'all numbers', for, as already extensively argued, there is no such thing as 'all numbers' (*PR* 147);[190] rejecting the extensionalism already discussed in the previous chapter entails conceiving of the meaning of unbounded quantifiers in a different way, specifically as being given by an induction, which itself is the limitless possibility of proof construction (given by a rule). It requires rejecting thinking of the unbounded quantifier on the model of the bounded quantifier.[191] Generality, in this case, is the infinite possibility of proof construction in accordance with a rule.[192]

5.9 Wittgenstein's Response to Skolem: Marion's and Shanker's Positions Considered

Based on what has been argued, it is apt here to note that I am in agreement with Mathieu Marion (1998, 105-106: n. 36), who challenges Stuart Shanker's claim that Wittgenstein engaged in a 'sustained attack… on Skolem's definition of "recursive proof"' (Shanker 1987, 199). As Marion points out, doubtless Skolem was unaware of Wittgenstein's (idiosyncratic) saying/showing distinction and, thus, did not attempt to apply it to his own proof. However, Marion also convincingly argues that Wittgenstein did not wish to undermine Skolem's notion of generality, the primary goal of which he would have been in agreement. That is, it is clear that Skolem wanted to avoid extensionalism with his own proof and certainly Wittgenstein would have been sympathetic to the motive behind the proof techniques, which was to avoid the use of the unbounded quantifier (as evidenced by the title of Skolem's paper).[193] Shanker (1987, 200) convincingly points out only one area of 'prose' in Skolem's proof that is

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[190] See p. 101 for the extensive explanation of this quotation.
[191] And Wittgenstein even recommends reserving the use of the universal quantifier to cases where the domain is finite, as Skolem also advocates (Marion 1998, 104).
[192] All of this is useful for understanding why a question about an algebraic law (or its generality) has no sense. When one strips away the ordinary language descriptions such as 'generality' or 'all numbers' it is not clear what the question means. If one asks, 'Does (a + b) + c = a + (b + c)?', is this asking whether the law applies? And, as has been extensively discussed, if one simply wants to know if there is the possibility of an induction, the only possible solution is the induction, which itself determines a new application for the law. In this way, it is impossible to ask about the very thing the proof establishes prior to the proof.
[193] For more information on the commonalities between Wittgenstein's views and Skolem's, see Marion (1998, 94-105).

confused (that is: Skolem's claim to be providing a 'foundation for elementary arithmetic'),[194] and, as we have seen, there is little, if anything, in the way of criticism of Skolem's proof itself (as opposed to possible confusions that arise as a result of a misinterpretation of Skolem's proof). This is supported by the fact that Skolem does not make the error of speaking of the proof as proving something with respect to 'all numbers' – a possible interpretation of the proof Wittgenstein rejects. In fact, the comments about the different uses of the variable 'c' in the inductive proof (with an emphasis on the first from the *Philosophical Remarks*) would seem to be the most direct criticism that can be found about Skolem's presentation of his proof. And, even in this case, it is hardly clear that Wittgenstein is criticizing Skolem's proof, or the notation he employs (or advocating for a different one), rather than simply arguing against a confusion that *could arise as a result of a misinterpretation* of Skolem's proof. It is clear that, at the very least, he thought that the proof had the potential to lead to conceptual confusions, even if these confusions, for the most part, were not themselves directly evident in, or a result of, Skolem's work. Assuming this is the case, all of this clearly agrees with Wittgenstein's distinction between calculus and prose, the latter which properly belongs to the province of philosophy – a position that Shanker emphasizes was Wittgenstein's and one he clearly thinks Wittgenstein otherwise generally abided by.

5.10 Tensions in Wittgenstein's Thought: Proof, Family Resemblance, and Meaning

It would appear that considerations related to inductive proof play an important role in the development of Wittgenstein's idea of the family resemblance concept. Already in the early intermediate period Wittgenstein is aware that a variety of logically different things can be referred to using the same name. Wittgenstein says:

[194] Marion (1998, 105-106: n. 36) does not mention Wittgenstein's claims against the impossibility of providing a foundation for arithmetic, which is one way in which Wittgenstein did 'attack' something Skolem claimed about his proof. In contrast, Shanker (1987, 200) makes reference to this problem with the 'prose', which at least lends some support to his claim about Wittgenstein's 'attack'. It is apparent that Wittgenstein would object to giving a foundation to a system generally (see footnote 165 for further details).

> The trouble is that our language uses each of the words ‘question’, ‘problem’, ‘investigation’, ‘discovery’ to refer to such basically different things. It’s the same with ‘inference’, ‘proposition’, ‘proof’. (MS 108, 93; translated in *PR* 190)

Even at this early stage (early in 1930), given the comments we have looked at, it makes sense that consideration of inductive proof played a role in this idea. Wittgenstein explicitly mentions ‘inference’ and ‘proof’ in the above quotation. And, as we have seen, he already sees the variety of meanings of what goes by ‘proof’ when he considers the inductive proof. The equations here have a different relationship to each other than in an algebraic proof. Moreover, I have attempted to show this also relates to how the variables are used themselves. And when he contemplates whether or not something is or is not a proof, this depends on the role the proof has (in contrast to the one he sees when dealing with equations).[195] That is, there is a tension between denying that algebraic laws can be proved and giving other explanations of the relationship between an induction and what is ‘proved’ (cf. Schroeder 2021, 52). When he considers the associative law and its ‘proof’ *solely* in relation to the system of algebra he can (rightly) only think of the law as a ‘stipulation’; and this calls into question giving a ‘proof’ for the law (its role can’t be that of a foundation). But when he is forced to think of the law in relation to arithmetic, he can begin to imagine a legitimate role for the proof. For example, he proposes, as we have seen, that it is better thought of as a ‘justification’. But, as Schroeder has rightly asked, ‘Isn’t that [a justification] a kind of proof?’ (2021, 52). Thus, it makes sense that this would come to be Wittgenstein’s position. Inductive proof can’t be a ‘justification’ *in the sense of a foundation*, but it can be a ‘justification’ for its *application to arithmetic,* which forges a connection with the mathematics we do use in everyday life. In this way, algebra acquires an extra-mathematical application, which contributes to an aspect of its meaning (the importance and origin of the idea of an extra-mathematical application is further discussed in Chapter 6). And it is partly on this basis that the associative law itself can be seen to acquire a meaning (further discussed below).

Although Wittgenstein never speaks of ‘proof’ specifically being a family resemblance concept in the *Philosophical Investigations*, he does speak of ‘number’ being one; and ‘number’ *is* referenced together with ‘proof’ in the *Philosophical*

[195] As late as October 1931, Wittgenstein suggests that there is an ‘ordinary grammar’ of proof (MS 112, 57v) and, given the other comments in the area of this one, suggests that inductive proof does not fit this ‘ordinary grammar’ (e.g. MS 112, 65r-65v). He even explicitly notes the tension in his thought: ‘Whence this conflict: “This isn’t a proof!” “That surely is a proof.”?’ (MS 112, 60r; translated in *PG* 415).

Grammar. Moreover, it is in this work that his idea of a family resemblance concept starts taking shape; he makes reference to members falling under a concept being like a 'family' and outlines some of the main points that characterize the family resemblance concept (*PG* 75). The following is pertinent:

> The definition of the word 'proof' is in the same case as the definition of the word 'number'. I can define the expression 'cardinal number' by pointing to examples of cardinal numbers; indeed instead of the expression I can actually use the sign '1, 2, 3, 4, and so on ad inf'. I can define the word 'number' too by pointing to various kinds of number; but when I do so I am not circumscribing the concept 'number' as definitely as I previously circumscribed the concept cardinal number, unless I want to say that it is only the things at present called numbers that constitute the concept 'number', in which case we can't say of any new construction that it constructs a kind of number. (*PG* 300)

The fact that family resemblance ideas are being referenced here (albeit not in a fully developed form) should be clear; certainly, the ideas, even in a more developed form, are taking shape in the *Philosophical Grammar*. In the context of explaining what is involved in the family resemblance concept, Wittgenstein appeals to certain psychological concepts such as 'understand', as well as 'game' and 'ball-game' as examples (*PG* 74-75, 68).[196] Moreover, that mathematics itself is a 'family' (of activities and purposes) is suggested later in the *Remarks on the Foundations of Mathematics* (pp. 273 and 399). In addition, inductive proof would not only involve a consideration of the concept of 'proof', but also 'proposition' (and 'question'). It is arguably the case that Wittgenstein comes to see the result of an inductive proof as a proposition too. Wittgenstein initially denied this because he thought the result of an inductive proof could not be stated, and was shown by the 'proof' itself (thus what was 'proven' stands in, or is a name, for the proof itself); and, as we have seen, he thought the result of the proof is not a properly derived equation, but rather requires additional insight to understand its infinite applicability. With the development of the family resemblance concept, his rejection of the saying/showing distinction (its role of filling an explanatory hole will ultimately not be needed), as well as the development of his philosophy of mathematics (further explained below), Wittgenstein would not be forced to accept this. As he moved away from his theory of meaning in the *Tractatus*, together

[196] This is not, however, evident in the early part of the intermediate period. In the *Philosophical Remarks* he speaks neither of examples such as 'game' nor of the diversity of things called 'numbers', which becomes clear later. It is only in *The Big Typescript*, where we first see these ideas taking shape (see 'Proposition. Sense of a Proposition. §15 – "Sentence" and "Language" Blurred Concepts' – most important 55e), which continues to develop into his more nuanced views in the *Philosophical Grammar*.

with his broader notion of proof, there is no reason for him to think that what is proved by an inductive proof doesn't have meaning (bipolarity isn't viewed as essential any longer). It thus becomes possible to *say* that the associative law applies to any number – albeit with the understanding that this still does not give credence to extensionalism. For all of these reasons, it should be apparent that Wittgenstein's investigation of inductive proof importantly relates to the evolution of the family resemblance concept, albeit in relation to the philosophy of mathematics.[197]

Finally, given how early reflections on inductive proof occur in the intermediate period, and the benefit of being able to see how Wittgenstein's philosophy will develop, it is doubtless these considerations that also contribute to the evolution of Wittgenstein's philosophy of mathematics. For example, it is consideration of inductive proof that contributes to the notion that the answer to a substantial mathematical problem (one that goes beyond algorithmic decidability) essentially gives meaning to a proposition. This is one of the conclusions he comes to specifically about what is proved in an inductive proof. Although, at times, as we have seen, he doubts whether what is proved is indeed a proposition, and whether it is best considered to be 'proved' at all, at other times he is lenient to this idea, and even suggests that the inductive proof gives meaning to what is proved (*BT* 675). It is on the basis of this that he would have reconsidered his regimentation of concepts such as 'problem' and 'question' also. This fits well with our discussion of the development of Wittgenstein's conception of mathematical proof in Section 3.2.4. Indeed, consideration of inductive proof likely was a contributing factor in the idea that mathematical proof involves stipulation: 'the proof shews me a new connexion, and hence it also gives me a new concept' (*RFM* 297f; cf. Schroeder 2021, 144). This is reinforced by Wittgenstein's continued adherence to the idea that infinity only exists in relation to a rule; an inductive proof will supply the necessary rule that provides for the infinite applicability of a proposition. And the

[197] Engelmann traces one explicit use of the family resemblance idea to Wittgenstein's final break (in 1933) with one of his central ideas from the *Tractatus* (into his intermediate work): that we can delimit meaningful propositions from senseless ones on the basis of some general considerations that, as Wittgenstein will come to see, are actually imposed upon language. Engelmann expertly reconstructs how Sraffa's famous Neapolitan gesture led to Wittgenstein to give up the idea that 'language' or 'sentence' can be usefully circumscribed as he had thought in the *Tractatus* (into the intermediate period). In fact, these concepts are a 'family of structures' (Engelmann 2013, 160). See Engelmann (2013, 151-160) for further details. From our reflections here, it should be clear that while Engelmann's account involves one of the most important applications of the family resemblance concept, especially as this applies to the philosophy of language, consideration of inductive proof would have played an important role in the development of the family resemblance idea in general.

detailed investigation of this kind of proof in particular, and how Wittgenstein examines this against the background of his understanding of derivation of equations and check procedures, are contributing factors to his more general conclusions in his philosophy of mathematics of this time. Indeed, consideration of inductive proof *challenged* Wittgenstein to give an account of mathematics that *went beyond mere algorithmic decidability*. This had important consequences for his notion of a 'calculus' or 'system' in mathematics (inductive proofs create new conceptual connections and create or join systems of mathematics), his development of the function and role of proof, as well as his realization of the diverse mathematical practices that are actually employed in mathematics (in contrast to limiting mathematics to equations or decision procedures). The importance of these reflections to his *mature* philosophy of mathematics, and even his philosophy generally, should be readily apparent.

We have, in this chapter, examined Wittgenstein's claims about inductive proofs in the intermediate period. We have seen that, for Wittgenstein, a universal generalization is to be understood in terms of an induction. This induction is the general rule for the construction of proofs (a proof-schema) through which one can survey the possibility of the construction of an individual proof in an infinite number of cases. This possibility is *shown* by the proof and is not itself something that can be proved. We have examined the various confusions that arise as a result of either conflating an inductive proof with a proof that essentially involves the transformation of an equation according to strict rules, or ones that arise through accompanying descriptions of the proof in ordinary language. With reference to relevant secondary literature, we have seen the very limited extent to which Wittgenstein's observations constituted a refutation of anything in Skolem's work. In all cases he is either arguing against a specific explanation (given in ordinary language) for what Skolem was doing (with his mathematics) or a possible confusion that could result on the basis of Skolem's 'prose' or his choice of notation. Finally, we examined the important role considerations of inductive proof played in his philosophy generally and his philosophy of mathematics.

6. Set Theory

Another extensive topic that is the focus of Wittgenstein's reflections during the intermediate period, and which directly relates to his reflections on the concept of infinity, is set theory. In what follows, we shall examine Wittgenstein's arguments on this topic. As will become clear, this topic too depends heavily on his conceptions of 'prose' and 'calculus', which will therefore get more extensive explanatory treatment in this chapter. Two points are worth mentioning at this stage. First, comments that go under the heading of 'set theory' can be (at least seemingly) diverse in subject matter and it may not immediately be apparent how they are related. Part of the focus of this exposition will be to show exactly what important insights thematically unify these seemingly disparate subject matters and how these comments have clear applications to the more technical aspects of set theory. Second, Wittgenstein's comments on set theory often involve the use of, and even occasional expansion upon, insights that have been already discussed in previous chapters (in particular, Chapters 2 and 4). In this way, Wittgenstein's comments on this topic importantly serve as an application/example of his extensive grammatical investigation into the nature of mathematics, proof, and infinity[198] in the intermediate period. In addition, comments that appear under the subject heading 'set theory' in later works of the intermediate period are already being composed right at the beginning of the intermediate period. Thus, it is important to realize that, while, given the presentation of this thesis, it will appear that his views on mathematics, infinity, etc. are subsequently applied to set theory, in reality the detailed development of the former were heavily dependent on his reflections on set theory (and related topics) at the time. We will begin with a very brief history of set theory followed by a relatively rudimentary discussion of some of its technical aspects. This will be followed by a brief explanation of Cantor's philosophy and, with this, his 'interpretation' of set theory. This will set the stage for Wittgenstein's criticisms of set theory, which will come to light by means of an examination of the following topics: Wittgenstein's philosophy of mathematics and the calculus/prose distinction, infinite sets and the categorial divide between the finite and infinite, the distinction between extensions with intensions, numbers and the number line, the uses of 'description' in

[198] As should be evident from Chapter 4, I can see no reason to think that Wittgenstein's arguments involving the concept of infinity in *Philosophical Remarks* are 'superficial' at all, as suggested by Shanker (1987, 166).

Wittgenstein's discussion of set theory, and the assessment of the calculus/prose distinction in relation to Gödel's first incompleteness theorem proof. The chapter will conclude with an examination of the descriptivist/revisionist debate within Wittgenstein studies (as this relates to set theory), using the work of Ryan Dawson and Victor Rodych as representative of the two positions, respectively. In the course of the evaluation of these positions, mention will be made of the axiom of choice.

6.1 Set Theory: Its History and Development

Before we enter into further discussion of Wittgenstein's views on set theory, it is important to give a brief synopsis of some details of its historical development and formal aspects.[199] A general interest in the concept of a set was present in the 19th century in various thinkers such as Bolzano, Dedekind, Frege, and Russell. Although some discussion of the infinite had taken place since antiquity, and a general notion of infinity was at least present in the periphery of the mathematical consciousness insofar as the different branches of mathematics all have reference to the idea, the notion that one could make the idea of infinite magnitudes as completed wholes with determinate sizes rigorous to mathematical treatment seemed undermined by important paradoxes (Moore 1991, 111).[200] And the orthodoxy, going back to Aristotle, was skepticism of the idea of the 'actual infinite'; insofar as mathematics makes reference to the infinite, it must be a 'potential infinite'.

It was Bolzano who first argued for the importance of the conception of a set in the understanding of infinity; when we speak of something being infinite we are saying that some set's number of members is infinite. A preoccupation at the turn of the century to make clear the fundamental notions of the discipline led to a greater interest in mathematical foundations. One of the pioneers in this area was Frege, who sought to reduce virtually all mathematical notions to pure logic. He too sought to make rigorous the notion of infinity, not only to make sense of 'infinite numbers' but to be able to

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[199] This and the following section are largely dependent on A.W. Moore's *The Infinite*, Chapter 8.

[200] For example, there is the paradox of the even numbers. By one criterion, there would appear to be fewer even numbers than total natural numbers since the subset of even numbers would appear to be half that of the natural numbers. On the other hand, the answer to the question of how many even numbers and natural numbers there are appears to be in both cases: infinite. And, in these two cases, one can correlate both sets indicating their infinite number. How one decides between these two criteria became of pivotal importance in the development of set theory.

make reference to the natural numbers (themselves infinite in number). And, hoping to do this without use of anything extra-logical, he was led to the concept of a set.[201] Russell's paradox famously undermined Frege's project, which in turn led Russell, still working in the spirit of Frege, to place limits on the concept of a set in order to avoid such problems. However, in order to do this, while retaining Frege's own concept of set generation, Russell needed to appeal to an axiom of infinity, an axiom whose status as purely logical was questionable and thus called into question the viability of the project as a whole (Moore 1991, 114-116).

Of most interest to us is how Cantor developed set theory and, with this, transfinite mathematics as we know it today. Here we will present some of the motivations for, and technical details of, Cantor's work, while delaying until later some of the (sometimes obvious, given what has already been written) Wittgensteinian responses to his work. Cantor, inspired by the possibility of a well-developed theory of sets, was undeterred by the new paradoxes that had arisen. And he was skeptical of the Aristotelian orthodoxy against the actual infinite. This partly arose because of his earlier work on real numbers and continuity, which led him to the view that mathematics must be able to deal with completed infinite sets of numbers, such as the set of natural numbers. As he famously said, 'each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite' (Cantor 1887, 410-411; Hallett's (1984, 25) translation). His argument is that the possibility of an infinite process[202] implies there must be a definable domain that is infinite. Thus, this domain, according to Cantor, must exist as a completed actually existing totality. In constructing set theory, Cantor persevered in the face of the various paradoxes and he considered himself vindicated in his perseverance by the effectiveness, clarity, and consistency of the resulting work. It is to some of the details of this work that we now turn.

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[201] One of the benefits of recourse to the notion of a set is the fact that one can abstract away from its members and thus deal with its purely formal properties (particularly apt for mathematics).

[202] Part of this argument, as it is reconstructed by Moore, involves the fact that the process can be said to exist *now*. This was in response to the Aristotelian orthodoxy that understood the (genuine) infinite in temporal terms and thus as always only a potential infinite (Moore 1991, 40). For Cantor it is the fact the process can be identified as infinite irrespective of it being carried out, and how this relates to the specifiable domain, that gives it its importance in the argument (Moore 1991, 117). And this too can be responded to using Wittgensteinian arguments.

6.2 Set Theory: A Brief Discussion of its Technical Elements

Set theory involves the iterative conception of a set. While sets can be seen as a collection of their members, one can abstract away from members of a set to talk merely of the size of the set. This allows one to deal with the abstract properties of sets, and it is dealing with such abstract properties that makes sets amenable to mathematical treatment. Thus, one can build up a hierarchy of sets using merely sets as members themselves. Beginning with the empty set, one can define ever bigger sets, which can, through techniques Cantor developed, lead one into sets with an infinite number of members. The empty set is the set with no members, and is represented with '∅'. A set with one member can thus be represented by '{∅}', two members by '{∅, {∅}}' and so on. These can be, in turn, identified with the natural numbers themselves: the empty set being identified with 0, the set with one member being identified with 1, and so on.[203]

Moreover, using even more elaborate techniques, Cantor was able to make rigorous a conception of 'set' that had infinite members, and, indeed, had different 'sizes' of infinite members. Cantor chose the correlation method[204] as the defining feature to determine the 'size' of a set (Moore 1990, 118). That is, one determines that two sets are the same 'size' by correlating one-to-one each member of the two sets (when impossible, this shows they are different 'sizes'). The equivalence (in terms of 'size') of two sets is thus defined by means of a one-to-one correspondence; two sets that can be put into one-to-one correspondence are said to the have the same 'cardinality' or 'power'. A finite set's size, then, is the size of the natural number to which the final member of the set is correlated. Within this method the set of natural numbers occupies a central place in Cantor's overall theory, since it is the first infinite set and the starting point for his creation of a hierarchy of 'sizes' of infinite sets (the first 'larger size' of infinite set is established by comparison with this one).

Under the definitions of Cantor's theory, a number of infinite sets are all the same size.[205] For example, the set of even natural numbers is the same size as the set of naturals. Moreover, Cantor also showed that the number of points on any size line

[203] Alternative identifications for the natural numbers have been suggested, such as (still) identifying the empty set with 0 and identifying each integer with the set containing the set identified with the predecessor of the integer. See footnote 212 for details regarding Cantor's reductionism.
[204] As Moore notes, another possible option for determining the size would be the 'subset' criterion: a set that contains every member of another set and members in addition is a bigger set.
[205] Here I use the terminology employed by Moore, as well as many other commentators. This is arguably the misleading 'prose' of set theory, and what Wittgenstein criticizes. Thus, I am not endorsing this way of speaking, and, as we shall see, Wittgenstein has good arguments against it.

segment is the same size as the set of points in all of space (all understood in terms of real numbers), a conclusion he was surprised by (Moore 1990, 118). However, as suggested, not all infinite sets are the same size. To establish this, Cantor created a new proof technique known as ‘diagonalization’.

To illustrate diagonalization, we take the example of the real numbers.[206] If we attempt to list all the real numbers (in the form of infinite sequences), starting with some commonly known ones and pairing each one with a natural number, using the diagonal technique, we can show that, no matter how many numbers we list, we can always define a real number that has not yet appeared on the list. And, if we subsequently add that number to the list, we can see that there will always be another real number that does not appear on the list (since the infinite expansion of the new number will also intersect the diagonal). We can illustrate the diagonalization using the following numbers, respectively: 1/3, $\pi - 3$, $\sqrt{2} - 1$, 1/2. One subtracts the whole numbers from the irrationals in order to be left with only the decimal expansion.

We can define the new real number by taking the decimal place of the corresponding real in the list (i.e., first place of the first number, second place of the second, etc.) and define it to be changed as follows: if the digit is a 3 make it a 4, if anything other than 3 make it a 3. Then it should be evident that we can always define a real number that won’t appear on the list no matter how many numbers are included. So, for example, we can represent this visually, the highlighted diagonal representing the decimal places from which the new real number can always be defined.

1 - 0.3333…
2 - 0.1415…
3 - 0.4142…
4 - 0.5000…
…
…
…

The newly defined irrational will thus be 0.4333… We can be certain this number will not be on the list for it will differ from the n-th number on the list at least in terms of the

[206] Here I use Moore’s exact example (1991, 119-120) because of its simplicity.

n-th digit, for every n. Thus, by the correlation criterion, the set of real numbers is 'bigger than' the naturals since it is impossible to correlate them with the natural numbers. And the results do not end here. One can show, for example, that there are more sets of natural numbers than natural numbers themselves. As a consequence of some of these preliminary results, Cantor further developed transfinite mathematics with his theory of ordinals.

Cantor was not only interested in how big a set was but in how that set could be ordered. In this context, the idea of a 'well-ordering' of a set plays a prominent role. In order for a set to be well-ordered the following condition(s) must apply: the imposition of order singles out one of the members of X as the first (unless X has no members); it singles out a member of X as the second, unless there is only one member; and so on, depending on how many members of the set there are. More generally expressed, for every member of X that has been singled out a well-ordering identifies another member as its successor, if there is one. And still more generally, this says that for any already identified set of members of X, a well-ordering specifies the first member to succeed them all (Moore 1990, 123).

With this idea in mind, it is clear that the set of natural numbers is a well-ordered set. Moreover, the non-standard order of the natural numbers that has the positive whole numbers followed by 0 is a well-ordered set. And the non-standard ordering of the naturals that has the infinite set of even numbers followed by the infinite set of odd numbers is well-ordered. In all of these cases, for any identified member of the set another can be identified as its immediate successor. In contrast, the standard order imposed on the whole numbers (infinite in the positive and negative directions) and the standard order imposed on the non-negative rationals are both not well-ordered. In the first case, no first member is identified; in the second case no second member is identified (Moore 1990, 123-124).

The idea of well-ordering allows one to not only talk about the size of a set, but also its structure. Ordinals (or ordinal numbers) allow one to represent this structure. Indeed, we can use ordinals to specify the exact structure or 'shape' of any well-ordered set. This is achieved through conditions that specify a well-ordering of the ordinals themselves as well as their infinite availability. Thus, the following conditions,[207] which constitute the hierarchy of ordinals, are essential:

(i) one ordinal is the first
(ii) for each ordinal, there is another ordinal which is its immediate successor
(iii) for each set of ordinals (finite or infinite), there is another ordinal which is the first to succeed them all

The first ordinals are the natural numbers themselves. By condition (iii), there is an ordinal which immediately succeeds the natural numbers. This is referred to as ω. It too has a successor which is referred to as $\omega + 1$. These ordinals represent the structure of the usual ordering on the natural numbers and the usual ordering on the natural numbers followed by another ordinal (e.g. another natural number), respectively. This ordinal is followed by $\omega + 2$, $\omega + 3$, and so on. The first ordinal to follow this infinite number of ordinals is $\omega \times 2$, followed by $\omega \times 2 + 1$, $\omega \times 2 + 2$, and so on (Moore 1990, 125-126).

As will be explained further below, an important factor in the development of the transfinite ordinals was the fact they could be ordered under the relation '>'. That is, this relation could be employed between the natural numbers, the natural numbers and the first transfinite ordinal (i.e., $\omega > v$ – where 'v' is the entire sequence of the natural numbers), and among the transfinite ordinals themselves (e.g., $\omega + 2 > \omega + 1$). Being able to use – what he took to be – the same relations and operations between the natural and transfinite numbers meant, for Cantor, establishing the consistency and, with this, the reality of the transfinite numbers – explained further below.[208] This is sufficient as an explanation of some of the technical details of set theory. It is to Cantor's

[207] For a more technical, yet accessible, discussion of Cantor's 'principles of generation' specifically, see Shanker (1987, 171-173).
[208] This point derives from Shanker (1987, 170-171). Shanker emphasizes that there is a categorial distinction between the infinite and finite which Cantor thinks he defines away, but is unable to. However, this claim does not seem to be borne out by the specific citations Shanker gives to Cantor (or Dauben). It is clear that Cantor saw the transfinite as an extension of the finite. But he may not have thought that he had escaped a categorial distinction with his definition of the '>' symbol. At times he suggests that the infinite concept 'splits up' into 'power' and 'number of elements' (Cantor 1883, 78) and he also suggests that the infinite can have different 'characteristics' than the finite (Cantor 1883, 77). And since at least some of these differences can have their 'basis in nature itself' (Cantor 1883, 75), Cantor may think he is *describing* these differences. This suggests he was likely aware of the difference in the meaning of the symbol (in the finite and infinite case), although his other metaphysical ideas may have meant that this was not a concern for establishing the transfinite numbers as a perfectly acceptable extension of the finite ones. This qualification should be kept in mind whenever this point is discussed.

interpretation of set theory, some of which has been implicit in my discussion of its technical elements, that we now turn.

6.3 The Interpretation of Set Theory: Its Purpose, Justification, and Application(s) According to Cantor

As we have seen, Cantor's interest in the infinite began with his work on real numbers and the continuum. Actual infinite sets, as he saw it, were necessary to the areas he worked in. For the purposes of what will subsequently be discussed in this chapter, it is worth elaborating further upon the principal reasons for the invention of set theory and some of the philosophical/metaphysical views that surrounded it. Although detailed analysis of Cantor's work is beyond the scope of this chapter, a short examination of how Cantor interpreted his work (in terms of its purpose, justification and application) in the context of his broader metaphysical views is of great relevance to understanding Wittgenstein's criticisms (as shall be made clear). Wittgenstein was familiar with Cantor's work and even considered devoting an appendix to his theory of infinity in the *Philosophical Investigations* (*RFM* 30). And, even more importantly, it should be evident that many of Wittgenstein's comments on the philosophy of mathematics, and set theory in particular, while not necessarily being a direct reference to Cantor, have clear implications for Cantor's work.

First, it is worth noting that Cantor's mathematical project was inextricably linked with a philosophical one. His creation of set theory, which includes the ability to, as he took it, coherently represent and calculate with infinite quantities, was intimately connected with his metaphysical views, which was explicitly stated (and clearly shown by the presentation of his work – he wrote on philosophy to justify his project (Cantor 1883, 74-79)). By formulating his calculus Cantor sought not just to do mathematics, but to bring legitimacy to the idea of the actual infinite (which, as we have seen, was largely seen as illegitimate because of the criticisms of both philosophers and mathematicians) (Dauben 1979, 120-124).

For Cantor, mathematics itself requires the formalization of a calculus of infinite sets. This is because, as already mentioned, every potential infinite is dependent on the existence of the actual infinite. Dauben elaborates this argument slightly. Assuming the 'objective existence' of the integers, we are compelled to also accept the transfinite numbers. For even 'finitists' would readily admit that for every arbitrarily large number

N, there is another number n such that n > N. And this, according to Cantor, presupposes the existence of an actual infinite. As Dauben explains, Cantor used the metaphor of a path: the potential infinite is like the 'wandering limit', which always 'leads to' the actual infinite, the idea being that the former can't be thought/understood without the latter. These insights apply to domains, which are especially important for areas like algebra, number theory, and analysis (Dauben 1979, 127).

However, as already suggested, Cantor's reflections were not limited to mathematical considerations. His metaphysical views, which relate to the nature of mathematics, the mind/ideas, and the world, as well as how all three relate to each other, determine the justification and applications of mathematics (and set theory in particular).

For Cantor, the transfinite numbers followed immediately from the abstraction from infinite sets.[209] If we know infinite sets exist, we are justified in abstracting from, and representing, the order of these sets. And, as we have seen, this culminates in the ordinals. Cantor followed this up with an argument for mathematicians to justify the adoption of these numbers: the way in which these numbers are defined is importantly similar to the way in which the irrationals were introduced (both depend on infinite sets of numbers). Thus, Cantor argues, if we are justified in using irrationals, we are justified in using the ordinal numbers (Dauben 1979, 128-129).

This last argument was meant to be persuasive, but was not necessary for Cantor to establish his point. In this respect only one thing was necessary: the consistency of the mathematical theory to which the respective numbers belonged (Dauben 1979, 128-129). Hallett argues this sense of 'consistency' had less to do with non-contradiction and more to do with the well-defined and integrated nature of the concepts under consideration (see Cantor 1883, 76). Thus, Hallett prefers using the term 'coherence' (Hallett 1984, 19). While Cantor did sometimes concern himself with non-contradiction, for example in his response to the type of arguments that served as historical criticisms of the actual infinite (Cantor 1883, 75-77; Hallett 1984, 19), he was more often concerned with the idea that a newly introduced concept properly involved

[209] As will become apparent, Cantor gives a variety of different arguments to justify the transfinite, all of which ultimately depend on some metaphysical ideas. I have tried to reconstruct his reasoning, as best as possible, using the detailed scholarship of Dauben and Hallett. However, Shanker (1987, 170) is essentially right, albeit perhaps a little too charitable, when he notes that Cantor 'in his zeal to leave no stone unturned' gave a number of justifications with 'little regard for mutual compatibility'. Space limitations prohibit noting all of their incompatibilities.

‘individuation’, ‘specification’, and ‘ordination’ (Hallett 1984, 22). While it strikes me as obvious that a concept or theory must not contradict itself, and Cantor saw it in this way, it is true, as Hallett suggests, that Cantor did not have an idea of consistency as a formal program (such as Hilbert’s). Thus, ‘coherence’, as Cantor called it, had more to do with the aforementioned properties rather than (mere) non-contradiction. It was on the basis of this notion of coherence that Cantor rejected the infinitesimals and thought the transfinite numbers legitimate objects with the full reality of any other readily accepted number system (e.g. the whole numbers) (Dauben 1979, 129). Shanker (1987, 170-171), it should be noted, has emphasized that of the utmost importance for Cantor establishing the ‘reality’ of the transfinite numbers was whether they could be seen as an ‘extension’ of the natural numbers. Shanker emphasizes that preserving the relationship indicated by ‘>’ between the natural numbers and the transfinite numbers was essential/central to this (see footnote 208 for an important qualification of this point).

And this notion of coherence, which was of ultimate importance for Cantor, had a corresponding metaphysical justification. Cantor argues that there are two types of reality: ‘intrasubjective’, or ‘immanent’ reality, and ‘transsubjective’, or ‘transient’ reality (Cantor 1883, 79). Immanent reality is what comes with being ‘well-defined in the mind, distinct, and different from all other components of thought’ (Dauben 1979, 132). With this, immanent reality modifies the ‘substance of our mind’ (Cantor 1883, 79) in a ‘connectional or relational’ way (Dauben 1979, 132). Transient reality, in contrast, has to do with the concrete ways in which numbers manifest themselves in the world (Cantor, 1883, 79; Dauben 1979, 132). Moreover, Cantor argued that one of the principal tasks of metaphysics was to understand the connection between immanent and transient reality. Cantor argued that everything with immanent reality had some degree of transient reality (they are ‘found together’) (Cantor 1883, 79). In this way, mathematics could concern itself with only immanent reality and disregard whether this had confirmation in the world. As Hallett explains, at least two justifications were given for this. Initially Cantor argues for a metaphysical idea of unity, which ensures everything (including these two modes of reality) is importantly connected (Cantor 1883, 79). This already deep and mysterious notion eventually evolved into a more elaborate, yet hardly philosophically satisfying, appeal to God (Hallett 1984, 20-21). The appeal in this context is somewhat predictable: mathematical concepts that are shown to be coherent must already, for metaphysical reasons, exist in the mind of God.

Coherence guarantees that these 'objects' have reality in the mind of God (Hallett 1984, 21); and given their reality, the rational process of ensuring their coherence ends up being a type of discovery. Thus, Cantor tries to combine idealism with realism and, with this, deviates from formalism.[210] In addition, this idea of coherence is linked with the concept of possibility: what is coherent and has being in the immanent sense already explained has the possibility to be actualized in the world. At first, Cantor suggested that what is coherent must exist in the mind of God since he subscribed to the idea of maximal possibility. What is coherent, given God's perfection, must exist somewhere (Hallett 1984, 21). And then Cantor is even, at least at (other) times, inclined to take this principle further: given God's perfection it makes sense to think that God must also *create* all that is possible (Hallett 1984, 23). In this way, Cantor assumed that his notion of the transfinite also had realization in the physical world in the form of 'aether' and 'monads' (Hallett 1984, 23). Cantor's appeal to God is not surprising given the tradition he was writing in. Numerous thinkers were opposed to the idea of the increasable infinite simply on the ground that the only true actual infinite was God, which fundamentally is unknowable. Cantor gave credence to the idea of an increasable actual infinite that man could 'take part in' (the transfinite), but his theory also reserved a place for God as the 'absolute' we can never fully understand (or 'rationally subjugate') (Hallett 1984, 13-14).

We needn't get any further bogged down in the details of Cantor's philosophy. Given the focus of this chapter, and especially considering some of the exegetical problems with giving a comprehensive account of Cantor's theory, raising some of the ambiguities and potential problems with Cantor's theory should be sufficient. As Hallett notes repeatedly (e.g., 1984, 24), the appeal to God is hardly satisfying as an *explanation* of anything. In addition, there is an ambiguity in Cantor's two views of reality. As Cantor explains it, transient reality is manifested in corporeal or intellectual nature. It is unclear what *exactly* is meant by 'intellectual nature', but given the explanation of the two types of reality, it makes sense for Hallett to think that transient reality also applies to what is in the mind of God.[211] After all, it is what is in the mind of

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[210] Cantor is even explicit about the simultaneous realist and idealist elements at the 'foundation' of his investigations (1883, 79). Dauben characterizes Cantor's position as one of formalism, but it is obvious that his position at most contains formalist elements (cf. Hallett 1984, 18-19).

[211] Hallett says: 'It is now clear why Cantor considered mathematics as so free. It does concern itself with objective truth and an independent (Platonic) realm of existents in so far as its objects of study are transiently real. But it need not attempt to investigate this transient reality directly, or even worry about the precise transient 'significance' of a concept. All that mathematics need worry itself with is

God that must contain some objective reality (that's seemingly the point of introducing it). It would then seem appropriate to have an explanation of what is common between what exists in the mind of God and what is a manifestation of this in the world, or to properly differentiate another sense of 'reality' for what exists in the mind of God. Problems related to this abound. For example, what exists in the mind of God: concepts or objects (of some sort)? If the former, what does existing in the mind of God add to what already exists? And why is it that Cantor himself, at least at times, seems to treat what exists in the mind of God as something more than a concept? Given some of Cantor's other views, it would seem to have to be the latter. Concepts are meant to pick out or generate mathematical objects and the whole point of introducing God as an explanation is to ensure the existence of infinite totalities (which themselves are conceived of as objects). But then it is neither clear why something instantiating/corresponding to this 'object' would need to exist in the world, nor how these 'objects' would relate to objects in the world. As suggested above, regardless of which explanation Cantor gives, it would appear further distinctions must be made to make sense of Cantor's philosophy. Moreover, given Cantor's reductionist views,[212] various abstract objects (e.g. sets) would need to exist somewhere, and since they are abstract objects, and given Cantor's self-avowed realist convictions, it would seem that sets must exist in the mind of God. Although Cantor always has the option of saying

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'intrasubjective' reality, and once this is established it is guaranteed that the concepts are also transiently real. There may be all kinds of ways in which transient reality is manifested; in particular concepts might be represented or instantiated in the physical world' (1984, 18). The first part of this clearly suggests that the ideas/concepts themselves are transiently real because they exist in the mind of God, while the second part suggests transient reality exists in things that are instantiated in the world. At times Cantor himself seems committed to the first interpretation, even though he more clearly seems to support the second interpretation with several of his explicit comments.

[212] It seems that Cantor subscribed to reductionism of some sort (relating to the different areas he worked on) throughout his work, the precise details of which are open to interpretation and, seemingly, change throughout his work. Sets play an important role in this reductionism, although the precise role is difficult to determine. In the area of foundations of analysis (where Cantor's work began), Cantor's new definition of real number involved a 'reduction to domains (collections)' (Hallett 1984, 31); it was this reduction that led to the idea that sets are collections but simultaneously a single thing – thus this idea is at the root of set theory itself. And Cantor's early work itself ultimately culminated in suggesting sets 'represent' numbers (which at least anticipated a form of reductionism – Hallett 1984, 49-50), before he moved to his more explicit reductionist position with his 'abstractionist' account of number in his later work (Hallett 1984, 120-121), However, as Hallett notes (1984, 120), although his 'abstractionist' position naturally leads one to want to say that sets, or 'number-classes', are themselves numbers, other parts of Cantor's thinking seem to suggest otherwise. Given Cantor's leaning towards some version of reductionism, as well as how he is historically situated, I take it that some version of foundationalism (in addition to his work on the foundations of analysis) was also at least a motive of Cantor's work. It was obviously a major consideration for other foundationalists who employed set theory. Due to space considerations, I have not dealt with foundationalism or Wittgenstein's extensive refutation of this position. See Schroeder (2021, Ch. 3) for an excellent discussion of Wittgenstein's view.

that every ‘object’ in the mind of God is a concept (although not without compromising our ordinary concepts of ‘object’ and ‘concept’), one would think an explanation of why concepts are more robust in the mind of God would be needed. Once again, the appeal to God does not serve as a very convincing explanation. Finally, using ‘concept’ as roughly equivalent to ‘object’, as is quite possibly Cantor’s position, would seem to, without explanation, run contrary to at least one influential philosophical position according to which objects are what are denoted by concepts (indeed, Cantor’s position may at other times depend on the idea that objects ‘fall under’ concepts – for example, the relation between our concepts of the numbers and the numbers themselves (in the mind of God)).

This is a sufficient to illustrate some of the ambiguities and potential (internal) problems with Cantor’s position. Wittgenstein’s comments on set theory, to which we now turn, constitute a more thorough-going and fundamental assessment of Cantor’s interpretation of his calculus.

6.4 Stage-Setting: Wittgenstein’s Philosophy of Mathematics and the Calculus/Prose Distinction Revisited

A unifying focus of Wittgenstein’s comments on set theory involves undermining the descriptivist account of mathematics. It is tempting, as many philosophers have done, to think of mathematics as descriptive, whether this be understood as a description of empirical objects, ideas in the mind, or ideal objects. As we have already seen in Chapter 2, Wittgenstein extensively argues against this conception in the intermediate period. Fundamentally, mathematics involves invention rather than discovery. We *use* intensions (i.e., rules) and extensions (i.e., lists, signs, strokes, etc.) to invent mathematics bit-by-bit (Rodych 2000, 284). Nonetheless, for reasons that will become apparent, set theory, more than other areas of mathematics, inclines one to the descriptivist position. Indeed, in what follows, we shall examine Wittgenstein’s use of ‘description’ in his comments on set theory in more detail to shed light on his views in general. Moreover, as Wittgenstein suggests, the pernicious idioms representative of, and inclining one in, these directions of thought, which are commonly found in set theory, are also readily found in other areas of mathematics. These temptations are already being addressed in Wittgenstein’s earliest comments upon his return to philosophy and will be addressed in more detail later in this chapter. Whenever possible, I will illustrate in what way confusions in set theory relate to, or are encouraged by, descriptivist tendencies in the philosophy of mathematics.

A brief review and discussion of the calculus/prose distinction will be made at this point in the chapter. It is central to understanding Wittgenstein's criticisms of set theory, as well as the 'descriptivist'/'revisionist' debate in Wittgenstein studies (discussed below). Thus, we will make reference to this distinction often in the sections that follow, and the distinction will be of the utmost importance in determining exactly how far Wittgenstein's criticisms of set theory extend. Wittgenstein makes much of the calculus/prose distinction in relation to set theory. Important quotations regarding the calculus/prose distinction appear in the early intermediate period. It is explained in a general fashion (with applications to 'consistency' and 'independence') as follows:

> It is a strange mistake of some mathematicians to believe that something *inside* mathematics might be dropped because of a critique of the foundations… [W]hat is caused to disappear by such a critique are names and allusions that occur in the calculus, hence what I wish to call *prose*. (*WVC* 149)

Wittgenstein begins elaborating the distinction in relation to set theory around June 1930. Further explanations of the distinction are given in the context of inductive proof and set theory, respectively:

> An explanation in word-language of the proof (of what it proves) only translates the proof into another form of expression: because of this we can drop the explanation altogether. And if we do so, the mathematical relationships become much clearer, no longer obscured by the equivocal expressions of word-language. (*PG* 422)

> In set theory what is calculus must be separated off from what attempts to be (and of course cannot be) *theory*. The rules of the game have to be separated off from inessential statements about the chessman. (*PG* 468)

Wittgenstein uses the analogy of chess to make his point. The rules of the game are the use of the signs within the calculus. However, it is also possible to dwell upon inessential elements of the calculus (e.g. in chess, what the pieces are made of, or the aesthetic features used to distinguish the pieces). For Wittgenstein the 'calculus' is everything connected with the actual use of the mathematical symbolism in all of the various forms in which it makes up the discipline. In contrast, the prose is what is given in the ordinary language descriptions or explanation of what is being done in the proofs. Wittgenstein is emphatic: to fully understand what a proof proves one looks carefully at the proof itself. This avoids conceptual confusion that arises because of the ordinary language descriptions. In some cases, the prose is innocuous, but in other cases not. And regardless of how pernicious the turn of phrase, the best way to gain clarity is to always look at the proof (and attempt conceptual clarification regarding any of the accompanying prose). Thus, for Wittgenstein, the prose is what has the possibility to mislead and what must always be clarified.[213] Even what could appear as relatively insignificant choices can lead to various confusions; for example, even the use of 'theory' itself, when one talks about set theory, can lead one to think that one is somehow trying to discover or give the best account of infinite sets (in the sense in which this could 'correspond to' a reality) or the infinite more generally.[214]

6.5 Wittgenstein on Infinite Sets and the Categorial Divide Between the Infinite and Finite

Based on what has already been discussed in Chapter 4, there are obvious objections to set theory. Here I shall limit myself to outlining Wittgenstein's objections to its conception of the infinite set as well as his positive characterization of the concept.

Wittgenstein denies the meaningfulness of the notion of the actual infinite in all of its forms.[215] More precisely, he denies the intelligibility of a certain interpretation of the infinite: the actual or extensional infinite. As we have seen, the idea of the natural numbers, or infinite sets generally, existing as a totality is a meaningless one. The idea of a set that contains *all* natural numbers is not intelligible (and similarly with any other infinite set); this is not an empirical claim to the effect that we have not yet discovered one, but rather an insight concerning the bounds of sense concerning the concept of the infinite. Talking of infinite sets *existing as a totality* has no meaning. Insofar as it is

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[213] It should also be noted that the prose needn't always be explicit. Rather, it is the *anticipated accompaniment* of the proof that is sometimes Wittgenstein's concern (in order to *avoid* conceptual confusion).

[214] It is noteworthy that the term 'hypothesis' is also used in the context of set theory (i.e., the 'continuum hypothesis'), even though this is one of the few uses of the term (another is Riemann's hypothesis – *BT* 619). One would have expected, in line with the usual terminology employed by mathematicians, that the term 'conjecture' would have been employed. This is another possible cause, or result, of the descriptivist tendency when it comes to set theory.

[215] As evident from Section 4.5, this is the position Wittgenstein eventually adopts in the intermediate period, and arguably continues to hold in his later work. As we have seen, the movement away from the consideration of (any) matters of fact in preference for exclusively dealing with matters of sense was an important development in the intermediate period, and one that importantly relates to his consideration of the concept of infinity.

legitimate to refer to infinite sets in mathematics at all, anything infinite is given by a rule which allows for the unlimited construction of what will make up the set. This is the intensional characterization that Wittgenstein thinks is legitimate.

Set theory, at least in terms of its prose interpretation of its calculus, assumes the intelligibility of completed infinite sets. It assumes the meaningfulness of talking of infinite sets according to the extensional interpretation, the comparison of the size of these sets, and the ascription of specific sizes to these sets. As we shall see in more detail when we start to list specific confusions, the latter confusions, conceived in a general way, always involve understanding the infinite on the model of the finite.

The above characterization serves as an introduction to Wittgenstein's more general point concerning the infinite and finite: there is a categorial divide between the two. As we have seen, this fundamentally involves the fact that the infinite can't be understood in terms of the finite and that any explanation of the infinite fundamentally involves reference to the infinite itself.[216] One is not understandable or definable in terms of the other. This can be further understood by the fact that what can be meaningfully said about the 'infinite' or 'finite' are not necessarily meaningfully applicable to the other. 'Infinite' does not indicate a quantity. One imagines it on the model of the finite, and then is tempted to think that it identifies a gigantic quantity (*PR* 157) or that a gigantic quantity is closer to infinity than a smaller one. And, while it is true that both 'infinite' and 'finite' can be meaningfully given in answer to the question 'how many?', the answers indicate something very different in both cases. 'Infinite' does not indicate an enormous, virtually unimaginable amount (*PR* 306), but instead, at least when it comes to mathematics, the possibility of an unending construction in accordance with a rule. This unending condition is essential to the concept and thus it is logically impossible to think that all of the numbers could be listed. This categorial difference between the infinite and finite is exemplified in the further confusions that will be outlined.

[216] Similarly, although not explicitly mentioned by Wittgenstein, the finite clearly can't be understood just in terms of the infinite, for, although one could argue that the finite is simply a part of the infinite, clearly an understanding of the finite is even required to start listing a given infinite set to begin with. Any enumeration/list for a set one actually gives will always be finite. More precisely: one shows how a rule yields an unending extension by listing parts of it with the addendum 'and so on'.

6.6 Confusing Extensions with Intensions

Exemplifying the categorial divide between the finite and infinite are the different ways in which these are represented. Finite sets can be given with a list, but infinite sets can only be given with an intension. The prose associated with set theory assumes that the intension, or rule, is merely an abbreviation or placeholder for the extension. In actuality, infinite and finite sets are radically different things, and this difference must be captured in an accurate symbolism. Wittgenstein says:

> We cannot imagine the same class finite at one time and infinite at another. The truth of the matter is that the word ‘class’ means completely different things in the two cases. It is not one and the same concept at all that is qualified by the addition of ‘finite’ or ‘infinite’. (*WVC* 228)
>
> A correct symbolism has to reproduce an infinite class in a completely different way from a finite one. Finiteness and infinity of class must be obvious from its syntax. In a correct language there must not even be a temptation of raising the question whether a class is finite or infinite. (*WVC* 228)
>
> ‘Infinite class’ and ‘finite class’ are different logical categories; what can significantly be asserted of the one category can’t be significantly asserted of the other. (*PG* 464-465)

One may be inclined to think that Wittgenstein denies the intelligibility of infinite sets altogether. I think that, with an understanding of his philosophy generally, it is evident that he is simply drawing attention to the fact that an infinite set, insofar as it can be rightly so described, is something completely different from a finite one. The term ‘set’ or ‘class’, when prefaced by the different adjectives,[217] can seem as if they were the same thing; thus, it is tempting to model the infinite set on the finite one. But an ‘infinite set’ is not given by an enumeration, but rather with an intension. An ‘infinite set’ is not a completed totality and instead is made possible by (our use of) a rule to continuously generate the infinite series without end. Indeed, it is impossible that an intension for an infinite class could be replaced with an enumeration (as one may think when it comes to, for example, descriptions made about reality – e.g. the number of objects falling under a concept). As Wittgenstein says, ‘there isn’t a dualism: the law and the infinite series obeying it; that is to say, there isn’t something in logic like description and reality’ (*PR* 221). As he continually emphasizes, it is the law that *yields*,

[217] Even though I refer to both of these words as ‘adjectives’, as this is used in ordinary grammar, it is clear that Wittgenstein, as already argued in Chapter 4, does not think ‘infinite’ *functions* as an adjective, but rather an adverb.

through our use of it, the extension. This extension can only ever be finite, albeit extended as long as we like. Set theory can lead one to assume that it makes sense to give an intension instead of an enumeration, and that this intension can more or less ‘describe’ the infinite series (as if the infinite series existed prior to the intension and then we needed a way to refer to it). But, as we have seen in Chapter 4, the intension allows us to generate extensions and is not something that exists prior to this generation.

It is apt to address here a problem with Rodych’s skepticism regarding Wittgenstein’s views about extensionalism. Given how important Rodych’s contributions are for understanding Wittgenstein’s philosophy of mathematics, it is surprising to read:

> the greatest disagreement between Wittgenstein and the received view, namely the existence of infinite mathematical extensions, is essentially *irresolvable*. Whether the classical interpretation of TST [Transfinite Set Theory] uses a platonist conception of actual (or existent), infinite mathematical extensions or a modalist conception of possible (or potential), infinite mathematical extensions, in the absence of a mental or perceptual means of checking for the *existence* of a nonphysical, mathematical realm, there is simply no way to decide (and no point in arguing for or against) the existence of actual (and perhaps also possible) infinite mathematical extensions. (Rodych 2000, 308)

This, especially with Rodych’s use and italicization of ‘existence’, importantly relates to a point already mentioned.[218] It is noteworthy that Rodych thinks that Wittgenstein’s most decisive argument in this context relates to the fact there is ‘nothing infinite in the [mathematical] symbolism’ (Rodych 2000, 308). This he thinks is beyond dispute. Where Rodych remains skeptical is with regard to *any* existing infinite totality. However, this does not serve as a convincing criticism of Wittgenstein’s arguments.

[218] Already in this chapter, there was occasion to recall that Wittgenstein was not interested in making statements about what exists or not, but rather what can be sensibly said about a given topic. While he could occasionally make errors about what was a matter of fact or not, his position, which was consolidated by the middle intermediate period, was only to deal with matters of sense and not matters of fact. Thus, Wittgenstein is interested in distinguishing between what can be said about an infinite set in contrast to a finite one. He does not ever seem to deny a given infinite set exists, but instead qualifies what can be meaningfully said about them (e.g. whether words like ‘totality’, ‘extension’ etc. apply to them, as well as making some further distinctions when it comes to the set of real numbers – which relates to the number line). It is therefore important to note that, at least in terms of his more developed position in the intermediate period (continuing on into his later work), it makes no sense to talk about the infinite on the extensional model and thus it is misleading to say, for example, that there is ‘no such thing as an infinite mathematical set *in extension*’ (Rodych 2000, 286; Rodych’s italics) and to base this on what Wittgenstein can ‘tell’ or ‘see’. Rodych’s choice of wording makes it seem as if Wittgenstein is making an empirical claim, which is further encouraged by his explicit use of ‘existence’ later in the paper (Rodych 2000, 308).

With what has been discussed in this chapter (an analysis of Wittgenstein's uses of 'description'), together with Chapters 3 and 4, it should be evident that Wittgenstein's arguments equally apply to any conception of the extensional (or actual) infinite (not just ones that are part of a mathematical symbolism). As argued extensively in Chapter 4, on the basis of the verification principle which is presented in Chapter 3, after imagining numerous scenarios, Wittgenstein concludes that there is no experience that would ever justify one in applying the term 'infinite'. Not being a possible object of experience, and with the meaning of a statement being its method of verification, he concludes that the idea of an infinite totality existing in the world is meaningless. Moreover, as we have seen, he extensively argues against the idea of the extensional conception in mathematics also; an 'infinite set' is not an existing totality, but rather the possibility of always constructing further members of a set in accordance with a rule (*PR* 313-314). In this case 'infinite' means 'unending'. In the various passages where Wittgenstein argues for this (at least some of which Rodych even refers to – e.g. *PR* 164), there is no reason to think that Wittgenstein is only referring to the 'symbolism', as opposed to numbers themselves.[219] Thus, the extensional model of the infinite in both mathematics *and* the world is a logical impossibility. Seemingly, Rodych's claim is intended to be empirical, for surely for it to be intelligible (involving the 'mathematical realm') it would have to be *empirically possible* that this realm could be perceived. This is further supported by his use, and italicization, of 'existence', which suggests that it could at least be discovered to exist. If the possibility of perception of this realm is ruled out by definition by Rodych, then the idea of verification is impossible and the idea of this realm unintelligible. This would make Rodych's claim a metaphysical one in precisely the sense Wittgenstein attacks throughout his later philosophy. Of course, Wittgenstein also extensively argues against the idea that mathematics involves description at all (*a fortiori* it is not a description of another mathematical realm). If mathematics does not involve description, it is not clear what role this other 'mathematical realm' is supposed to play. And Wittgenstein's

[219] Indeed, Wittgenstein uses 'numbers' in various passages (e.g. *PR* 162, 164). 'Symbolism' seems to be used to refer to the totality of symbols used in at least a (basic) part of our mathematical practices (e.g. numbers and certain function symbols, such as addition). If 'symbols' is conceived very broadly (to also mean spoken words), it is apparent there could be no numbers without symbols and that it is only the possibility of endless construction of symbols in accordance with rules that gives rise to the intensional infinite. And Wittgenstein observes that this can be expressed with a 'single symbol' (*PR* 162).

comprehensive discussion of the concept of meaning proves that the meaning of a numeral isn't an object (and therefore is not one in another realm).[220]

Wittgenstein's arguments strip the extensional infinite of the logical function it plays in the Platonist explanation (and, indeed, show it to be meaningless). Contrary to what Rodych suggests, whether understood as an empirical claim or as a logical one, the Platonist conception of the actual infinite remains unintelligible. And it is up to Rodych to explain in what other way we are to make sense of this Platonist conception, as well as why some other 'nonphysical mathematical realm' should be posited at all (whatever is precisely meant by this). Ironically, this actually bolsters the position Rodych is arguing for, since he is (ultimately) attempting to defend the plausibility and coherence of Wittgenstein's position on set theory. I consider it to be even more defensible than Rodych does.

6.7 Related Problems: Wittgenstein on Numbers and the Number Line

With these preliminary clarifications in mind, it is possible to make better sense of some of Wittgenstein's other comments. It is evident that Wittgenstein was contemplating topics that he took to be related to set theory immediately upon his return to philosophy. This supports the idea, already mentioned, that Wittgenstein's reflections on set theory influenced the development of his views when it came to other areas of his thought (e.g. infinity). A comment that appears in *The Big Typescript* and the *Philosophical Grammar* under the title 'Set Theory' is the first comment he makes in the manuscripts upon his return to doing philosophy in 1929.

> Is a space thinkable that contains all rational points, but not the irrational ones?
> That only means: don't the rational numbers set a precedent for the irrational numbers? (MS 105, 1; translated in *PG* 460)

The implied answer, as is made explicit in later comments by Wittgenstein (MS 111, 29; translated in *PG* 460) is 'no'. Wittgenstein, where he expands upon this, uses the example of chess and draughts; nothing about the rules of chess is contained, or presupposed, in the rules of draughts. Arguably, there is more of a relationship between

[220] Going further, one could say that by definition a number is not perceived, which further undermines Rodych's position.

rational and irrational points (than the games), since, as Wittgenstein suggests elsewhere, the latter are defined in (some) relation to the former; [221] of course, this does not mean that the rules for rational points already contain the irrational ones. In contrast, the two games are, in terms of essentials, entirely independent of each other.

Just before the above quotation in MS 111, Wittgenstein says:

> Is a space thinkable that contains all rational points, but not the irrational ones? Would this structure be too coarse for our space, since it would mean that we could only reach the irrational points approximately? Would it mean that our net was not fine enough? No. What we would lack would be the laws, not the extensions. (MS 111, 28-29)

It may not be immediately clear how these passages are related to the topic of set theory. The problem, as Rodych notes, *originates* with the attempt to 'describe continuity'. In thinking of continuous motion from A to B, we are tempted to think that an object, in travelling this distance, must travel not only through the (mere) density of the rational numbers, but over points not marked by the rational points (Rodych 2000, 291). As briefly mentioned below, a similar temptation develops with respect to the irrationals (and seemingly so on, as new number systems are constructed). Thus, various mathematicians, such as Cantor, thought it was useful to 'describe' this continuity (as Wittgenstein notes: *PR* 208). The problems with this conception are explained below.

We can illustrate Wittgenstein's point in the above quotation using a line itself (something in/importantly related to space – as it is ordinarily conceived). The above passage is related to set theory because of the extensionalism the image evokes, as well as the idea of talking of sets of numbers as completed totalities. There is no such thing as 'all numbers' of any infinite set; and any particular set of numbers does not exist somewhere prior to our actually giving signs a certain meaning by employing meaning-endowing rules for both the signs and what constitutes the infinite possibility of their

[221] Wittgenstein suggests that irrational numbers are defined against rational numbers (*PG* 460); the laws for the two kinds of numbers are different (and the irrational number can be rightly considered a different kind of number because its corresponding law is different). But the former's law importantly relies on a contrast with the law for rational numbers (irrational numbers can't be expressed as a fraction of two integers). Irrationals involve a new rule. Wittgenstein may also be making reference to the fact that irrationals, in contrast to rationals, in their decimal expansion form are non-terminating and non-periodic (although this isn't a definition of them). But, of course, there is also the fact that real numbers can be, for example, defined in reference to 'sequences of rationals or a cut' (Hallett 1984, 31), a point Wittgenstein also discusses (and thus is obviously aware of) (*PG* 460-461). This last way would clearly actually make use of rationals in the definition, as opposed to being a new rule that is in no way dependent on the rationals.

combination; and in this way there is no ‘gap’ in this ‘space’. Irrational numbers are introduced in relation to the rational numbers. Our invention of them involves giving a new rule that is essentially dependent on the rules of the other system. Thus, we do not lack extensions, since any number system does not exist anywhere prior to the rules being laid down that give rise to the number system. Prior to the invention of any given number system there is nothing that could even count as the extension of the number system. The possibility of setting out to list (but not completing) an infinite set of numbers only arises with the development of the (infinite) rules for that number system.

It is natural to think of the number line as an adequate visual representation of the way the various number systems are related. This is, after all, one of the ways the number systems (and their relationship to each other) are typically explained when they are taught. But, at best, Wittgenstein wants to emphasize, this should be understood as a pedagogical aid and the image the aid evokes should not be taken too literally. In fact, constructions like a line in space are precisely that: *constructions*. And in this way they share an important similarity with the number systems themselves, and can, indeed, be logically dependent upon them.[222] The existence of actual physical objects or phenomena (e.g. sticks, logs, streaks of light etc.) do not give any primacy to a line (over numbers) *as understood by the mathematician*. The following further clarifies Wittgenstein’s position:

> Mathematics is ridden through and through with the pernicious idioms of set theory. *One* example of this is the way people speak of a line as composed of points. A line is a law and isn't composed of anything at all. A line as a coloured length in visual space can be composed of shorter coloured lengths (but, of course, not of points). And then we are surprised to find, e.g., that ‘between the everywhere dense rational points’ there is still room for the irrationals! What does a construction like that for √2 show? Does it show how there is yet room for this point in between all the rational points? It merely shows that the point *yielded* by the construction is *not rational*. (*PR* 211)

A line, as an object of study of the mathematician, is not composed of anything. A line, understood this way, is a law. Even the visual representation of a line, Wittgenstein argues, is not composed of points. Seemingly, Wittgenstein’s idea is that points are without length (thus a line still isn’t *composed* of them), and, even if they were taken to be the smallest possible line segment, there would be no way to determine this with

[222] Definitions of a line can require the use of numbers as well as various mathematical operations. This is similarly the case for *constructing* various points on a line or curve (e.g. the highest point – an example Wittgenstein uses: *PG* 462-463).

precision based (merely) on perception and no way to do so mathematically without already presupposing our various number systems.

There are important parallels between lines and the number system. Both are constructions given by laws; in this way, representing points on the number line presupposes the various number systems that, under the extensionalist conception, must be already contained in the number line itself. But, far from points already being contained in a line (in any logical 'space') and then being given numbers to represent them, listing points on the line is only possible because of the constructed numbers systems. That is, the possibility of listing points, in some cases the unlimited possibility of doing so, only arises with the already developed number systems. As also already argued, some number systems themselves are based logically on other number systems. There is no such thing as the irrationals without the rationals, and no such thing as the reals without both. This makes it fairly explicit in what ways points must be *constructed*. Moreover, there is no sense in which a number line could comprise an infinite collection of points (even if points were conceived as tiny line segments). A line, just like an infinite set of numbers, isn't an actual infinity, since there is no such thing. The possibility of continuous division of a line is a possibility (made possible by mathematical symbolism), and not something that exists already in the line. Indeed, various ways of continually dividing a line (e.g. using the rational number system) would only be possible given the rational number system (and couldn't even be done for very long when actually undertaken using a physical line).

Finally, Wittgenstein emphasizes that one of the roots of the mistakes in set theory is to think of irrational numbers as necessarily infinite extensions. It is this conception, Wittgenstein claims, that gives rise to the idea of the actual infinite, which is one of the principal reasons for the development of set theory. This is further encouraged by the idea that 'irrational number' is a concept without any strict limit. Viewing the concept of an irrational number as necessarily an extension gives rise to the idea of lawless irrationals and pseudo-irrationals 'filling in the gaps' left by irrationals. This terminology is Rodych's (1999), but the distinctions between the two are reflections of Wittgenstein's examples. Wittgenstein outlines two examples that go under Rodych's label of 'lawless irrational' (Rodych 1999, 283-284). As Rodych characterizes it, the first conception is that of a '*non*-rule-governed, non-periodic, infinite expansion in some base' (Rodych 1999, 283). Indeed, Wittgenstein wonders whether we can imagine an irregular infinite decimal expansion that would be 'brushed under the carpet' if we are only allowed ordinary, rule governed, irrationals (*PR* 224). In response, most generally, Wittgenstein notes that only what is governed by a law can 'reach to infinity', which deprives the question of sense (*PR* 224). In addition, connected with this, Wittgenstein notes that there would be no way to notice the absence of the non-rule governed irrational (*PR* 223-224). For there would be no point at which the expansion of the non-rule governed irrational would leave a finite expansion behind (i.e., a rational number) and one can't compare a rule with no rule (*PR* 224). Neither can one place dots after a part of the lawless irrational (with the idea that the dots would stand in for the expansion) (*PR* 224), for these can only be employed when we have a way of continuing to construct the expansion (as far as we wish). 'Dots of laziness' thus require a law (*PR* 224). The second type of lawless irrational is one generated either by a free-choice sequence or by some non-mathematical means. In this context (*PG* 483), Wittgenstein considers what is the difference between an infinitely complicated law and no law (implying there is none; elsewhere this is made explicit: *PR* 148). Wittgenstein uses the example of dicing: every place of a decimal expansion is determined by a roll of the die (*PG* 483). However, in this case 'no final result ever comes out' (*PG* 483). Wittgenstein's idea seems to be that no expansion is thereby *completed*. After every throw of the die, the 'point' (which would correspond to the completed expansion of the lawless irrational) is still infinitely indeterminate (*BT* 759). And since this expansion is essential to individuating this lawless irrational *as a lawless irrational*, it becomes impossible once again, in the absence of a law, to actually give the lawless irrational. Moreover, an incomplete lawless irrational can't be used like a genuine irrational such as π. π can be used to generate an infinite extension and it can be used, like ordinary numbers, in actual calculations, but a lawless irrational cannot. Perhaps most importantly (as Rodych thinks: 1999, 282), and a consequence of the immediately preceding point, a genuine irrational can be compared (in terms of ordering) with a rational number, but a lawless irrational cannot (*PR* 236-237). Similar problems exist for the idea of a pseudo-irrational. The details of how Wittgenstein argues against pseudo-irrationals would take us too far off the topic at hand;[223] for our purposes, the main upshot of these reflections,

[223] An example of a pseudo-irrational is π with the additional rule that every instance of '7' in its expansion be replaced with '3' (this can be symbolized with '$\pi^{7\rightarrow 3}$'). Wittgenstein, as could be expected, argues that this number is not well-defined (Rodych 1999, 285-286). In addition to the fact that there could be no '7' in the expansion, in which case it 'means nothing' (*PR* 228), there is no *law* that identifies where each 7 will be in the expansion (*PR* 235) and thus the only way to identify where the '7's are is by

given everything that has been said, is that genuine irrationals are *essentially* laws and not extensions. Arguing against the unjustified proliferation of irrationals curbs the idea that there are numbers that are essentially infinite extensions (instead of essentially rules),[224] which is one of the driving forces behind set theory.

6.8 'Description' in Wittgenstein's Comments on Set Theory

With the aforementioned confusions in mind, we can start to make sense of Wittgenstein's criticisms of set theory in their specifics. In some of Wittgenstein's most decisive comments, he makes reference to the concept of 'description'. And, given all of its occurrences, 'description' is used in a few different ways by Wittgenstein in this time period. The first way is the sense already discussed in Chapter 2. 'Description' is here used as a way of distinguishing between the *a priori* methods of mathematics and the *a posteriori* methods of the scientific (empirical) disciplines. Only the latter properly involve description. This is reflected in the fact that the statements are bipolar (capable of being true and false) and ultimately compared with reality in order to establish their truth-value.

Closely related to this, yet possessing its own unique use in the set theory sections, is the use of 'description' where it is used in contrast to 'representation'.[225] Wittgenstein says:

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working out the expansion of π, which can't be done to infinity. This rule is not appropriately 'self-contained' (Rodych 1999, 286), since it relies on expanding π (what Wittgenstein views as akin to an 'experiment' and not properly mathematical) and one can't *recognize* a law, from any part of the expansion, that determines where all '7's will appear (*PR* 235). Another reason that Wittgenstein does not consider this a genuine rule is that it makes the *decimal* expansion of π essential to the new rule (*PR* 231-232), which makes this new rule, in contrast to π, not system invariant (that is, relative to a certain base). Indeed, this number creates a new system (*PR* 231). This pseudo-irrational is thus 'homeless' and, again, not comparable to any given rational (*PR* 228, 236) (a necessary criterion for being a genuine number). For further explanation of Wittgenstein's position on irrationals (including further explanation about the criteria for what constitutes a proper irrational), as well as his related views against pseudo-irrationals, see Rodych (1999).

[224] Dawson has a slightly different interpretation: he suggests that Wittgenstein merely suggested there were other 'conceptions' of real numbers (than as expansions) (Dawson 2015, 11-12). Taking 'real number' to include irrational numbers, I think it is apparent that Wittgenstein was much more critical about the extensional 'conception' than Dawson suggests. For further details, see Rodych (1999).

[225] These are translated slightly differently in *Philosophical Remarks* and *Philosophical Grammar,* even though it is the identical German. The choice to translate '*darstellen*' as 'presenting' instead of 'representing' (to make what is essentially the same point) in the *Philosophical Remarks* (*PR* 208) is, given its aptness for suggesting an opposition to 'describe', the better choice. If one 'describes' something, one does not give the thing itself, and Wittgenstein's point is that in mathematics one must give the thing itself. It should be noted that in an early version of the above passage (MS 106, 84) Wittgenstein does use '*erfassen*', but opts for '*darstellen*' in most, if not all, subsequent passages.

> In logic we do not have an object and the description of that object. You will say for example, 'To be sure, we cannot enumerate all the numbers of the set, but we can give a description'. That is nonsense. You cannot give a description instead of an enumeration. The one is not a substitute for the other. What we can give, we can give. We cannot reach the same target from behind. (*WVC* 102)

> Set theory attempts to grasp the infinite at a more general level than the investigation of the laws of the real numbers. It says that you can't grasp the actual infinite by means of mathematical symbolism at all and therefore it can only be described and not represented. The description would encompass it in something like the way in which you carry a number of things that you can't hold in your hand by packing them in a box. They are then invisible but we still know we are carrying them (so to speak, indirectly). One might say of this theory that it buys a pig in a poke. Let the infinite accommodate itself in this box as best it can. (*PG* 468; *PR* 206)

Here 'presented' instead of 'represented' would be the more accurate word-choice. As mentioned in footnote 225, what Wittgenstein means to emphasize is that mathematics gives the thing itself, which is better captured by 'presented'.[226] Whatever is presented is actually enumerated. In this context, 'description' is thought of as an abbreviation or intermediary for giving this enumeration. An incorrect interpretation of the calculus is to think that a description can be given in place of an (infinite) representation. An 'infinite representation' is a logical impossibility and hence there is no such thing as giving a description in its place. As we have seen, this commits the calculus to the extensional conception that Wittgenstein has argued against. In this context, Wittgenstein's motto would be: no description without representation! Using an abbreviation in place of, for example, a set, requires being able actually to enumerate all the members of the set. And this is precisely what can't – in the logical sense – be done with the infinite sets used in set theory.

This is further explained using material already discussed in Chapter 4. In combating the intelligibility of the actual, extensional infinite, Wittgenstein not only combats the intelligibility of an actual infinite as, for example, something existing in another realm which we are describing with our mathematics, but the intelligibility of the idea that we are 'describing' *mathematics itself*. In mathematics there is not actuality and possibility in relation to the signs themselves, but only actuality. The following quotation bears repeating in its entirety in this context:

[226] I am grateful to Severin Schroeder for explaining this point.

> The feeling is that there can't be possibility and actuality in mathematics. It's all on *one* level. And is, in a certain sense, actual.
>
> And that is correct. For what mathematics expresses with its signs is all on *one* level; i.e. it doesn't speak sometimes about their possibility, and sometimes about their actuality. No, it can't even try to speak about their possibility. On the other hand, there is a possibility in its signs, i.e. the possibility found in genuine propositions, in which mathematics is applied.
>
> And when (as in set theory) it tries to *express* their possibility, i.e. when it confuses them with their reality, we ought to cut it down to size. (*PR* 164-165)

And also related:

> In mathematics there is no 'not yet' and no 'until further notice' (except in the trivial sense that we haven't yet multiplied two 1,000-digit numbers together). (*PR* 187)

I take it that the second quotation exemplifies a way in which possibility can be connected to an application of mathematics (as suggested by the first quotation). Possibility, insofar as it exists at all in mathematics, originates with *our use* of the signs; that is, that we can use a rule to generate a sequence as far as we would like. Possibility exists because of intensions or (as we have seen in the previous chapters) decision procedures. It is the possibility of our constructing some mathematical result on the basis of already given rules that allows for speaking of 'possibility' to begin with. There are not possible signs, for example, those that would make up an infinite set before it is actually listed. And hence there is no way to somehow 'describe' them instead of using the signs themselves to represent an infinite class. The use of the term 'possible' can be misleading because it can incline us to think that it is then capable of being actual (Wittgenstein suggests we are tempted by the slogan 'Let what is possible now become actual' – *PG* 466).[227] But, as argued extensively in Chapter 4, this is to misunderstand the nature of mathematics (and the infinite in particular). Thus, one of the uses of the term 'description' is in relation to what Wittgenstein thinks is to be found in the prose of Cantor's theory: the possibility of an infinite totality of objects, whether this be in reality *or as part of the mathematical symbolism itself*. It is not that our powers are insufficient to represent such a set, but rather that such a set is a logical impossibility. The prose of set theory wrongly suggests otherwise. It follows that there is no such thing as comparing the size of an infinite set with another infinite set (if this is understood on the finite model). It is only possible to compare a rule with another rule,

[227] Shanker (1987, 168) sees this confusion as the root of Cantor's invention of set theory.

and hence any mention of 'size' in this context must be a newly constructed concept on the basis of properties of rules.[228]

In addition to the above quotations, there is another way in which Wittgenstein uses the term 'description': as it refers to describing a 'logical form' or 'mathematical structure'. This is closely related to the first use, since it involves the idea that one can use mathematics to describe at all. In this case, Wittgenstein is disputing a specific way of understanding this 'description' in mathematics. It is somewhat unclear exactly what is meant by 'logical form' *in its details at this point in his thought* (partly because it is rapidly changing), but it is apparent that, for example, Wittgenstein takes the finite and infinite to be different logical forms. Arguably the idea of this technical notion of 'logical form' goes back to the *Tractatus*, with the 'logical concept' being the linguistic correlate of this shared form (between language and the world). The idea of a 'logical form' is used in a few instances in the intermediate period (the most important being in reference to the visual field (MS 106, 55), the subject-predicate form (which he ultimately argues isn't a logical form) (MS 106, 109), and in relation to the infinite); the idea of 'logical concept' is used in two ways, both of which are related to the infinite: the idea of a 'variable' and the axiom of infinity. Although, as we have seen, Wittgenstein holds that there is a shared form between language and the world at least until late 1929, even in the aforementioned examples where this interpretation could still be held (e.g. with respect to the visual field), Wittgenstein does not emphasize this aspect of the concept. In all of these cases, Wittgenstein is essentially indicating the existence of an internal relation, or, as he will subsequently call it, a grammatical rule, even though this latter description is not consistently used at this point in his work. This specific use of 'logical form' seems to indicate a general category under which at least a few related internal relations (or grammatical rules) could be subsumed. When describing a logical form, as in the case of the infinite and finite, one recasts it as if one could give a description (of the form) to determine whether something (e.g. a class) does or does not fall under the logical forms being described. But this is to treat internal properties as if they are not internal and, with this, act as if one could, on the empirical model, determine whether an object possesses the property in question. Given what

[228] Of course, this is exactly what one sees when one looks at the mathematical machinery of set theory. Most of the proofs involve rules concerning functions (e.g. 'injective', 'surjection', 'bijection' etc.), and deriving results from these rules. Lists are rarely made in these cases, and, when they are, are used to show a *limitation* of a 'number system'. See Section 6.11 and Rodych (2000, 293-297) for more information.

Wittgenstein says, he takes a logical form to be in some sense irreducible; it relates to the ‘ultimate grammatical given’ (Frascolla 1994, 96).

Thus, to try to describe or define a logical form will always lead to circularity and vacuity. Circularity because it is impossible to define these properties without making reference to the primitive properties themselves, and vacuity since, insofar as the definition is accurate at all, nothing is gained with the definition. An example of this is Dedekind’s definition of an infinite set. Wittgenstein says:

> Of course the web of errors in this region is a very complicated one. There is also e.g. the confusion between two different meanings of the word ‘kind’. We admit, that is, that the infinite numbers are a different *kind* of number from the finite ones, but then we misunderstand what the difference between different kinds amounts to in this case. We don’t realise, that is, that it’s not a matter of distinguishing between objects by their properties in the way we distinguish between red and yellow apples, but a matter of different logical forms. – Thus Dedekind tried to *describe* an infinite class by saying that it is a class which is similar to a proper subclass of itself. Here it looks as if he has given a property that a class must have in order to fall under the concept ‘infinite class’. Now let us consider how this definition is applied. I am to investigate in a particular case whether a class is finite or not, whether a certain row of trees, say, is finite or infinite. So, in accordance with the definition, I take a subclass of the row of trees and investigate whether it is similar (i.e. can be co-ordinated one-to-one) to the whole class! (Here the whole thing has become laughable.) It hasn’t any meaning; for, if I take a ‘finite class’ as a sub-class, the attempt to coordinate it one-to-one with the whole class must *eo ipso* fail: and if I make the attempt with an infinite class – but already that is a piece of nonsense, for if it is infinite, I cannot make an attempt to coordinate it. – What we call ‘correlation of all the members of a class with others’ in the case of a finite class is something quite different from what we, e.g., call a correlation of all cardinal numbers with all rational numbers. The two correlations, or what one means by these words in the two cases, belong to different logical types. An infinite class is not a class which contains more members than a finite one, in the ordinary sense of the word ‘more’. If we say that an infinite number is greater than a finite one, that doesn’t make the two comparable, because in that statement the word ‘greater’ *hasn’t the same meaning* as it has say in the proposition 5 > 4! (*PG* 463-464)

In defining an ‘infinite class’ one presents the essential difference between the infinite and finite in a roundabout manner, such that it appears as if one has identified a property possessed by the infinite class that isn’t possessed by the finite one. This then seems to be akin to discovering a property (which would correspond to the use of ‘description’) that one can then use to define any set one way or the other. But any such definition already relies on the recognition of the different forms (or what I have called the ‘categorial divide’) between the infinite and finite. For it is obvious that one can’t coordinate a part of a finite class with the whole of a finite class; this is news from nowhere (what Wittgenstein also calls a ‘tautology’ – *PG* 465). And, based on the finite understanding, to coordinate two infinite sets 1-1 must have no meaning whatsoever (since, in the infinite case, there is no such thing as even attempting such a coordination, given there is no possibility of success). So correlation must be understood differently, and in this case requires a function (e.g. Fx=2x). But such a function is already understood to have an infinite application. Thus, the definition itself presupposes the very understanding it wishes to somehow stipulate/explicate. This nicely exhibits the circularity and vacuity of the definition. In the context of set theory, as could be expected, 'size', 'coordination', 'class', and 'number' all have different meanings depending on whether these terms are applied to the finite or the infinite (also see, for instance, *PG* 468-469). And, connected with this, it should be noted that Cantor's attempt to bring legitimacy to the transfinite by making mathematical operations and relations (especially '>') that were well-defined in the finite case applicable to the transfinite fails to properly account for this categorial difference as well. Mere identity of sign does not guarantee identity of symbol.[229]

Thus, most generally, set theory mangles our understanding of internal relations and the relationship between the concepts of 'sense' and 'proof'. Wittgenstein says:

> This is always a case of the mistake that sees general concepts and particular cases in mathematics. In set theory we meet this suspect generality at every step…
>
> …
>
> The distinction between the general truth that one can know, and the particular that one doesn't know, or between the known description of the object, and the object itself that one hasn't seen, is another example of something that has been taken over into logic from the physical description of the world. (*PG* 467)

One acts as if the one could discover whether a given structure possesses the property that is clearly essential to it. Naturally, this is to assimilate internal relations to the model of physical descriptions of the world. A similar confusion applies to the

[229] Shanker explains this well. Any attempt to define '>' to reflect the same relationship between the transfinite ordinals that exists between the natural numbers will end up reflecting the categorial divide between the finite and the infinite. It is impossible to define this away. Indeed, Cantor's two principles of generation can be seen as what corresponds to this divide (one could say they already reflect the different 'logical forms'). Thus, by way of these principles, the relation '>' symbolizes a different relationship depending on whether it applies to the natural numbers or the transfinite. Roughly expressed, under the former it conveys a relationship of magnitude, where under the latter it indicates a 'shape' or 'length' – as Moore refers to it (1991, 125-126) – of a 'rule governed series'. To give one example: $\omega + 2 > \omega + 1$ means something categorially different from $n + 1 > n$. The former relationship does not involve magnitude. For further explanation of this, see Shanker (1987, 170-175). However, as already mentioned (footnote 208), it is not clear that Cantor was indeed committed to the view that Shanker is attacking and, thus, that he thought that he, through his mathematical work, had managed to define away the categorial divide between the finite and the infinite.

relationship of sense to proof. Set theory makes it appear as if a meaningful question can be applied about a given set and then a proof seeks to prove one of the two options to be true and the other false. And this is, as we have seen (Section 3.2.4), to conceal the true relationship between proof and sense: the proof gives meaning to a proposition. Thus, set theory, *at least in its prose interpretation*, assumes the conceivability of the inconceivable (Frascolla 1994, 98).

Finally, as we shall see in more detail later, at least in the terms employed by Rodych (2000), 'description' is used in contrast to 'revision'. In this context, the 'descriptive' position on Wittgenstein's philosophy of mathematics claims that Wittgenstein is only giving an account of what mathematics consists of and what mathematicians actually do. Not concerning itself with 'correct' or 'incorrect' mathematics, 'descriptivism' limits itself to conceptual concerns or interpretations that accompany a calculus, rather than evaluating or interfering in the calculus itself. 'Anti-revisionism' could also be used to characterize this position, although this term would not, without further explanation, carry with it the idea of Wittgenstein's conceptual analysis or his 'description' of what the mathematician does, which is included in Rodych's use of the term. 'Revisionism' is used in contrast to this to suggest that the calculus itself is being evaluated; on the basis of its lack of application, Rodych argues that Wittgenstein thought it wasn't a fully meaningful mathematical calculus and is, therefore, to be rejected. While Rodych's use of 'descriptivism' has some textual basis (e.g. *RFM* 210), it is potentially misleading especially in the context of a discussion of set theory where Wittgenstein regularly employs 'description' in a very critical way. Therefore, I shall generally, in what follows, opt to avoid it, except as a reminder of how Rodych framed the debate. In the final section, I shall show that *what is correct* about these two positions are not contradictory elements of Wittgenstein's philosophy, but rather complementary. It should be apparent from our discussion that Wittgenstein, the vast majority of the time in the early intermediate period, *only* wished to deal with the interpretation of set theory. From his views about mathematics not being a descriptive activity (and thus his opposition to Cantor's realism), to his repudiation of the extensional infinite, Wittgenstein is providing thoroughgoing criticisms of Cantor's *interpretation* of his calculus.[230] That is, Wittgenstein does not think set theory can have the application Cantor thinks it does.[231]

6.9 The Calculus/Prose Distinction in Question: Grève and Kienzler on Gödel's Proof

This chapter involves a greater interest in how Wittgenstein's thought evolves in his later thinking (in contrast to previous chapters). Thus, it is worth mentioning here that it has been argued by Grève and Kienzler, in the context of discussing Wittgenstein's comments on Gödel's first incompleteness theorem proof, that Wittgenstein eventually gives up the calculus/prose distinction. This is connected with his evolving philosophy of mathematics. As we have seen, in the early intermediate period, Wittgenstein envisions a mathematical calculus as autonomous and 'pure' (constituted wholly by its rules). Later in the intermediate period (around 1933) Wittgenstein starts developing his notion of language-games. With this, he now realizes that linguistic and non-linguistic items can't be easily separated, but are importantly interconnected. Thus, according to Grève and Kienzler, in the context of mathematics, the calculus/prose distinction loses its fundamental importance and Wittgenstein begins to investigate how linguistic and non-linguistic activity are importantly intertwined (Grève and Kienzler 2016, 80-82).

Grève and Kienzler do not provide much textual evidence that Wittgenstein gives up this distinction, although there are very few references to 'prose' in Wittgenstein's later work and their explanation of this does certainly fit very well with the overall development of Wittgenstein's philosophy. It is difficult to understand exactly why Wittgenstein would not still primarily be interested in the prose (whether

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[230] There are, of course, many less important parts of Cantor's philosophy that Wittgenstein would also disagree with. For example, as we have seen, he would also dispute Cantor's view that there is 'reciprocal and unique correspondence' between real numbers and points on the real line (Dauben 1979, 131); this is the case assuming Cantor did not say that 'real line' simply meant the real numbers, which is contradicted by the fact Cantor also thought the finite real numbers were 'complete' (Dauben 1979, 131). Wittgenstein suggests that Cantor wrongly tried to 'describe the continuum' (*PR* 208), which Wittgenstein also considers to be a 'form'.

[231] From everything discussed, it is clear that Wittgenstein would dispute the interpretation of Cantor's calculus as a description of the infinite. Due to space considerations, I have not dealt with set theory's role in foundationalism (which would include reductionism); even if foundationalism wasn't the principal purpose for the invention of the calculus for Cantor, it was this (potential) application that was the principal reason for set theory's employment by other foundationalists at the turn of the century. And, it should be noted, Wittgenstein developed a battery of arguments against this possible application also. For further information, see Rodych (2000, 291-293; 311-312) and the excellent discussion of these issues in Schroeder (2021, Ch. 3).

or not he uses this term) of Gödel's proof (i.e., the conceptual elements, in the way of an interpretation, that accompany the proof), since, as Grève and Kienzler admit, he isn't much concerned with the proof at all (its 'formal correctness') and, indeed, confines himself to certain 'explanations' *about* the proof. At least, one would think he would be still interested in the linguistic (and conceptual) elements of Gödel's proof (which he arguably is), even if he realizes this is not easily separable from Gödel's project as a whole. That is, insofar as he gives up the calculus/prose distinction, arguably this could be because of a realization about the role the prose plays, even though one can still distinguish between the mathematics and the linguistic elements (the change is in the idea that the 'prose' does *not merely* 'accompany' the calculus). Regardless of the exact details, which are nuanced and thus inspire various interpretations,[232] the following should be kept in mind. There are similarities, but also, arguably, less important differences, between Wittgenstein's treatment of set theory and Gödel's proof. It is important to emphasize that most, if not all, of the criticisms Wittgenstein starts voicing in his early intermediate period about set theory live on into his later work. Wittgenstein continues, into his later work, to question the *interpretation* of Cantor's work, all based on the intensionalism he has already developed in his early intermediate period. Nothing he does subsequently undermines his intensionalism or any of the prose (or what would subsequently be 'explanations' – as Grève and Kienzler opt to call them) connected with the interpretation he has already questioned on the basis of it. In contrast, Wittgenstein's focussed discussion of Gödel's proof only begins later in his thought, so it is more difficult, if not impossible, properly to place the use of the calculus/prose distinction in the evolution of his thought on the basis of his comments on Gödel's proof. It can simply be noted: what use of the distinction there is in relation to Gödel's proof would seem to be similar to what Wittgenstein has done with regard to set theory (since Wittgenstein arguably also wishes to dispel surprises that arise with Gödel's proof, and what he says about this in his early intermediate period isn't contradicted/undermined by his later work – see below).

[232] See Schroeder's (2021, Ch. 12) excellent discussion of this topic, which nicely illustrates just how complicated the issues are. Nothing Schroeder argues for suggests that Wittgenstein is concerned about anything but the interpretation of Gödel's proof. And there is no reason to think that Wittgenstein misunderstood the technical elements (as sometimes claimed) of Gödel's proof (at least in broad outline), even though he does not, because there is no need, deal in any depth on these matters. (This, of course, suggests that one can separate mathematical from non-mathematical matters, even if one doesn't choose to call these 'prose' and 'calculus', and even if the linguistic elements aren't separable from the project of which the calculus is a part). Schroeder's work also helpfully makes reference to numerous different interpretations of Wittgenstein's work on this topic.

Cantor's set theory and Gödel's proof are arguably both based on their inventor's respective philosophical agendas. Both, Grève and Kienzler note, involve proofs that are surprising or have a feeling of paradox. Wittgenstein's analysis of Cantor's diagonal proof, in his later work, revolves around looking at the proof itself and offering a new interpretation for the results (and thus dispelling any feeling of surprise).[233] Thus, in contrast to Gödel's proof, Wittgenstein very much *does* pay attention to the calculus of set theory and, in general, is able to separate conceptual matters from the pure calculus itself. This is happening even after the calculus/prose distinction is supposedly given up. In addition, Wittgenstein arguably does not *ignore* the details of Gödel's formal calculus either, but only deals with what is *philosophically necessary* to get clear about the *interpretation* of Gödel's proof. Ultimately, he deals with a justified sketch of what the Gödel proof shows, given by Gödel himself, in order to call into question the interpretation of the proof. In contrast to what Grève and Kienzler claim, arguably Wittgenstein dispels surprises in the interpretations of both set theory and Gödel's proof, although it is plausible that he uses Gödel's proof as an opportunity also to clarify that it is not possible for mathematical practice to have surprises or paradoxes (this could be connected with what is unique to Gödel's proof). Thus, while Wittgenstein may have given up on *using* the calculus/prose distinction (at least under that name) for the reasons Grève and Kienzler give, arguably he did not give up on the distinction between conceptual clarification (which relates to the interpretation of the calculus) and pure mathematics (which the philosopher can't interfere with). According to his later philosophy of mathematics, it may be more difficult *at times* to differentiate the two, but it is conceptual clarification that remains the proper province of philosophy. While the 'prose' may not be as easily recognizable (at least when it comes to Gödel's proof), it is essential to the interpretation/application of a proof or calculus, which is the focus for Wittgenstein. I don't think Wittgenstein's comments on Gödel's proof contradict this position, but rather support it. Moreover, set theory serves as an obvious example of this. The customary application, as a description of the infinite, is nonsense. But there can be, and have been, other interpretations of the calculus that would provide a perfectly respectable application for the calculus. Set

[233] Discussion of Cantor's diagonal proof occurs later in Wittgenstein's thought (*RFM*), and therefore isn't dealt with in detail here. *RFM* 125-142 contains many of the most important passages on the subject. Rodych (2000, 293-297) and Schroeder (2021, 151-158) are helpful examinations of the topic.

theory is used, for example, in computer science and this would require new prose to specify the new application for the calculus.

6.10 How Far Does Wittgenstein Go?: Dawson and Rodych on Wittgenstein's Criticisms of Set Theory

There is a persistent debate amongst Wittgenstein commentators as to the extent of Wittgenstein's criticisms of set theory. As characterized by Rodych, the debate revolves around whether Wittgenstein held a 'descriptive' or 'revisionist' position in relation to set theory. Debate about this, as well as *what each position exactly means*, has continued into the modern day, with two of the most important proponents of each position being Dawson and Rodych, respectively. After summarizing both of their positions, I will attempt to delimit what is best accepted and rejected from each, which will essentially show that the positions can, in broad outline, be reconciled with each other.

Dawson can be said to hold the 'descriptivist' interpretation of Wittgenstein's position (according to Rodych's use of that term). Based on the fact that mathematics is essentially invention, the (related) fact that mathematics, and set theory in particular, is not a description of anything (which includes the infinite itself), and finally that the infinite is a concept that essentially relates to a possibility and not an actuality, Dawson argues that Wittgenstein essentially holds to the position that mathematics should be 'left as it is' and that Wittgenstein's position on infinity is reconcilable with his views about set theory. This position is not at odds with what has been presented in this thesis. As has been argued many times, Wittgenstein's clarification of the concept of the infinite does not preclude understanding the mathematical work involving the infinite according to Wittgenstein's interpretation (of the potential infinite). Dawson, likewise, focuses on this interpretation first by outlining (and agreeing with) the calculus/prose distinction and then by presenting Wittgenstein's views on infinity. Dawson then gives convincing arguments to the effect that Wittgenstein merely disagreed with the prose interpretation of Cantor's proof, rather than the proof itself. Although the details of Dawson's account are beyond the scope of this paper, this is borne out by what Wittgenstein says (and it generally agrees with the account given in this chapter). Instead, as an example of Dawson's interpretation of Wittgenstein's work, we'll briefly look at what he says about the power set axiom.

Briefly explained, in ordinary prose, the power set axiom says that for any set there exists another set composed entirely of that set's subsets. This, as Dawson notes, is readily understood in the finite case: a set's elements can be readily divided up into subsets and then taken together as itself forming a set. The case is more complicated when it comes to infinite sets. In that case, Dawson argues, the power set axiom can be understood as a rule licensing certain inferences. He uses the proof that a set can't be put into one-to-one correspondence with its power set as an example of a way in which the power set can be seen as a rule within the calculus that licenses certain inferences.

Dawson does a good job of making sense of Wittgenstein's comments about set theory, although I take issue with his interpretation of the axiom of choice. Wittgenstein's claim about the axiom of choice is much more critical than Dawson suggests. Indeed, the quotation Dawson himself uses, from the *Remarks on the Foundations of Mathematics*, is critical enough, although Dawson tries to explain this away and does not even address Wittgenstein's most critical comments from his intermediate period. Detailed discussion of the axiom of choice is beyond the consideration of this chapter, but skepticism regarding Dawson's interpretation is warranted.

Dawson usefully characterizes the axiom of choice as follows:

> Informally put, the axiom of choice states that for any collection of non-empty sets, it is possible to select one element from each set. More formally:
> If S is a family of sets and $\emptyset \notin S$, then a *choice function* for S is a function
> f on S such that
> (V.1) $f(X) \in X$
> for every $X \in S$.
> The Axiom of Choice postulates that for every S such that $\emptyset \notin S$ there exists a function
> f on S that satisfies (5.1).[234]
>
> This applies in infinite as well as the finite cases, and the axiom allows that there is no need for a method of selection to be specified. Wherever a definite choice function can be specified, there is no need to use the axiom of choice. Rather, the axiom is of use when a choice function cannot be specified. Because the function cannot be specified, the axiom of choice allows that an object (a choice function or choice set) can exist without being fully specified. (Dawson 2015, 18)

[234] For the part that is more formal, Dawson quotes from Thomas Jech's *Set Theory: The Third Millennium Edition* (p. 47).

Dawson then goes on to argue that the comment Wittgenstein makes in the *Remarks on the Philosophy of Mathematics* has typically been interpreted to be more critical than it actually is. Here is that comment:

> Mathematics is, then, a family; but that is not to say that we shall not mind what is incorporated into it.
>
> We might say: if you did not understand *any* mathematical proposition better than you understand the Multiplicative Axiom, then you would *not* understand mathematics. (*RFM* 399-400)[235]

Dawson takes this to mean that Wittgenstein considers the axiom of choice to be a perfectly understandable axiom. Dawson argues for a family resemblance concept of 'mathematical proposition' and, with this, that mathematics is to be understood as exemplifying a core/periphery contrast. Parts of mathematics are to be considered central mathematical practice, while other parts are not to be considered paradigmatic mathematics. The axiom of choice, according to Dawson, occupies the peripheral position. It is not a core case of mathematics, but precisely because of this distinction it is possible to suggest it is still a meaningful part of mathematics (Dawson 2015, 19-20).

In response to Dawson, a few things should be noted. First, it is not even clear that the quotation Dawson chooses (together with an understanding of Wittgenstein's philosophy at this time) proves what he thinks it does. Indeed, the 'but that is not to say that we shall not mind what is incorporated into it' *by itself* suggests that Wittgenstein is calling into question the legitimacy of the axiom of choice – that is, he is likely saying it is questionable that it would be incorporated into mathematical practice. The 'but' in the above quotation seems to be used to emphasize this point, one that may not have been immediately evident to the superficial reader of Wittgenstein (and thus worthy of emphasis).[236]

Moreover, Dawson ignores Wittgenstein's other more critical comments about the axiom of choice. The reason for the rejection of a part of mathematics (in this case, the axiom), as it should be clear, would be on philosophical grounds. In this case, these

[235] 'Multiplicative Axiom' is what Wittgenstein uses to refer to the axiom of choice.

[236] The text Dawson relies on for his interpretation (which immediately precedes what he quotes) indicates that Wittgenstein, *in this context*, does think that a use of the axiom of choice can be justified (given its connection to other 'core' elements of set theory), even though it is not a paradigmatic example of a mathematical axiom. By not including this in the quotation, Dawson makes it appear as if he thinks this core/periphery contrast directly follows from the family resemblance concept. In contrast, I think it deserves mention that the core/periphery contrast *at best* can be understood together with the family resemblance concept, and it is not the case that the former follows from the latter.

grounds are developed in Wittgenstein's intermediate period, although these results are still accepted in his later work. Thus, the rejection of the axiom of choice happens in the context of discussing whether the infinite (in this case, in the specific forms of an unending row of trees or infinite decimal) can be understood without the specification of a rule. It thus appears to follow necessarily from some of the conclusions Wittgenstein reaches regarding the infinite (extensively discussed in Chapter 4 and this chapter). Given the decisive nature of this material, it is especially concerning that Dawson doesn't even acknowledge its existence, let alone provide any adequate response to it. Regarding the axiom of choice specifically, Wittgenstein says:

> What gives the multiplicative axiom its plausibility? Surely that in the case of a finite class of classes we can in fact make a selection [choice]. But what about the case of infinitely many sub-classes? It's obvious that in such a case I can only know the law for making a selection.
>
> Now I can make something like a *random* selection from a finite class of classes. But is that *conceivable* in the case of an infinite class of classes? It seems to me to be nonsense. (*PR* 167)

Infinity, as he suggests here, is only understood in relation to a rule. Infinity refers to the endless possibility of construction in accord with a rule. The rule is what gives meaning to the idea of infinity, since it is *by the rule* that the set is *constructed*. In the absence of a rule, as could be anticipated by our detailed examination of Wittgenstein's philosophy, the axiom of choice can't be properly understood. For the axiom of choice acts as if there is a rule, even in the case none can be provided. Without a rule, as Wittgenstein explains immediately preceding the previous quotation (*PR* 166-167), one is then forced to assume the series is random, but then all there is to the series is the list. And then what makes the infinite possible is missing (since there is no such thing as an infinite list or infinite *random* series – 'nothing can be known apart from the fact I can't know it' (*PR* 167)). Wittgenstein takes the idea of an 'infinite selection' in the absence of a rule to be meaningless.

I see this as quite decisive evidence against Dawson's interpretation. While the axiom of choice can be used (even if it is only a fanciful use of mathematics), it is far from 'readily understandable'. In the intermediate period Wittgenstein is especially harsh on the axiom, suggesting it is clear nonsense, but with other developments in his philosophy of mathematics he arguably becomes more charitable to fanciful uses of mathematics as long as they are properly connected to 'core' examples of mathematics (in this context, set theory viewed as a formally correct calculus is what is meant) (*RFM* 399). It should also be noted that it is unsurprising that on the assumption of the axiom of choice various unexpected results follow (as understood even by mathematicians). While Dawson of course has a point that it is still possible for a mathematician to use it in his system,[237] it is equally unsurprising that how it is understood, its use, application, and the results of its use have all been debated in the mathematical community. For if it can't be readily understood (explained by Wittgenstein's stronger claim that it is indeed nonsense), as some mathematicians claim, it is to be expected that results derived from it could similarly not be readily understood or even be paradoxical in nature.[238]

In addition to Dawson's interpretation of the axiom of choice, it is also noteworthy that he *explicitly* chooses not to deal with the question of set theory's application.[239] This is what fuels the 'revisionist' interpretation, which is best defended in Rodych's (2000) work. Rodych, in contrast to Dawson, emphasizes Wittgenstein's negative appraisal of set theory. That is, he is arguing the calculus itself is to be rejected. Rodych attempts to argue that this position begins in the early intermediate period, but none of the passages he marshals in support are convincing for this interpretation. In each case (*PR* 166, 211; *WVC* 102; *PG* 464, 470), Dawson's interpretation would appear to hold: Wittgenstein is criticizing various parts of the prose that accompanies the calculus, but not condemning the calculus as a whole. Indeed, one of the passages is obviously not even *directly* about set theory, but rather the extensionalist interpretation of the infinite (*PR* 166).[240] And the passages that are

[237] The axiom can still be laid down and used in reasoning about 'infinite sets'. It will guarantee the existence of an 'infinite set' (made up of one member from an infinite number of sets), even when one can't specify a choice function that defines the set (Dawson 2015, 18). However, that it can be used in this way in drawing inferences about infinite sets hardly guarantees that this makes sense or that we can make sense of the results we use it to derive (which makes it understandable that the axiom of choice and the results derived from it have been contentious topics among mathematicians – see, for example, Thomas J. Jech's *The Axiom of Choice*, pp. 2-3).

[238] Indeed, later in Wittgenstein's work he expresses skepticism about the axiom of choice having an application that can be made sense of at all (*RFM* 282-283). In his later philosophy Wittgenstein calls into question the applicability of some specific propositions and proof procedures, such as the diagonal method. His views on the axiom of choice seem to be the most critical, since he suggests it does not have any application (as opposed to a different or more limited one – as is seemingly the case with the diagonal procedure). The especially critical treatment of the axiom of choice is anticipated by Wittgenstein's intermediate period views.

[239] Dawson (2015) mentions this possible objection on two occasions: in footnote eight and at the end of the paper. In both cases he indicates that, while this is a possible line of objection, he is intentionally not dealing with it in the paper. But, insofar as this is a major part of Wittgenstein's later philosophy and his thought in the philosophy of mathematics and set theory in particular, it is important to give it the focus it deserves.

[240] Of course, this comes with the qualification made at the beginning of the chapter that doubtless these reflections were tightly interconnected, reflections on set theory prompting more extensive examinations

applicable are easily reinterpreted using an intensionalist framework (which leaves the calculus as it is). *At best*, the *relevant* examples could be seen as both a comment on the prose and a negative appraisal of a possible application for the calculus.[241] Rodych rightly draws attention to the fact that Wittgenstein begins to develop, what Rodych refers to as, his ‘extrasystemic requirement’ in the late intermediate period.[242] By that, Rodych means the following. In the context of examining what constitutes rule-following and, in particular, mathematical rule-following, Wittgenstein suggests that what is necessary to mathematics, that is, what makes mathematics into mathematics (beyond a mere game) is the possibility of the application of a given calculus in empirical descriptions (Rodych 2000, 300-302). On this basis, Rodych argues that Wittgenstein did not see set theory as a *fully meaningful mathematical* calculus and thought it was to be rejected. More generally, this movement in his thinking corresponds to his shift from the calculus conception of language (which relates – loosely – to the verification principle) to his anthropological viewpoint about language (which relates to his new view that ‘meaning is use’). To fully appreciate how language functions, one must not merely examine the relations between words and sentences, but one must also examine how these are embedded in human life; linguistic utterances must be seen as human actions and words and sentences as tools for such actions (cf. Schroeder 2021, 182-183). Thus, we see different aspects of what constitutes the meaning of mathematical calculi (proof and application) in the intermediate period, and the ultimate realization that *both of these* are different aspects of their meaning.

Along these lines we can see that Dawson and Rodych emphasize different aspects of Wittgenstein’s philosophy without properly appreciating that *both aspects* are equally important to understanding Wittgenstein’s developed philosophy. Dawson is surely right that Wittgenstein, the majority of the time, was not criticizing mathematics (the calculus) itself. We have seen that in the intermediate period Wittgenstein is almost

of the infinite (and *vice versa*). Nonetheless, this comment that Rodych relies on does not involve set theory (nor does it originate among comments on set theory in the manuscripts – MS 106, 244-246).

[241] *PR* 211 and *WVC* 102 both seem directed at prose, although both passages, but especially *WVC* 102, also call into question the idea that set theory could have a certain application (i.e., ‘describing’ infinite sets in place of enumerating them). *PG* 470 involves the same concerns, but clearly emphasizes the prose interpretation (‘clouds of thought’). *PG* 464 emphasizes that a definition can’t have the application it is taken to have (as, for example, a decision procedure), although it seemingly is part of establishing a new meaning for concepts such as ‘>’.

[242] As suggested (footnote 241), Wittgenstein, at least implicitly, expresses his opposition to one possible application early in the intermediate period. However, his concern about lack of applications becomes explicit later in the intermediate period (*BT* 747; *PG* 467).

entirely arguing against the extensionalist interpretation of the calculus. These criticisms of various parts of the interpretation of the calculus, whether this is considered ‘prose’ or ‘explanations’, continue on into Wittgenstein’s later work with, for example, the new interpretation he provides for Cantor’s diagonal proof. The one obvious objection to this, as we have seen, is the axiom of choice; Wittgenstein insists on it being nonsense in the early intermediate period and even his more tentative discussion of it in his later thought is critical of the axiom. The best way to explain this is that the axiom occupies a unique position where prose and calculus meet. In this case, Wittgenstein’s clarification of the concept of infinity requires a skeptical look at the axiom. Whereas typically Wittgenstein is identifying an interpretation or (seeming) application that accompanies the calculus, in this case Wittgenstein points out that, given the detailed arguments for the idea that infinity can only be given with a rule, it is difficult, if not outright impossible, to make sense of the axiom (since it simply assumes there is such a rule, without being able to specify one). As we have seen, Wittgenstein becomes more charitable to what he will later consider (at best) a fanciful use of mathematics. However, as I believe Rodych rightly observes, the axiom of choice, much like set theory itself, could be seen ultimately to have applications in, for example, an empirical area such as physics, which would give it complete mathematical legitimacy (Rodych 2000, 310-311). Of course, I think it important to note that this new application would almost certainly come with a different interpretation of the axiom.

Rodych, in turn, emphasizes something that becomes explicit for Wittgenstein only beginning later in his intermediate period: that an aspect of the meaning of a calculus derives from the possible applications it has in empirical descriptions. Rodych is right to draw attention to this part of Wittgenstein’s thinking although he exaggerates its role in Wittgenstein’s thought. Together with the misinterpretation of various passages from the early intermediate period, he thereby encourages the especially harsh pronouncements that set theory is not mathematics and that Wittgenstein’s comments can be understood as prescriptions for the mathematician. As we have seen, not all of the passages he uses as evidence are even directly about set theory (*PR* 166), and those that are are obviously primarily about the prose interpretation of the calculus. And it would appear that Wittgenstein only really starts thinking about (the lack of) applications of the calculus later in the intermediate period. Nonetheless, Rodych is right to emphasize this important development in Wittgenstein’s thought. Whereas Dawson wishes to avoid the question of an extrasystemic application altogether, Rodych places great importance on it. In contrast, I think it important to emphasize (roughly equally) *both* parts of Wittgenstein's thought about this issue and be very clear about the use of 'descriptivism' and 'revisionism' (as employed by Rodych). Arguably a central part of Wittgenstein's challenge to set theory is rooted in his conceptual clarification of the prose accompanying set theory (this is what Rodych, rather confusingly, refers to as 'descriptivism'). The purpose/function of the calculus is the result of conceptual confusion. Thus, *this application* is not a possible one. Wittgenstein is, in the vast majority of cases, not interfering with the rules, proofs, or notation employed by the mathematician. Set theory is 'wrong' or 'nonsense' *because* there are important problems with its prose descriptions (or the 'explanations' connected with parts of the calculus). Wittgenstein's objections are still *fundamentally directed at a certain interpretation of the calculus*, which necessitates a *certain attitude* towards the achievements[243] and, related to this, future expectations for the calculus. However, at the same time, Wittgenstein does not pronounce upon any future applications the calculus may have. Indeed, as even Rodych admits, Wittgenstein suggests that both the calculus and the axiom of choice specifically could find perfectly clear applications (Rodych 2000, 311). For this reason, it is difficult to think that Wittgenstein is actually trying to prescribe what mathematicians should or should not do. At most he wants to pronounce on what mathematicians should think of *set theory in relation to its standard interpretation*. Moreover, as we have already seen (here and in Section 3.2.4), in Wittgenstein's later philosophy the meaning of a calculus is derived from two sources: the rules, decision procedures, proofs, etc. and the extrasystemic application. It is the latter that Wittgenstein, primarily through the evaluation of its prose, emphasizes set theory lacks. But it does not lack the former. Other than the interpretation of set theory, there is nothing special about this calculus in comparison to any other that lacks (or lacked) an application. The benefit of my interpretation is it seems best able to explain the fact that both of these elements of meaning for a mathematical calculus are emphasized in Wittgenstein's later philosophy of mathematics as well as the fact that a great deal of what is generally considered mathematics would be in precisely the same situation as set theory. That is, much work in mathematics lacks an empirical application, but would not, I think, inspire the same negative pronouncements from

[243] This explains why he ends up reinterpreting some of the proofs, or advocating for different notations or interpretations, as opposed to outright condemning individual proofs or the calculus as a whole.

Wittgenstein that set theory does. And, indeed, we even find passages in Wittgenstein that suggest a calculus may still be considered mathematics even without an application, given mathematics too forms a ‘family’ (*RFM* 399e); paradigmatic cases of mathematics require an application, but this is not to say that every single calculus, or piece of mathematics, does. And, at the very least, there is no reason to think a calculus would *immediately require* an application (as has been borne out by the mathematics used in physics). *Properly conceived*, Wittgenstein’s conception of an extrasystemic application is complementary to his idea of conceptual clarification.[244] It is only by placing undue emphasis on the extrasystemic criterion, together with a distorted interpretation of Wittgenstein’s comments that are directed primarily at the prose of set theory, that one is led to Rodych’s ‘revisionism’, which has the possibility to mislead further.[245]

6.11 Leaving Everything as it Is?: Responding to Concerns of Revisionism

It would seem that Wittgenstein’s claim to have primarily been interested in, and restricted his criticisms to, the prose of set theory is contradicted by the details of his criticisms. Although space considerations preclude a detailed analysis of this, here I will attempt to address the most obvious concerns with Wittgenstein’s position as they relate to what has already been presented. Perhaps most obvious is the question of whether Wittgenstein’s idea that infinity is the property of a rule contradicts the idea that there are different ‘sizes’ of infinity, as claimed by Cantor. This would apply most obviously

[244] This is not to say that every single comment Wittgenstein makes perfectly supports the interpretation given here. It should be pointed out that a few comments may be especially harsh, critical, or ‘revisionist’ in nature. This would not be surprising since Wittgenstein’s thought was developing. Nonetheless, virtually all, if not all, of Wittgenstein’s comments can be readily understood as actually an argument against a certain interpretation of the calculus.

[245] It is Rodych’s aforementioned mistakes, together with his choice of wording when discussing his ‘revisionist’ interpretation, that likely leads Valérie Lynn Therrien (who cites Rodych) to her absurd claim: ‘Wittgenstein’s grammatical analysis has the effect of an axe dropping on the concretism of the Continuum Problem and Cantor’s Theorem: Cantor’s Theorem is logically false and the Continuum Hypothesis is not an ‘unsolved problem’ but, rather, merely a nonsensical pseudo-problem – which solves Hilbert’s first problem’ (2012, 62). There is no reason to think that Wittgenstein’s ‘revisionism’ extends to determining that Cantor’s theorem is ‘logically false’ or that the Continuum Hypothesis is a nonsensical pseudo-problem (that is, that it is any more nonsense than any other conjecture that has not yet been proved false – i.e., nonsensical). Of course, Cantor’s Theorem and the Continuum Hypothesis should not be viewed through the lens of extensionalism, but Wittgenstein’s criticisms about set theory do not question the formal correctness of any individual proof or conjecture. At most, Wittgenstein’s position only applies to how one views an entire calculus (and the odd piece of mathematics at the intersection between the prose and calculus), and has no bearing on individual parts of the calculus.

to Cantor's idea of 'power' or 'cardinality' [*Mächtigkeit*], which was supposed to represent this notion 'size'. Whereas in the finite case the cardinality corresponds to simply the number of members of the set, in the infinite case the 'cardinality' of a set needs to be defined through the idea of one-to-one correspondence. A set is of the same cardinality as another set if it can be put into one-to-one correspondence with that set. The 'smallest' infinite set through which the idea of one-to-one correspondence is employed is the set of natural numbers. It is by means of this set and the idea of one-to-one correspondence that Cantor *creates* a new concept of 'size'; this idea of 'size' has nothing to do with magnitude (i.e., the number of members of a set) but instead involves properties of functions. As Bruno Whittle (2015) has argued in more detail, Cantor's arguments merely involve functions and not the number of members of sets.[246] This is intuitively convincing since, when establishing properties of infinite sets, arguments usually concern functions and defined sets and not lists of members of sets. Even where there appears to be a list involved (e.g. proving the cardinality of the set of real numbers is 'bigger' than the natural numbers), Wittgenstein points out that the technique does not actually involve completing a list, but rather showing a procedure by which, *however long you make the list*, a 'number' not on the list can be constructed (and also is never completed). That is, since any actual list has to be finite, Cantor shows that, given any *method* for trying to *generate* a well-ordered list of the real numbers, there will be a way of constructing a 'number' (by way of the diagonal procedure) so that it can be shown not to be on the list regardless of how long the list is extended. And even if you add the diagonal to the list one can create a new diagonal number not on the list. Thus, Wittgenstein argues that this technique too is best understood as not proving anything about the size of sets (as this applies in the finite case), but rather that the real numbers are of a different kind from the natural or rational numbers (where an ordering can be given) (*RFM* 132). Thus, he gives another interpretation for how the diagonal procedure is best understood. The inability to give an ordering is precisely what is meant by 'non-denumerability' and is defined by means of the diagonal procedure (*RFM* 129-130).

[246] Although there are important limitations to Bruno Whittle's paper (see footnote 248), he does a good job of showing that any infinite conception of 'size' must deviate from the finite conception. He does this by carefully examining some of the more technical arguments in set theory, such as Cantor's argument that there is no one-to-one function from the powerset of a set A to A itself, as well as how this relates to Russell's paradox.

It is not the case that we have a concept of size applicable to both finite and infinite sets (*RFM* 132; *PG* 464; cf. Schroeder 2021, 157). For the very concept of 'infinite' means that there is no size (in the ordinary sense). It may be convenient to speak of open-ended rules for generating series as 'infinite sets' (as if they were a totality), as we do, for example, in the case of natural numbers or series of rational numbers (i.e., the real numbers). But this is already a departure from our ordinary concept of 'set', since there is no such thing as an infinite series existing as a totality (as suggested by 'set'). Having accepted this way of speaking, we may also, as Cantor did, look for a way in which something like the concept of size can be applied to infinite sets. Cantor chose to use the one-to-one correspondence criterion of size to the exclusion of the criterion that a subset is smaller than the set of which it is a part (cf. Moore 1991, 111; 114). This is Cantor's concept of 'cardinality', although, as has been shown, there are alternative ways of developing the idea of 'infinite size'.[247] This is concept creation, and it is unsurprising that this concept of 'infinite size' should come into conflict with our ordinary concept of size.[248] Wittgenstein, it should be noted, has no objection to concept creation, but only the misunderstandings that can arise without the understanding that concept creation is what is at issue. The most important point is that, in this case, Wittgenstein's arguments do not involve attacking the mathematics but the way Cantor chooses to describe the mathematics he has done together with at

[247] Paolo Mancosu (2009, 627-636) has noted that there are recent mathematical developments that develop the concept of size in relation to the infinite on the basis of part-whole criterion and not the one-to-one correlation criterion.

[248] Based on this, it is unsurprising that Bruno Whittle finds the concept of size (as it is understood in the finite case) to be an inadequate (and unneeded) explanation in the case where a one-to-one correspondence can't be given between two infinite sets. Given 'size' has a new meaning in the infinite case (something Whittle does not acknowledge), Whittle is at most drawing attention to a feature of Cantor's definition: it only shares similarities with the finite notion of size (and excludes other aspects of the finite) and can't be entirely justified or explained by reference to the finite idea of size (as Wittgenstein already emphasizes). Mathematical definitions of 'size' in relation to the infinite deviate importantly from our finite understanding of that concept, so much so that paradoxes that can arise on the basis of its introduction have served as a reason to consider alternative definitions. But this is not to say any definition of 'size' in relation to the infinite can be reconciled with our understanding of this concept in the finite case, nor that it needs to be. By not acknowledging that Cantor has provided a new definition of 'size' in relation to the infinite, Whittle can justify doing metaphysics and examining whether the concept of size (as it is used in the finite case) can justify the idea of different 'sizes' in the infinite case. Thus, Whittle (2015, 18) also mistakenly concludes that there is only one size of infinity, whereas, as we have seen, Wittgenstein would insist that the infinite is not a quantity at all and, following from this, the *unique meaning* of 'size' and 'different sizes' in the infinite case. Whittle's project depends, incorrectly, on assuming there is one concept of size applicable to the finite and infinite that can be used to determine whether or not there are different 'sizes' of infinity.

least part of his reason for its creation. However, as Wittgenstein suggests, we may discard this interpretation (which includes Cantor's motivation) and, indeed, leave room for other interpretations to connect up with mathematics that has a proper empirical application. Arguably this has already happened with developments in computer science and certain work in number theory. For example, Goodstein's theorem is a statement about the natural numbers, specifically that any Goodstein sequence of numbers will terminate at 0. Its proof makes use of ordinal numbers, vindicating the idea that set theory can have applications to areas of mathematics that ultimately relate to an empirical application (see Goodstein (1944) for the technical details of the proof and Miller (2001) for a more accessible presentation). Moreover, Goodstein aimed to do this in a purely finitist way, which included giving an account of the ordinals that did not presuppose Cantor's 'theory of infinite classes' (Goodstein 1944, 33). It was subsequently proven to be another statement not provable in Peano arithmetic. It is apt to note that the person after whom the theorem is named, Reuben Goodstein, was a PhD student and admirer of Wittgenstein, and continued to hold a Wittgensteinian position in the philosophy of mathematics throughout his career, part of which was spent at the University of Reading. His theorem, considered more generally, thus also serves as a vindication of the idea that the calculus can be found to have an application (and thus separated from the 'prose'); the topic of mathematics and its application is briefly discussed by Goodstein himself (Goodstein 1972, 282-283).

Another argument from Wittgenstein that would seem *initially* to contradict the mathematics that Cantor uses involves the set of real numbers. This set can't be put into one-to-one correspondence with the natural numbers and for this reason is sometimes thought to be bigger in size than the set of natural numbers. This interpretation of this set in terms of magnitude is to be rejected. However, given the importance of this set to Cantor's work, it is important to address another way in which Wittgenstein's philosophy of mathematics may be thought to contradict Cantor's mathematical work. As we have seen, the infinite is to be identified with a mathematical rule. As it is understood in terms of the natural numbers, there is a rule by which one can always construct the next natural number in the sequence such that, if one constructed the sequence far enough, one would reach any natural number in the sequence. This is an example of an induction, which can itself be used when constructing other inductions. It is on the basis of this induction that we are justified in speaking about properties of *all* the natural numbers. As we have seen, Wittgenstein, especially in the intermediate period, questioned whether irrational numbers not given by a rule are actually numbers. Arguably, in his later work, he became more lenient to the idea, sometimes thinking of these ‘numbers’ as extensions of the concept of number (e.g. *PG* 113, 115, 300; *RFM* 130; cf. Schroeder 2021, 155). Specifically, he suggests that there is a use for Cantor’s diagonal procedure of listing infinite expansions (which includes expansions not given by a rule) (*RFM* 129) and, going beyond this, that there may even be a use for the diagonal number as well as the fact that this number fulfills some of the criteria of what he considers paradigmatic numbers (*RFM* 126). It is debatable whether the diagonal procedure produces a ‘number’, although Wittgenstein is more open to this idea in his later work as long as the differences between all of the things called ‘numbers’ are realized (e.g. *LFM* 15). In any case, such irrational numbers make the possibility of a well-ordering of the set of real numbers impossible. This was decisively shown with Cantor’s diagonal argument. When understood as non-terminating, irregular decimal extensions, any possible ordering one chooses can never include all real numbers; whereas, in the case of the natural numbers, there is a method of listing its members that preserve a well-ordering of this set, there is not in the case of the real numbers. Understood this way, there is no rule by which one can set out to list *all* of the real numbers, i.e. whereby one will ultimately, given the extension of the series far enough, list any real number; any potential ordering one chooses will never be a well-ordering. In this way the concept of counting is not only inapplicable in the sense of not reaching an end (as it is with all infinite sets) (cf. *PG* 285), but also in the sense that any list will not be a well-ordering according to any rule (*RFM* 129-130). For this concept Cantor used the term ‘non-denumerable’. Given that the use of ‘all natural numbers’ is given meaning by induction, it is clear that Wittgenstein is critical of the idea of the set of *all* real numbers (along with the idea of *the set* of such numbers) (*RFM* 129, 132). This is, indeed, one aspect of Wittgenstein’s thinking on this topic at this time, although, paralleling what he says about the lawless irrational numbers, I think it also possible that Wittgenstein would have seen the idea of a ‘set’ of real numbers as an extension of the concept of an infinite set. There are reasons that speak for and against referring to it as a ‘set’ (e.g. it includes other perfectly legitimate infinite sets of numbers, but can’t be well-ordered itself, respectively). Regardless of the specifics, this can be seen as another example of the intersection of calculus and prose, and arguably has a similar solution to what Wittgenstein says about the axiom of choice. Although not a paradigmatic case of an infinite set, we needn’t outlaw elements of the calculus on this basis. Even if it is not justified to see real numbers as an extension of the concept of ‘infinite set’, the signs that are meant to represent this set can be seen to have a clear connection with other well-defined parts of the calculus, which bestow them with a meaning that shouldn’t be simply dismissed on the basis of the prose. Even if Wittgenstein would not accept the idea that the real numbers are best understood as an infinite ‘set’, he can still envisage a respectable use for the corresponding signs (even a more ‘fanciful’ use of mathematics needn’t be rejected – *RFM* 399).

To conclude, to set the stage for understanding Wittgenstein’s criticisms of set theory, we have examined the history of set theory, some of its technical elements, as well as Cantor’s philosophy/metaphysics (insofar as this inspired the calculus). We saw that Wittgenstein applies his notion of a categorial divide between the finite and infinite in order to emphasize the distinction between extensions and intensions and, with this, his notions of ‘description’ and ‘representation’. As we have seen, Wittgenstein’s various criticisms of the prose (as he calls it in the intermediate period) of set theory all ultimately relate to this categorial divide. In the course of doing this, we saw several different, but related, ways in which Wittgenstein uses the concept of ‘description’ in the context of his remarks on set theory. We then examined Kienzler’s and Grève’s claim about Wittgenstein’s diminished use of the calculus/prose distinction in his later work, which led to a fruitful comparison of Wittgenstein’s treatment of Gödel’s proof with his position on set theory. We then turned to a debate within Wittgenstein studies as to the extent of Wittgenstein’s criticisms of set theory. We saw that *what is correct* in these two positions are not contradictory, but rather are complementary aspects of Wittgenstein’s mature views on set theory (reflecting his philosophy of mathematics more generally). Finally, we anticipated concerns that Wittgenstein’s position in fact leads to revisionism and showed that this is not the case.

7. Conclusion

In broad outline, the work of Wittgenstein’s intermediate period picks up right where the *Tractatus* left off. Two of the main subjects that occupy him upon his return to philosophy, the phenomenological language and infinity, are the result of criticisms made by, or discussions he had with, Ramsey and/or concerns he already had about his earlier work. We have even seen that Wittgenstein’s few remarks on the philosophy of mathematics in the *Tractatus* are explicitly addressed upon his return to philosophy. Thus, the idea that his philosophy moved in incremental steps, with changes happening only gradually, including some important realizations not being fully grasped or consistently applied until later, is vindicated. Strikingly, a part of his early philosophy that is rejected in the early intermediate period, the necessity of an extra-mathematical application for a paradigmatic mathematical calculus, was only to be, upon further reflection, resurrected in the later part of the intermediate period. Several ideas that constitute new directions in his thought have some continuity with his earlier thinking. For example, skepticism about the actual infinite is already expressed in his earlier work and the verification principle possesses some continuity with the *Tractatus* (where ‘verification’, in the empirical case, means the positive comparison of a proposition with the world).

We have seen that the most original directions in this new phase of his thought are the following: his initial use of the verification principle to make logical distinctions in the philosophy of mathematics (including between mathematics and the empirical disciplines) and his explicit comparison of mathematical propositions with empirical ones (instead of tautologies)[249]. At least partly on the basis of his developing verificationism, along with his consideration of inductive proof, he also comes to the conclusion that individual words, such as ‘proposition’ and ‘proof’, can have a number of distinct senses. This idea first makes its appearance slightly later in the intermediate period and is best thought of as a result of his quickly developing thought at the time. It serves as an early statement of what will become the family resemblance concept.

In this new phase, there were important interrelationships between areas of his thought. Indeed, given his greater interest in the philosophy of mathematics in the early

[249] Even prior to his discussion of mathematical verificationism, and the suggestion that mathematical propositions have sense, Wittgenstein considers the idea that logical propositions themselves have sense [*Sinn*] and meaning [*Bedeutung*] (MS 105, 129).

intermediate period, it is unsurprising to find several major insights in his philosophy originating in his philosophy of mathematics. As we examined in Chapter 2, the idea of a comprehensive grammar develops with the consideration of mathematical systems (including using ‘calculus’ and ‘system’ to describe it). Properties that apply to this comprehensive grammar are anticipated with his consideration of mathematical systems. As we saw in Chapter 3, early formulations of what was to become the verification principle appear in his philosophy of mathematics soon upon his return to philosophy. Already we have the appearance of ‘way’ [*Weg*] (MS 105, 8 and 10), and ‘method’ [*Methode*] (MS 105, 10), and even one of the more famous early formulations of verificationism: ‘Every proposition is a cheque for its verification’ (MS 105, 16).[250] Although verificationism is generally mentioned (in the secondary literature) in relation to Wittgenstein’s treatment of empirical propositions, it is important to note that these ideas begin earlier in the context of distinguishing between the meanings of mathematical propositions. This serves as a perfect illustration of how closely interconnected his mathematical reflections were with other areas of his philosophy.

Although he was skeptical of the actual infinite even in his early work (as we saw in Chapter 1), it is only in the intermediate period that he comes to investigate this idea in detail. As we saw in Chapter 4, initially he is tempted to reduce all infinity to mathematical infinity but, as a consequence of his interest to investigate words in the context in which they are employed, comes to think there are a variety of distinct uses of ‘infinity’. Thus, his conclusions about the infinite in 1931 arguably serve as a combination of his calculus conceptions and an early example of his anthropological viewpoint. These alternative analyses of infinity are arguably not correct philosophically, which he seems to have realized later on. The idea of non-mathematical uses of ‘infinite’ does not occur in his later work. Nonetheless, this perfectly exemplifies a general development in his philosophy. Moreover, his philosophical approach concerning the concept of the infinite involves seeds of what will become the genetic method. He not only wants to identify confusions related to the concept of the infinite, but also the trains of thought that lead to them. It is only in this

[250] This is Schroeder’s translation (2021, 38: n. 3); he convincingly argues that ‘cheque’ is a better choice than the standard ‘signpost’ because of the metaphor it conveys. A meaningful mathematical proposition stands in the place of, or is a promise for, the proof which stands behind it and gives it meaning. Of course, this is an important component of Wittgenstein’s philosophy of mathematics that lives on into his later work. Insofar as it is an early expression of the idea of a proposition being given meaning by something else that it stands in place for (or in relation to), it is also apparent how this can serve as part of the origin of the calculus conception of language.

way that a philosophical problem can be completely eradicated. We identified several underlying tendencies of thought that lead to confusion in Chapter 4 and showed that the unbounded concept of an irrational number is as well (discussed in Chapter 6).

In the final two chapters, 5 and 6, we saw the extensive use Wittgenstein made of his reflections into infinity. In relation to his reflections on inductive proof, first mention is made of the rule-following considerations, as well as an argument about foundational rules being essential for other propositions to be 'tested'. The form of this argument very much resembles ones that will eventually be given in *On Certainty* regarding 'hinge' propositions. Moreover, it is his consideration of inductive proof which arguably is one of the driving forces in his philosophy of mathematics. It is one of the crucial topics in his philosophy that prompts him to extend his philosophy of mathematics beyond giving an account of just equations or algorithmic decidability. His reflection on inductive proof forced Wittgenstein to account for the diverse nature of mathematical practices, including what counts as a 'proof'. Directly related to this, it is his scrutiny of the concepts of 'proof' and 'proposition' in the context of his thoughts on inductive proof, together with his evolving philosophy of mathematics and verificationist considerations, that contributes to the origin of the family resemblance concept. With the development of this concept, together with his eventual rejection of the saying/showing distinction, he will neither be forced to reject the idea that an inductive proof is a proof, nor that it proves a proposition about any number. One can consider the proposition as what is proved by the proof and hold that its infinite applicability can be *stated* by the proposition.

Finally, in Chapter 6, we saw there were important parallels between Wittgenstein's philosophy of mathematics and his philosophy generally. It is unsurprising to find his calculus conception of language developing out of his reflections of mathematical systems and that this intrasystemic characterization of mathematics, together with his mathematical verificationism, characterized Wittgenstein's philosophy of mathematics into the 1930's. But, paralleling the developments in his philosophy of language, Wittgenstein came to think that application, in addition to proof, is an important part of mathematics. This is connected with his new anthropological view of language; seeing language as essentially embedded in our lives, interconnected with other non-linguistic behaviour, he comes to see meaning as use. In mathematics this use is its application, which is essential for at least paradigmatic mathematics. It is further supported by his more detailed thoughts on rule-following and, in particular, what characterizes mathematical rule-following, together with his reflections on the insufficiency of the mere (correct) manipulation of signs to mathematical understanding. Thus, just as verificationism is restricted in the philosophy of language, and even the calculus conception is seen only to represent *a part* of the use of language, verificationism is correspondingly restricted in the philosophy of mathematics and Wittgenstein admits two sources of meaning for mathematics: proof and application. This serves as a further example of how closely interconnected Wittgenstein's reflections were.

With all of this in mind, it is worth considering what lessons can be learned from Wittgenstein's intermediate period more generally. First, the amount of effort he now devotes to the philosophy of mathematics is noteworthy. We have already suggested that this resulted from historical contingencies, such as his meetings with the Vienna Circle and his attending of Brouwer's lecture. But it must also be that in this new work Wittgenstein found problems that genuinely bothered him. The philosophy of mathematics was no longer an afterthought, a mere footnote to another project, but worthy of his sustained attention. Given that he made advancements towards solutions to these problems, study of the intermediate period is necessary also for a proper understanding of Wittgenstein's philosophy of mathematics even into his later work. For example, his views on mathematical infinity are largely, if not entirely, worked out in the intermediate period, and continue to be of lasting importance.

What Wittgenstein reports about his own work does indeed seem to be borne out by the study of his intermediate period: he laboured over many different topics, and his thought was often forced rapidly between them (*PI*, preface). While the *Philosophical Investigations* can be seen as the condensed, developed thoughts of a mature thinker, the intermediate period shows all of the work that led up to them. Study of the intermediate period not only shows the development of many of his new insights, but also all of the tentative discussions, ideas that weren't pursued, and failed lines of thought. This is not only important for Wittgenstein scholars but, as evidenced by his reflections on inductive proof (in light of his comments on *On Certainty*), there are even potential insights that could still be applied to other philosophical problems. Failed lines of thought can stand as possible challenges to philosophers pursuing similar lines of thinking today (e.g. phenomenology and the calculus conception of language). Moreover, given the importance of the genetic method to Wittgenstein's later work, the intermediate period can also serve as a record of exactly how the genetic method was employed by one of the most accomplished thinkers of all time. Wittgenstein always thought it important not to spare another thinker the trouble of thinking, but one’s own thinking can be doubtless improved through the careful study of someone who struggled with many of the most important philosophical problems and who systematically rooted out confusions in his own thought.

Appendix: Engelmann and Hacker on the Origins of the Verification Principle

The analysis given in Section 3.1.5 calls into question both Engelmann's and Hacker's interpretation of the development of the verification principle. We will summarize their respective positions before we proceed to evaluate them.

Engelmann claims that, while maintaining the thesis that one compares propositions with reality and in order to preserve the determinacy of sense of the *Tractatus* (that every proposition must describe reality completely and its truth value be true or false), it now becomes necessary to speak of *methods of verification*. It was thought that logical relationships such as those involving the 'forms'/'space' of colour or space could be analysed using function and argument, but this was not possible. Instead, a complete analysis/verification of a meaningful proposition takes place in the phenomenological language which is designed to adequately capture the logic of the 'forms'. But this phenomenological language calls for *particular methods* of investigation into all the different logical relationships that hold for each 'form'. And these, in turn, are 'made explicit in the way that we verify a proposition' (Engelmann 2013, 27). Propositions about colours require methods of verification unlike those about sounds and, at the same time, will show what can't be verified at all (because they are nonsense, e.g. 'A is red and A is blue') (Engelmann 2013, 27-28). What counts as nonsense and sense no longer is encompassed by a general *a priori* logic, but requires specific understanding of all the different 'spaces' to which propositions can relate. For ascertaining a statement's truth will always be relative to the 'space' that is being investigated and what in principle can and can't be verified will be relative to that 'space' also. When methods of verification are taken seriously the rules governing sense and nonsense in relation to the different 'spaces' become clear; when it becomes clear why something is unverifiable, new rules of the individual 'spaces' are made perspicuous (Engelmann 2013, 28). The principle of verification, together with the other developments in Wittgenstein thought, is a necessary supplement to the purely *a priori* logic of the *Tractatus*, which failed to adequately represent the 'forms'. It preserves the determinacy of sense by clearly ruling out that which can't be verified (i.e., nonsense) (Engelmann 2013, 28).

In his explanation of what accounts for the 'great gulf' between *Tractatus* 4.024 and verificationism, Hacker lists two fundamental points: the demise of *both* logical atomism and the metaphysical harmony between language and the world. Here the 'gap' Hacker has in mind is an explanatory one. It is the challenge to give an accurate account of the continuity between 4.024 and the verification principle while also explaining the important difference(s) between the two. Hacker's explanation rightly turns on the role of both in the context of Wittgenstein's overall philosophy at the relevant times. In the *Tractatus*, to understand a proposition was to know what is the case if it is true. And to tell if the proposition was true would have seemingly been a simple matter: one simply checks whether the meanings (the objects in the world) of the simple names in the proposition are concatenated as the proposition says they are. But once the metaphysical scaffolding of the *Tractatus* collapsed, it was admitted that, along with the world not consisting of facts and those facts not being made up of objects that are meanings, simple names are now to be explained by reference to samples that belong to the method of representation. In order still to maintain that propositions are to be compared with reality, exactly *how* they are compared must be a matter pertaining to their meaning. If meanings of words are established by rules within a language (often with reference to samples) and are not objects in the world (which would allow for a connection between language and the world), then a new explanation is needed for what counts as *p* being the case and what counts as knowing *p* is the case. For talking about the concatenation of meanings in the world can no longer function as an explanation. This new account is provided by knowing exactly how a proposition is to be compared with reality (i.e., its method of verification). 'The method of comparing a sample with reality must be internally related to the meaning of the proposition in question' (Hacker 1986, 140). To know what is the case for *p* to be true now becomes at least part of the explanation of *p*'s meaning. If red were an object, then to know the meaning of 'red' (i.e. that object) would be to already know how the proposition 'A is red' is to be compared with the world. But if the meaning of 'red' is given by a rule within language (e.g. that 'This [point at red] is red'), then, in addition, to know what 'A is red' means one must know how to verify it. There must be an internal connection between meaning and verification in order to preserve the possibility of establishing what it is for *p* to be the case (Hacker 1986, 139-140).

In Engelmann's case, it is doubtful that the principal reason for the equivalence of meaning with a method of verification was to deal with the problems raised by Ramsey's criticisms of the *Tractatus* (at least if he means this in the context of the phenomenological language) (Engelmann 2013, 27).[251] As Engelmann himself states (2013, 41), the phenomenological language was given up by October 1929. Thus, if Engelmann were correct, we would expect numerous passages in which the verification principle is mentioned in the context of the phenomenological language. However, in 1929, the vast majority of the comments relate to the philosophy of mathematics. Indeed, even the quotation Engelmann uses (2013, 26) to justify his position is found in the context of a discussion of the philosophy of mathematics (MS 107, 143). The one comment that does relate to a very general discussion of the phenomenological language (MS 105, 120) is too general to conclude what Engelmann does.[252] Another problem for Engelmann's interpretation is the fact that it is not easy to understand why Wittgenstein, while employing the phenomenological language, would speak of different *methods* of verification. As Engelmann himself admits (2013, 43-44), the phenomenological language was meant to be *the* method/notation in which all ordinary language sentences would be verified.[253] There would be no need then, at this point, to speak of different methods.[254] Thus, *at the very least*, the verification principle was not the initial way of dealing with Ramsey's criticisms but, as argued, a subsequent way of dealing with the problems *after* the demise of the phenomenological language and the initial development of grammar. Thus, it is a way of dealing with the problems of

[251] There is an ambiguity as to whether Engelmann does see the verification principle being used in the construction of the phenomenological language. On the one hand, he deals with verificationism first in the general context of explaining the phenomenological language (Section 1.2.4, 'Verification and Sense' appears between the sections 'A Complementary Notation Grounded in the Structure of Experience' and 'Phenomenological (primary) language: a draft and method'). On the other hand, he clearly states that the 'post-1929 equivalence of a proposition having sense and having a method of verification should be seen as a response to the problems related to Ramsey's objection' (Engelmann 2013, 27). Since he suggested this equivalence also happened in 1929 itself, I am inclined to think that the 'post' in this case is meant to *include* 1929. Given where he chooses to discuss the topic in his book, it is natural to suppose that he does see the verification principle as being a part of, or at least happening simultaneously with, the development of the phenomenological language.

[252] Indeed, in this context, all Wittgenstein suggests is that a phenomenological language would be verified in the present.

[253] Engelmann could distinguish between what is required for verification within the notation, and what is required in order to set up the notation itself (the latter which might require different 'methods'). Of course, the details of this would have to be explained. He admits a similar distinction when he discusses different uses of 'description' in the context of the phenomenological notation (2013, 14-15), but Engelmann does not make precisely this distinction when he discusses verification within the context of the phenomenological notation (only seemingly referring to the former). And Wittgenstein is aware of the problems with establishing the phenomenological notation (in terms of its status as *a priori*/*a posteriori*), so it is unlikely that he is countenancing a similar approach when he starts presenting the changes to his philosophy after the break with the phenomenological notation.

[254] Hacker makes this same point in relation to the *Tractatus*. While it is correct that there is only one method of verification in the *Tractatus*, he overlooks the fact that it is very likely the phenomenological language that Wittgenstein is reacting to at this point in his thought (although the *Tractatus* was also wrong).

different ‘spaces’, but only after the phenomenological language had already been attempted as a solution.

Hacker’s view (1986, 134-145) also contains problems. First, like Engelmann, Hacker does not note the connection between the verification principle and the philosophy of mathematics. This leads him to put undue focus on the verification principle as the main feature meant to preserve parts of the *Tractatus*. While it is apparent that the verification principle is meant to preserve certain parts of the *Tractatus*, Hacker does not correctly note what features these are, or that practical applications of the principle are made well before this. While Wittgenstein, in the context of his early reassessment of his *Tractatus* views, gave up the possibility of specifying the shared form between language and the world (and arguably the possibility of specifying the connection between language and the world), he did not give up the idea of a connection between language and the world, and likely even the shared form between language and the world, until 1930. As suggested, this happens with the autonomy of grammar arguments (which develop starting in August 1930). Thus, as late as early 1930, we find Wittgenstein saying that the signs must ultimately have a connection with reality in order to be meaningful. Yet, practical applications of the verification principle have already been used since early in 1929 and the appeal to, and centrality and use of, the verification principle is already in decline by the time Wittgenstein starts to give up the connection between language and the world. Thus, Hacker’s interpretation that gives the verification principle central importance in preserving the *Tractatus* framework in the context of Wittgenstein’s change from simple objects to seeing those objects as actually samples that belong to language is not plausible.[255] However, it does make sense that a method of verification would be appealed to as a way of logically distinguishing between propositions; and, while this began in the philosophy of mathematics, it makes sense that it would also be used when it comes to viewing propositions as forming systems, in order to distinguish between the different systems and the different rules that relate to the systems. Moreover, it makes sense that the verification principle would be required to preserve the application of language and, together with this, what is necessary for language to be more than a

[255] Hacker considers the verification principle in relation to the autonomy of grammar, which is anachronistic. As I see it, since the centrality of the verification principle happens in late 1929, at this point Wittgenstein still holds to the idea that language and the world are connected (albeit the connection – and the form – may not be possible to specify). Thus, the verification principle would seem to be a tool for bypassing this problem.

mere game. In contrast to Hacker's position, it is the verification principle that importantly preserves the applicability of language, regardless of whether the connection between language and the world or shared form (which Wittgenstein still likely holds at this point) is specifiable. While the connection between language and the world is still thought to be a necessary part of language, it is clear that the verification principle is what now guarantees the applicability of language to the world.

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